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Nonvanishing of $\mathrm{Gl}(2)$ automorphic L functions at $1/2$

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Let k be a number field with ring of adeles $\mathbb{A}$, let B be a quaternion algebra defined over k, and let $G = B^\times$. Let π be an infinite dimensional irreducible cuspidal automorphic representation of $G(\mathbb{A})$. Then the vanishing or nonvanishing of $L(1/2, \pi)$ has been conjectured or shown to be equivalent to conditions of considerable interest in number theory or automorphic representation theory. For example, if $k = \mathbb{Q}$, $B = \mathrm{M}_{2\times 2}$ and π corresponds to an elliptic curve E defined over $\mathbb{Q}$, then Birch and Swinnerton-Dyer conjectured that the order of vanishing of $L(s, \pi)$ at $1/2$ is the rank of the torsion free part of $E(\mathbb{Q})$. To take another example, if the central character of π is trivial, then Waldspurger showed in [W1] and [W2] that the nonvanishing of $L(1/2, \pi)$ is equivalent to the nonvanishing of the theta lift of π to $\mathrm{Mp}(2, \mathbb{A})$, the metaplectic cover of $\mathrm{Sl}(2, \mathbb{A})$. In this paper, again when the central character of π is trivial, we show how another condition is related to the nonvanishing of $L(1/2, \pi)$. We also consider the implications of our results for modular forms.

Our first main result relates the nonvanishing of $L(1/2, \pi)$ to the existence of another irreducible cuspidal automorphic representation σ of $G(\mathbb{A})$ along with an embedding of π in $\sigma \otimes \sigma^\vee$. For a precise account we need some notation. If σ is an infinite dimensional irreducible cuspidal automorphic representation of $G(\mathbb{A})$, define the trilinear form

$$T(\sigma \otimes \sigma^\vee \otimes \pi) : \sigma \otimes \sigma^\vee \otimes \pi \to \mathbb{C}$$

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by

$$f_1 \otimes f_2 \otimes f \mapsto \int_{\mathbb{A}^\times G(k)\backslash G(\mathbb{A})} f_1(g) f_2(g) f(g)\, dg.$$

For the remainder of this introduction, assume that the central character of π is trivial. In Theorem 1 we prove that if there exists an infinite dimensional irreducible cuspidal automorphic representation σ of $G(\mathbb{A})$ such that $T(\sigma \otimes \sigma^\vee \otimes \pi) \neq 0$, then $L(1/2, \pi) \neq 0$. We also show that the converse holds in the case $G \neq \mathrm{Gl}(2)$. To prove Theorem 1, we use the above mentioned criterion of Waldspurger and theta correspondences in the form of certain seesaw pairs. Theorem 1 is proven in Sect. 1. In Sect. 1 we also discuss some possible similar results and the connection of Theorem 1 to the Jacquet conjecture.

In the case $G = \mathrm{Gl}(2)$, the first part of Theorem 1 has a consequence for modular forms. As an illustration of the more general result of Sect. 3, suppose that N is a nonnegative integer, k is a positive even integer and $F_1 \in S_{k/2}(\Gamma_0(N))$ is an eigenform of the Hecke operators $T(p)$ for $p \nmid N$. Then $F_1^2 \in S_k(\Gamma_0(N))$. The above result implies that if $F \in S_k(\Gamma_0(N))$ is a new form and $\langle F, F_1^2 \rangle_{\Gamma_0(N)\backslash\mathfrak{H}} \neq 0$ then $L(k/2, F) \neq 0$.

Under some hypotheses, in the case $G = \mathrm{Gl}(2)$, our second main result gives a necessary and sufficient condition for $T(\pi \otimes \pi(\chi) \otimes \pi(\chi)^\vee) \neq 0$ for some χ, where χ is a Hecke character of a quadratic extension E of k that does not factor through N_k^E, and $\pi(\chi)$ is the irreducible cuspidal automorphic representation of $\mathrm{Gl}(2, \mathbb{A})$ associated to χ. Suppose such a χ exists. By Theorem 1, we have $L(1/2, \pi) \neq 0$. Using Theorem 1 again, we show that $L(1/2, \pi \otimes \omega_{E/k}) \neq 0$. See Lemma 2. We prove in Theorem 2 that for many π, these two necessary conditions are also sufficient. To prove this result, we use another seesaw. See Lemma 1. By this lemma, our trilinear form is related to the product of two integrals over $\mathbb{A}^\times E^\times \backslash \mathbb{A}_E^\times$. These integrals can be analyzed using the main result of [W3], and an idea from [H]. This result is described in Sect. 2.

Our final main result applies Theorem 2 to modular forms. The key step in making the transition from the abstract situation of Theorem 2 to modular forms is to show that the local trilinear forms do not vanish on certain pure tensors formed from a combination of new and old vectors. In particular, we need more information than is contained in [GP], where the case of a triple tensor product of unramified representations or a triple tensor product of special representations is treated. We also need to generalize the description of the new vector in a Kirillov model from [GP] to the case when the central character is not trivial. The result on trilinear forms appears in Lemma 3, and the new vector in a Kirillov model is described in the discussion preceding the lemma.

We will use the following notation and definitions. Given a topological group, a character of the group is a continuous homomorphism from the group to $\mathbb{C}^\times$, and the trivial character of the group is denoted by $\mathbf{1}$. Throughout the paper, k is a number field, with group of adeles $\mathbb{A}$, B is quaternion algebra defined over k, and $G = B^\times$. The notation for trilinear forms will be as above. Let v be a place of k, and let π be an irreducible admissible representation of $G(k_v)$ or an irreducible cuspidal automorphic representation of $G(\mathbb{A})$. The central character of π will be denoted by ω_π, and the contragredient of π by $\pi^\vee$. If π is an infinite dimensional irreducible cuspidal automorphic representation of $G(\mathbb{A})$, let $\mathrm{JL}(\pi)$ be the the infinite dimensional irreducible cuspidal automorphic representation of $\mathrm{Gl}(2,\mathbb{A})$ associated to π by the Jacquet-Langlands correspondence, as in Theorem 10.5 of [Ge]. If τ is an irreducible cuspidal automorphic representation of $\mathrm{Gl}(2,\mathbb{A})$, and τ lies in the Jacquet-Langlands correspondence with respect to $G(\mathbb{A})$, let $\mathrm{JL}(\tau)$ be the associated infinite dimensional irreducible cuspidal automorphic representation of $G(\mathbb{A})$; otherwise, let $\mathrm{JL}(\tau) = 0$. If π is an infinite dimensional irreducible cuspidal automorphic representation of $G(\mathbb{A})$, then $L(s,\pi)$ is defined to be $L(s,\mathrm{JL}(\pi))$. Let E be a quadratic extension of k. We denote the nontrivial Hecke character of $\mathbb{A}^\times$ that is trivial on $k^\times\,\mathrm{N}_k^E(\mathbb{A}_E^\times)$ by $\omega_{E/k}$. If χ is a Hecke character of $\mathbb{A}_E^\times$ that does not factor through N_k^E, then $\pi(\chi)$ is the irreducible cuspidal automorphic representation of $\mathrm{Gl}(2,\mathbb{A})$ associated to χ as in Theorem 7.11 of [Ge]. If F is a nonarchimedean local field, then Sp is the special representation of $\mathrm{Gl}(2,F)$, i.e., the irreducible quotient of $\rho(|\;|^{-1/2},|\;|^{1/2})$; the last representation is defined as in [Ge]. If D is a quaternion algebra, the canonical involution of D will be denoted by $*$ and the reduced norm N and trace T of D are defined by $\mathrm{N}(x) = xx^*$ and $T(x) = x + x^*$. Let $(U, (\;,\;))$ be a nonzero, nondegenerate finite dimensional symmetric or symplectic bilinear space over a field F not of characteristic two. An F linear map $T : U \to U$ is called a similitude if there exists $\lambda \in F^\times$ such that $(Tu, Tu') = \lambda(u,u')$ for $u, u' \in U$; in this case, λ is uniquely determined, and we write $\lambda(T) = \lambda$. We denote the group of all similitudes by $\mathrm{GO}(U)$ or $\mathrm{GSp}(U)$, depending on whether U is symmetric or symplectic, respectively. If U is symmetric and of dimension $2n$, then we denote the subgroup of $T \in \mathrm{GO}(U)$ such that $\det(T) = \lambda(T)^n$ by $\mathrm{GSO}(U)$. The notation for modular forms will be as in [Sh]. Finally, if M is a positive integer, we let

$$W_M = \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix}.$$

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1. The general case

In this section we prove Theorem 1. At the end of the section we make some remarks about the proof and possible analogous results. We also discuss the relationship between Theorem 1 and the Jacquet conjecture.

To prove Theorem 1 we will use a certain seesaw from the theory of the theta correspondence. For an outline of the global theory of the theta correspondence for isometries and similitudes, the reader can consult [HPS] and Sect. 2 of [HST], respectively. For more about seesaws, see [K].

Theorem 1. *Let π be an infinite dimensional irreducible cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character. If there there exists an infinite dimensional irreducible cuspidal automorphic representation σ of $G(\mathbb{A})$ such that $T(\sigma \otimes \sigma^\vee \otimes \pi) \neq 0$, then $L(1/2, \pi) \neq 0$. Conversely, if $L(1/2, \pi) \neq 0$ and $G \neq \mathrm{Gl}(2)$, then there exists an infinite dimensional irreducible cuspidal automorphic representation σ of $G(\mathbb{A})$ such that $T(\sigma \otimes \sigma^\vee \otimes \pi) \neq 0$.*

Proof. To define the seesaw used in the proof, let X be the symmetric bilinear space defined over k with underlying space B and symmetric bilinear form $(\ ,\ )$ corresponding to $-\mathrm{N}$, where N the reduced norm of B. Let X_0 be the subspace of X of elements x such that $x^* = x$, and let X_1 be the subspace of X of trace zero elements. Then there is an orthogonal decomposition $X = X_0 \perp X_1$. Let Y be the nondegenerate two dimensional symplectic bilinear space over k. We write $\mathrm{Sl}(2) = \mathrm{Sp}(Y)$ and $\mathrm{Gl}(2) = \mathrm{GSp}(Y)$. Consider the symplectic spaces $W = X \otimes Y$, $W_0 = X_0 \otimes Y$ and $W_1 = X_1 \otimes Y$ defined over k. Via the obvious inclusions, $(\mathrm{O}(X), \mathrm{Sl}(2))$ is a dual pair in $\mathrm{Sp}(W)$. Via the inclusion coming from the orthogonal decomposition $W = W_0 \perp W_1$, $(\mathrm{O}(X_0) \times \mathrm{O}(X_1), \mathrm{Sl}(2) \times \mathrm{Sl}(2))$ is also a dual pair in $\mathrm{Sp}(W)$. Since $\mathrm{Sl}(2)$ is contained in $\mathrm{Sl}(2) \times \mathrm{Sl}(2)$ and $\mathrm{O}(X_0) \times \mathrm{O}(X_1)$ is contained in $\mathrm{O}(X)$, our two dual pairs are seesaw dual pairs, which is illustrated by the diagram:

$$\begin{array}{ccc} \mathrm{Sl}(2) \times \mathrm{Sl}(2) & & \mathrm{O}(X) \\ \uparrow & \times & \uparrow \\ \mathrm{Sl}(2) & & \mathrm{O}(X_0) \times \mathrm{O}(X_1) \end{array}.$$

Let q be the projection of the metaplectic group $\mathrm{Mp}(W(\mathbb{A}))$ onto $\mathrm{Sp}(W(\mathbb{A}))$. Since the dimension of X is even, it follows that the inverse images of $\mathrm{Sl}(2, \mathbb{A})$ and $\mathrm{O}(X(\mathbb{A}))$ in $\mathrm{Mp}(W(\mathbb{A}))$ are split. It follows that the inverse image of $\mathrm{O}(X_0(\mathbb{A})) \times \mathrm{O}(X_1(\mathbb{A}))$ is also split. However, the inverse image of $\mathrm{Sl}(2, \mathbb{A}) \times \mathrm{Sl}(2, \mathbb{A})$ is not split.

In addition, consider the dual pairs $(\mathrm{O}(X_0), \mathrm{Sl}(2))$ in $\mathrm{Sp}(W_0)$ and $(\mathrm{O}(X_1), \mathrm{Sl}(2))$ in $\mathrm{Sp}(W_1)$. The inverse images of $\mathrm{O}(X_0(\mathbb{A}))$ and $\mathrm{O}(X_1(\mathbb{A}))$

are split, while those of $\mathrm{Sl}(2,\mathbb{A})$ are commonly isomorphic to $\mathrm{Mp}(2,\mathbb{A})$. Moreover, there is an epimorphism p of $\mathrm{Mp}(2,\mathbb{A})\times\mathrm{Mp}(2,\mathbb{A})$ onto $q^{-1}(\mathrm{Sl}(2,\mathbb{A})\times\mathrm{Sl}(2,\mathbb{A}))$ such that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Mp}(2,\mathbb{A})\times\mathrm{Mp}(2,\mathbb{A}) & \xrightarrow{p} & q^{-1}(\mathrm{Sl}(2,\mathbb{A})\times\mathrm{Sl}(2,\mathbb{A})) \\
\downarrow & & \downarrow \\
\mathrm{Sl}(2,\mathbb{A})\times\mathrm{Sl}(2,\mathbb{A}) & \xrightarrow{\mathrm{id}} & \mathrm{Sl}(2,\mathbb{A})\times\mathrm{Sl}(2,\mathbb{A})
\end{array}.$$

Here, the vertical maps are projections.

We can summarize the situation by the following diagram:

$$\begin{array}{ccc}
\mathrm{Mp}(2,\mathbb{A})\times\mathrm{Mp}(2,\mathbb{A}) & & \\
\xrightarrow{p} q^{-1}(\mathrm{Sl}(2,\mathbb{A})\times\mathrm{Sl}(2,\mathbb{A})) & & \mathrm{O}(X(\mathbb{A})) \\
i\uparrow & \times & \uparrow \\
\mathrm{Sl}(2,\mathbb{A}) & & \mathrm{O}(X_0(\mathbb{A}))\times\mathrm{O}(X_1(\mathbb{A}))
\end{array}.$$

Next, we define the representations that will be used to construct our theta lifts. Fix a nontrivial additive unitary character ψ of $\mathbb{A}/k$. Fix a symplectic basis e,f for Y. Fix a basis $e_0=1$ for X_0, and fix an orthogonal basis e_1,e_2,e_3 for X_1. Then $e_0,\dots,e_3$ is an orthogonal basis for X, and

$$\frac{1}{(e_0,e_0)}e_0\otimes e,\dots,\frac{1}{(e_3,e_3)}e_3\otimes e, e_0\otimes f,\dots,e_3\otimes f$$

is a symplectic basis for W. This basis also contains obvious symplectic bases for W_0 and W_1. As usual, the above basis for W determines a complete polarization of W, and we have an identification of the Langrangian spanned by $e_0\otimes f,\dots,e_3\otimes f$ with X. Similar comments apply to W_0 and W_1. Let $(r_0,L^2(X_0(\mathbb{A})))$, $(r_1,L^2(X_1(\mathbb{A})))$ and $(r,L^2(X(\mathbb{A})))$ be the Schrödinger models of the Weil representations of $\mathrm{Mp}(W_0(\mathbb{A}))$, $\mathrm{Mp}(W_1(\mathbb{A}))$ and $\mathrm{Mp}(W(\mathbb{A}))$ defined with respect to ψ, and the above symplectic bases, respectively [Rao]. Denote the composition of r with the natural maps of $q^{-1}(\mathrm{Sl}(2,\mathbb{A})\times\mathrm{Sl}(2,\mathbb{A}))\times(\mathrm{O}(X_0(\mathbb{A}))\times\mathrm{O}(X_1(\mathbb{A})))$ and $\mathrm{Sl}(2,\mathbb{A})\times\mathrm{O}(X(\mathbb{A}))$ into $\mathrm{Mp}(W(\mathbb{A}))$ by ω and ω', respectively. Then the seesaw property holds: the restrictions of ω and ω' to $\mathrm{Sl}(2,\mathbb{A})\times(\mathrm{O}(X_0(\mathbb{A}))\times\mathrm{O}(X_1(\mathbb{A})))$ are identical. In addition, we have a decomposition of ω. Denote the composition of r_0 and r_1 with the natural maps of $\mathrm{Mp}(2,\mathbb{A})\times\mathrm{O}(X_0(\mathbb{A}))$ into $\mathrm{Mp}(W_0(\mathbb{A}))$ and $\mathrm{Mp}(2,\mathbb{A})\times\mathrm{O}(X_1(\mathbb{A}))$ into $\mathrm{Mp}(W_1(\mathbb{A}))$ by ω_0 and ω_1, respectively. Then the map from the tensor product of Hilbert spaces $L^2(X_0(\mathbb{A}))\otimes_{\mathbb{C}}L^2(X_1(\mathbb{A}))$ to $L^2(X(\mathbb{A}))$ determined by mapping $\varphi_0\otimes\varphi_1$ to φ with $\varphi(x_0\oplus x_1)=\varphi_0(x_0)\varphi_1(x_1)$ is an isomorphism of $\mathbb{C}$ vector spaces such that

$$\omega(p(g_0,g_1),(h_0,h_1))\varphi(x_0\oplus x_1)=\omega_0(g_0,h_0)\varphi_0(x_0)\omega_1(g_1,h_1)\varphi_1(x_1)$$

for $\varphi_0 \otimes \varphi_1 \in L^2(X_0(\mathbb{A})) \otimes_{\mathbb{C}} L^2(X_1(\mathbb{A}))$, $(g_0, g_1) \in \mathrm{Mp}(2, \mathbb{A}) \times \mathrm{Mp}(2, \mathbb{A})$ and $(h_0, h_1) \in \mathrm{O}(X_0(\mathbb{A})) \times \mathrm{O}(X_1(\mathbb{A}))$.

In fact, to define the theta lifts we will use certain subspaces of smooth vectors. If v is a finite place of k let $\mathcal{S}(X(k_v))$ be the $\mathbb{C}$ subspace of $L^2(X(k_v))$ of locally constant, compactly supported functions. If v is real, let $\mathcal{S}(X(k_v))$ be the subspace of functions of the form

$$p(x_0, \ldots, x_3) \exp(-\frac{|c|}{2}(x_0^2 + \cdots + x_3^2))$$

where p is a polynomial in four variables with $\mathbb{C}$ coefficients, and $\psi_v(x) = \exp(icx)$; this is the subspace corresponding to the polynomials in the Fock model of the Weil representation of $\mathrm{Mp}(W(k_v))$ [A]. If v is complex we make a similar definition. Let $\mathcal{S}(X(\mathbb{A}))$ be the restricted direct product of the $S(X(k_v))$. Similar definitions hold for W_0 and W_1. We note that under the above isomorphism, the ordinary tensor product $\mathcal{S}(X_0(\mathbb{A})) \otimes_{\mathbb{C}} \mathcal{S}(X_1(\mathbb{A}))$ is mapped onto $\mathcal{S}(X(\mathbb{A}))$.

We define the appropriate theta kernels. For $\varphi \in \mathcal{S}(X(\mathbb{A}))$, $(g', h') \in \mathrm{Sl}(2, \mathbb{A}) \times \mathrm{O}(X(\mathbb{A}))$ and $(g, h) \in q^{-1}(\mathrm{Sl}(2, \mathbb{A}) \times \mathrm{Sl}(2, \mathbb{A})) \times (\mathrm{O}(X_0(\mathbb{A})) \times \mathrm{O}(X_1(\mathbb{A})))$ let

$$\theta(g, h; \varphi) = \sum_{x \in X(k)} \omega(g, h)\varphi(x), \quad \theta(g', h'; \varphi) = \sum_{x \in X(k)} \omega'(g', h')\varphi(x) .$$

If $(g, h) = (g', h')$ is in $\mathrm{Sl}(2, \mathbb{A}) \times (\mathrm{O}(X_0(\mathbb{A})) \times \mathrm{O}(X_1(\mathbb{A})))$, these functions clearly agree. For $\varphi_0 \in \mathcal{S}(X_0(\mathbb{A}))$, $\varphi_1 \in \mathcal{S}(X_1(\mathbb{A}))$, $g \in \mathrm{Mp}(2, \mathbb{A})$, $h_0 \in \mathrm{O}(X_0(\mathbb{A}))$ and $h_1 \in \mathrm{O}(X_1(\mathbb{A}))$ let

$$\theta(g, h_0; \varphi_0) = \sum_{x \in X_0(k)} \omega_0(g, h_0)\varphi_0(x) ,$$

$$\theta(g, h_1; \varphi_1) = \sum_{x \in X_1(k)} \omega_1(g, h_1)\varphi_1(x) .$$

If $\varphi_0 \in \mathcal{S}(X_0(\mathbb{A}))$, $\varphi_1 \in \mathcal{S}(X_1(\mathbb{A}))$, $g_0, g_1 \in \mathrm{Mp}(2, \mathbb{A})$, $h_0 \in \mathrm{O}(X_0(\mathbb{A}))$, $h_1 \in \mathrm{O}(X_1(\mathbb{A}))$, and $\varphi \in \mathcal{S}(X(\mathbb{A}))$ corresponds to $\varphi_0 \otimes \varphi_1$, then

$$\theta(p(g_0, g_1), (h_0, h_1); \varphi) = \theta(g_0, h_0; \varphi_0)\theta(g_1, h_1; \varphi_1) .$$

There is a characterization of the right hand side of the above diagram that we will use. We have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{GSO}(X(\mathbb{A})) & \xleftarrow{\sim} & (G(\mathbb{A}) \times G(\mathbb{A}))/\mathbb{A}^\times \\ \uparrow & & \uparrow \\ \mathrm{SO}(X_1(\mathbb{A})) \cong \mathrm{SO}(X_0(\mathbb{A})) \times \mathrm{SO}(X_1(\mathbb{A})) & \xleftarrow{\sim} & G(\mathbb{A})/\mathbb{A}^\times \end{array} .$$

Here, the top map is defined by $\rho(g,g')x = gxg'^{-1}$, the bottom map is defined by $\rho(g)x = gxg^{-1}$, and the second vertical map takes g to (g,g); note that $\mathrm{SO}(X_0(\mathbb{A}))$ is trivial.

Next, we recall the theta correspondences and seesaw identity associated to our situation. Fix an $\mathrm{SO}(X_1(\mathbb{A}))$ right invariant measure on $\mathrm{SO}(X_1(k))\backslash \mathrm{SO}(X_1(\mathbb{A}))$. Let $f \in \pi$ and let $\varphi \in \mathcal{S}(X(\mathbb{A}))$. Let $\tilde{f}$ be the function on $\mathrm{SO}(X_0(\mathbb{A})) \times \mathrm{SO}(X_1(\mathbb{A}))$ such that $\tilde{f} \circ \rho = f$. Define $\theta(\tilde{f},\varphi)$ on $q^{-1}(\mathrm{Sl}(2,\mathbb{A}) \times \mathrm{Sl}(2,\mathbb{A}))$ by

$$\theta(\tilde{f},\varphi)(g) = \int_{\mathrm{SO}(X_1(k))\backslash \mathrm{SO}(X_1(\mathbb{A}))} \theta(g,(1,h_1);\varphi)\tilde{f}(h_1)\,dh_1.$$

If φ corresponds to $\varphi_0 \otimes \varphi_1 \in \mathcal{S}(X_0(\mathbb{A})) \otimes_{\mathbb{C}} \mathcal{S}(X_1(\mathbb{A}))$, and $(g_0,g_1) \in \mathrm{Mp}(2,\mathbb{A}) \times \mathrm{Mp}(2,\mathbb{A})$ then

$$\theta(\tilde{f},\varphi)(p(g_0,g_1)) = \theta(g_0,1;\varphi_0)\theta(\tilde{f},\varphi_1)(g_1),$$

where $\theta(\tilde{f},\varphi_1)$ is the theta lift defined in [W1], p. 25, and denoted there by $T_{\psi'}(\varphi_1,g_1,f)$ for the additive character ψ' defined by $\psi'(x) = \psi(x/2)$.

The second theta correspondence will require some more notation. By [HK1], the representation ω' extends to a representation of the group

$$R'(\mathbb{A}) = \{(g,h) \in \mathrm{Gl}(2,\mathbb{A}) \times \mathrm{GO}(X(\mathbb{A})) : \det(g) = \lambda(h)\}.$$

See also [R1]. Here, $\lambda(h)$ is the similitude factor of $h \in \mathrm{GO}(X(\mathbb{A}))$. With the aid of the extended representation we can lift representations of $\mathrm{Gl}(2,\mathbb{A})$. Fix a right $\mathrm{Sl}(2,\mathbb{A})$ invariant measure on $\mathrm{Sl}(2,k)\backslash \mathrm{Sl}(2,\mathbb{A})$. Define, as above, $\theta(g,h;\varphi)$ for $(g,h) \in R'(\mathbb{A})$ and $\varphi \in \mathcal{S}(X(\mathbb{A}))$. Suppose that τ is an irreducible cuspidal automorphic representation of $\mathrm{Gl}(2,\mathbb{A})$. Let $f' \in \tau$ and $\varphi \in \mathcal{S}(X(\mathbb{A}))$. Define $\theta(f',\varphi)$ on $\mathrm{GSO}(X(\mathbb{A}))$ by

$$\theta(f',\varphi)(h) = \int_{\mathrm{Sl}(2,k)\backslash \mathrm{Sl}(2,\mathbb{A})} \theta(g_1g',h;\varphi)f'(g_1g')\,dg_1.$$

Here, $g' \in \mathrm{Gl}(2,\mathbb{A})$ is such that $\det(g') = \lambda(h)$. Note that for $h \in \mathrm{SO}(X(\mathbb{A}))$, $\theta(f',\varphi)(h)$ is the same as the usual theta lift of f', regarded as a cusp form on $\mathrm{Sl}(2,\mathbb{A})$, with respect to φ. Let $\theta(\tau)$ be the $\mathbb{C}$ vector space spanned by the functions $\theta(f',\varphi)$ for $f' \in \tau$ and $\varphi \in \mathcal{S}(X(\mathbb{A}))$. Then $\theta(\tau)\circ\rho = \{F\circ\rho : F \in \theta(\tau)\}$ is the $\mathbb{C}$ vector space spanned by the functions $f_1 \otimes f_2$, where $f_1 \in \mathrm{JL}(\tau)$ and $f_2 \in \mathrm{JL}(\tau)^\vee$, and $f_1 \otimes f_2$ is the function on $(G(\mathbb{A}) \times G(\mathbb{A}))/\mathbb{A}^\times$ defined by $(f_1 \otimes f_2)(g_1,g_2) = f_1(g_1)f_2(g_2)$. For details, see [H] and [S1].

To state the seesaw identity for theta lifts, let τ be an irreducible cuspidal automorphic representation of $\mathrm{Gl}(2,\mathbb{A})$. Because our dual pairs form a seesaw, if τ is an irreducible cuspidal automorphic representation of $\mathrm{Gl}(2,\mathbb{A})$, then we have

$$\langle f', \theta(\tilde{f}, \varphi)\rangle_{\mathrm{Sl}(2)} = \langle \tilde{f}, \theta(f', \varphi)\rangle_{\mathrm{SO}(X_1)}$$

for $f \in \pi$, $f' \in \tau$, and $\varphi \in \mathcal{S}(X(\mathbb{A}))$. Here,

$$\langle f', \theta(\tilde{f}, \varphi)\rangle_{\mathrm{Sl}(2)} = \int_{\mathrm{Sl}(2,k)\backslash \mathrm{Sl}(2,\mathbb{A})} f'(g)\theta(\tilde{f}, \varphi)(g)\, dg,$$

and $\langle \tilde{f}, \theta(f', \varphi)\rangle_{\mathrm{SO}(X_1)}$ is similarly defined. Before beginning the proof, we give a simplified diagram which summarizes our notation.

$$\begin{array}{ccccccc} \theta(\tilde{f};\varphi) & \mathrm{Sl}(2)\times\mathrm{Sl}(2) & & \mathrm{O}(X) & & \theta(f';\varphi)\in\sigma\otimes\sigma^{\vee} & \\ & \uparrow & \times & \uparrow & & & . \\ f'\in\tau & \mathrm{Sl}(2) & & \mathrm{O}(X_0)\times\mathrm{O}(X_1) & & f\in\pi & \end{array}$$

Now assume that there exists an infinite dimensional irreducible automorphic representation σ of $G(\mathbb{A})$ such that $T(\sigma \otimes \sigma^{\vee} \otimes \pi) \neq 0$. Let $T = T(\sigma \otimes \sigma^{\vee} \otimes \pi)$. Then $T(f_1 \otimes f_2 \otimes f) \neq 0$ for some $f_1 \in \sigma$, $f_2 \in \sigma^{\vee}$ and $f \in \pi$. From above, we can write $f_1 \otimes f_2$ as a linear combination of functions $\theta(f', \varphi) \circ \rho$, where $f' \in \tau = \mathrm{JL}(\sigma)$ and $\varphi \in \mathcal{S}(X(\mathbb{A}))$. From $T(f_1 \otimes f_2 \otimes f) \neq 0$ it follows that there exist $f' \in \tau$ and $\varphi \in \mathcal{S}(X(\mathbb{A}))$ such that $\langle f, \theta(f', \varphi)\circ\rho\rangle_G \neq 0$. Here, the integral is over $\mathbb{A}^{\times}G(k)\backslash G(\mathbb{A}) \cong \mathrm{SO}(X_1(k))\backslash \mathrm{SO}(X_1(\mathbb{A}))$. Now:

$$\begin{aligned} \langle f, \theta(f', \varphi)\circ\rho\rangle_G &= \langle \tilde{f}\circ\rho, \theta(f', \varphi)\circ\rho\rangle_G \\ &= \langle \tilde{f}, \theta(f', \varphi)\rangle_{\mathrm{SO}(X_1)} \\ &= \langle f', \theta(\tilde{f}, \varphi)\rangle_{\mathrm{Sl}(2)}. \end{aligned}$$

Since $\langle f, \theta(f', \varphi)\circ\rho\rangle_G \neq 0$, we have $\theta(\tilde{f}, \varphi) \neq 0$. It follows that $\theta(\tilde{f}, \varphi) \neq 0$ for some $\varphi = \varphi_0 \otimes \varphi_1 \in \mathcal{S}(X_0(\mathbb{A})) \otimes_{\mathbb{C}} \mathcal{S}(X_1(\mathbb{A}))$. Hence, $\theta(\tilde{f}, \varphi_1) \neq 0$. By [W1], Théorème 1, and [W2], Proposition 22, this implies that $L(1/2, \pi) \neq 0$.

Next, assume that $L(1/2, \pi) \neq 0$ and $G \neq \mathrm{Gl}(2)$, so that $B(k)$ is a division algebra. Let $f \in \pi$ and $\varphi \in \mathcal{S}(X(\mathbb{A}))$. Assume that $\varphi = \varphi_0 \otimes \varphi_1 \in \mathcal{S}(X_0(\mathbb{A})) \otimes_{\mathbb{C}} \mathcal{S}(X_1(\mathbb{A}))$. We first show that $\theta(\tilde{f}, \varphi)|_{\mathrm{Sl}(2,\mathbb{A})}$ is a cusp form. Let $g \in \mathrm{Sl}(2, \mathbb{A})$, and let $(g_0, g_1) \in \mathrm{Mp}(2, \mathbb{A}) \times \mathrm{Mp}(2, \mathbb{A})$ be such that $p(g_0, g_1) = g$, where g is regarded as an element of $q^{-1}(\mathrm{Sl}(2, \mathbb{A}) \times \mathrm{Sl}(2, \mathbb{A}))$. A computation shows that

$$\begin{aligned} \int_{k\backslash\mathbb{A}} \theta(\tilde{f}, \varphi)(i(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} g))\, dn = \sum_{x\in X_0(k)} \omega_0(g_0, 1)\varphi_0(x) \\ W(\varphi_1, g_1, \tilde{f}, -(x,x)) \\ = \sum_{a\in k} \omega_0(g_0, 1)\varphi_0(a\cdot 1)W(\varphi_1, g_1, \tilde{f}, a^2), \end{aligned}$$

where, as in [W1],

$$W(\varphi_1, g_1, \tilde{f}, t) = \int_{k\backslash\mathbb{A}} \theta(\tilde{f}, \varphi_1)(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} g_1)\psi'(-tn)\, dn$$

for $t \in k$. We assert that $W(\varphi_1, g_1, \tilde{f}, a^2) = 0$ for all $g_1 \in \mathrm{Mp}(2, \mathbb{A})$ and $a \in k$. If $a = 0$, this follows as in [W1], p. 30. Since for all $g_1 \in \mathrm{Mp}(2, \mathbb{A})$ and $a \in k^\times$,

$$W(\varphi_1, g_1, \tilde{f}, a^2) = W(\varphi_1, (\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, 1)g_1, \tilde{f}, 1),$$

it now suffices to show that $W(\varphi_1, g_1, \tilde{f}, 1) = 0$ for $g_1 \in \mathrm{Mp}(2, \mathbb{A})$. As in [W1], p. 29, for $g_1 \in \mathrm{Mp}(2, \mathbb{A})$,

$$W(\varphi_1, g_1, \tilde{f}, 1) = \int_{\mathrm{SO}(X_1(k))\backslash \mathrm{SO}(X_1(\mathbb{A}))} \sum_{x \in X_1(k),\ (x,x)=1} \omega_1(g_1, h_1)\varphi_1(x)\tilde{f}(h_1)\, dh_1.$$

Since $B(k)$ is a division algebra, there exist no $x \in X_1(k)$ such that $(x, x) = 1$, and our claim follows.

Since $L(1/2, \pi) \neq 0$, again by [W2], there exist $f \in \pi$, $\varphi_1 \in \mathcal{S}(X_1(\mathbb{A}))$ and $g_1 \in \mathrm{Mp}(2, \mathbb{A})$ so that $\theta(\tilde{f}, \varphi_1)(g_1) \neq 0$. There exists $\varphi_0 \in \mathcal{S}(X_0(\mathbb{A}))$ and $g_0 \in \mathrm{Mp}(2, \mathbb{A})$ such that $p(g_0, g_1) \in i(\mathrm{Sl}(2, \mathbb{A}))$ and $\theta(g_0, 1; \varphi_0) \neq 0$. By the last paragraph, it follows that if $\varphi = \varphi_0 \otimes \varphi_1$, then $\theta(\tilde{f}, \varphi)|_{\mathrm{Sl}(2,\mathbb{A})}$ is a nonzero cusp form on $\mathrm{Sl}(2, \mathbb{A})$. Hence, there exists an infinite dimensional irreducible automorphic cuspidal representation τ of $\mathrm{Gl}(2, \mathbb{A})$ and $f' \in \tau$ so that $\langle f', \theta(\tilde{f}, \varphi)\rangle_{\mathrm{Sl}(2)} \neq 0$. Since $\langle f', \theta(\tilde{f}, \varphi)\rangle_{\mathrm{Sl}(2)} \neq 0$ for some $f' \in \tau$, we have $\langle \tilde{f} \circ \rho, \theta(f', \varphi) \circ \rho\rangle_G \neq 0$ by an identity from above. This implies that $T(\sigma \otimes \sigma^\vee \otimes \pi) \neq 0$, where $\sigma = \mathrm{JL}(\tau)$. This completes the proof of Theorem 1. $\square$

We make some remarks on the proof of Theorem 1 and a possible analogous result. The argument for the second part of Theorem 1 fails if $G = \mathrm{Gl}(2)$. In this case, we do not always have $W(\varphi_1, g, \tilde{f}, 1) = 0$. To see this, suppose that $B(k)$ is not a division algebra and that the notation is as in proof of Theorem 1. Then by [W1], p. 30,

$$W(\varphi_1, g, \tilde{f}, 1) = \int_{S(\mathbb{A})\backslash \mathrm{SO}(X_1(\mathbb{A}))} \omega_1(g, h_1)\varphi_1(x_1) \int_{S(k)\backslash S(\mathbb{A})} \tilde{f}(sh_1)\, ds dh_1.$$

Here, $S(k)$ and $S(\mathbb{A})$ are the groups of elements in $\mathrm{SO}(X_1(k))$ and SO $(X_1(\mathbb{A}))$, respectively, that fix

$$x_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now $S(k)$ is conjugate to the image under ρ of the subgroup of $\mathrm{Gl}(2,k)$ consisting of the elements of the form

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix},$$

where $a \in k^\times$. As in [W1], for a proper choice of $f \in \pi$ and $h_1 \in \mathrm{SO}(X_1(\mathbb{A}))$, we find that

$$\begin{aligned} \int_{S(k)\backslash S(\mathbb{A})} \tilde{f}(sh_1)\, ds &= \int_{k^\times\backslash\mathbb{A}^\times} f\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\, da \\ &= L(1/2,\pi)\ . \end{aligned}$$

Since we are assuming $L(1/2,\pi) \neq 0$, this implies that for some $f \in \pi$, $\varphi_1 \in \mathcal{S}(X_1(\mathbb{A}))$ and $g \in \mathrm{Mp}(2,\mathbb{A})$ we have $W(\varphi_1, g, \tilde{f}, 1) \neq 0$.

As for the similar result, it may be possible to prove a statement analogous to Theorem 1, with the quadratic base change of an irreducible cuspidal automorphic representation of $\mathrm{Gl}(2,\mathbb{A})$ in place of $\sigma \otimes \sigma^\vee$. This might be obtained by replacing X_0 from the proof of Theorem 1 with a different one dimensional symmetric bilinear space. For example, fix a quadratic extension $K = k(\sqrt{d})$ of k with Galois group $\mathrm{Gal}(K/k) = \{1,\gamma\}$, and consider the symmetric bilinear space X over k with underlying vector space

$$X'(k) = \{\begin{pmatrix} a & b\sqrt{d} \\ c\sqrt{d} & \gamma(a) \end{pmatrix} : a \in K, b, c \in k\}$$

and bilinear form coming from the restriction of $(-1/d) \cdot \det$. We see that $X'(k) = X'_0(k) \perp X'_1(k)$, where again $X'_0(k) = k \cdot I$ and $X'_1(k)$ is the subspace of elements of trace zero. Also, $X'_1(k)$ is isometric to $X_1(k)$ from the proof of Theorem 1. However, $X_0(k)$ and $X'_0(k)$ are not isometric, nor are $X(k)$ and $X'(k)$. The appropriate identifications of the groups of similitudes are now given by

$$\begin{array}{ccc} \mathrm{GSO}(X'(\mathbb{A})) & \xleftarrow{\sim} & (\mathbb{A}^\times \times \mathrm{Gl}(2,\mathbb{A}))/\mathbb{A}^\times \\ \uparrow & & \uparrow \\ \mathrm{SO}(X'_1(\mathbb{A})) \cong \mathrm{SO}(X'_0(\mathbb{A})) \times \mathrm{SO}(X'_1(\mathbb{A})) & \xleftarrow{\sim} & \mathrm{Gl}(2,\mathbb{A})/\mathbb{A}^\times. \end{array}$$

Here, the top map ρ is defined by $\rho(t,g)x = t^{-1}gx\gamma(g)^*$, the bottom map is defined by $\rho(g)x = gxg^{-1}$, and the second vertical map sends g to

$(\det(g), g)$. The inclusion of $\mathbb{A}^\times$ takes x to $(\mathrm{N}_k^K(x), x)$. With these objects playing the role of their counterparts, there may be a development like that in the proof of Theorem 1. However, the theta correspondence from the proof of Theorem 1 that involves the Jacquet-Langlands correspondence will now involve base change to $\mathrm{Gl}(2, \mathbb{A})$. In the general setting, this theta correspondence has not been developed as thoroughly as the one corresponding to the Jacquet-Langlands correspondence. See, however, [C] and [R2].

Finally, we make some remarks about the connection between Theorem 1 and the Jacquet conjecture. Recall that, in our case, the Jacquet conjecture states that if σ is an irreducible cuspidal automorphic representation of $G(\mathbb{A})$, and at every place of k, the local component of π embeds in the local component of $\sigma \otimes \sigma^\vee$, then $T(\sigma \otimes \sigma^\vee \otimes \pi) \neq 0$ if and only if $L(1/2, \mathrm{JL}(\sigma) \otimes \mathrm{JL}(\sigma^\vee) \otimes \mathrm{JL}(\pi)) \neq 0$. The Jacquet conjecture is known in many cases. See [HK2]. Now if σ is an irreducible cuspidal automorphic representation of $G(\mathbb{A})$, then

$$L(s, \mathrm{JL}(\sigma) \otimes \mathrm{JL}(\sigma^\vee) \otimes \mathrm{JL}(\pi)) = L(s, \pi) L(s, \mathrm{JL}(\pi) \otimes \mathrm{JL}(\sigma), r).$$

Here, r is the representation of the L-group $\mathrm{Gl}(2, \mathbb{C}) \times \mathrm{Gl}(2, \mathbb{C})$ of $\mathrm{Gl}(2) \times \mathrm{Gl}(2)$ with underlying vector space $\mathbb{C}^2 \otimes \mathrm{Sym}^2 \mathbb{C}^2$, and action defined by $r(g, g') = g \otimes (1/\det g') \mathrm{Sym}^2 g'$. The action on $\mathbb{C}^2$ is the standard one. As is pointed out in [GK], $L(s, \mathrm{JL}(\pi) \otimes \mathrm{JL}(\sigma), r)$ is entire. If the Jacquet conjecture is true, then the first part of Theorem 1 follows from the above equality of L-functions. It is not clear if the second part of Theorem 1 also follows from the assumption of the Jacquet conjecture. In addition to assuming the Jacquet conjecture, one would need a σ such that $L(1/2, \mathrm{JL}(\pi) \otimes \mathrm{JL}(\sigma), r) \neq 0$.

2. The case of Hecke characters

In this section we consider the case when $B = \mathrm{M}_{2\times 2}$ and $\sigma = \pi(\chi)$, where χ is a Hecke character of a quadratic extension E of k that does not factor through N_k^E, and $\pi(\chi)$ is the irreducible cuspidal automorphic representation of $\mathrm{Gl}(2, \mathbb{A})$ associated to χ. We show that for many π, there exists a χ such that $T(\pi(\chi) \otimes \pi(\chi)^\vee \otimes \pi) \neq 0$ if and only if $L(1/2, \pi) L(1/2, \pi \otimes \omega_{E/k}) \neq 0$. Using another seesaw, Lemma 1 reduces the analysis of such trilinear forms to the investigation of some period integrals over $\mathbb{A}^\times E^\times \backslash \mathbb{A}_E^\times$. These integrals can be understood using [W3] and an idea from [H]. The seesaws of Theorem 1 and Lemma 1 are quite analogous and of the same general type. However, in contrast to the seesaw in Theorem 1, in Lemma 1 the trilinear form appears on the symplectic side of the seesaw.

Lemma 1. *Let D be a quaternion algebra defined over k and let E be a quadratic extension of k contained in $D(k)$ as a k subalgebra. Let*

$\mathrm{Gal}(E/k) = \{1, \gamma\}$. *Let* χ_0 *and* χ_1 *be Hecke characters of* $\mathbb{A}_E^\times$ *that do not factor through* N_k^E. *Let* π *be an irreducible cuspidal automorphic representation of* $\mathrm{Gl}(2, \mathbb{A})$. *Assume that* $\omega_\pi \chi_0|_{\mathbb{A}^\times} \chi_1|_{\mathbb{A}^\times} = 1$. *If*

$$\int_{\mathbb{A}^\times E^\times \backslash \mathbb{A}_E^\times} f_1(x)\chi_0(x)\chi_1(x)\,dx \cdot \int_{\mathbb{A}^\times E^\times \backslash \mathbb{A}_E^\times} f_2(y)\chi_0(y)^{-1}\chi_1(\gamma(y))^{-1}\,dy \neq 0$$

for some $f_1 \in \mathrm{JL}(\pi)$, $f_2 \in \mathrm{JL}(\pi)^\vee$, *then*

$$T(\pi(\chi_0) \otimes \pi(\chi_1) \otimes \pi) \neq 0.$$

Proof. The proof of the lemma will be analogous to the proof of Theorem 1. Again, we will use a seesaw. To define the seesaw, regard D as a symmetric bilinear space X with symmetric bilinear form $(\ ,\ )$ induced by the reduced norm of D. Let X_0 be the subspace of X such that $X_0(k) = E$, and let X_1 be the orthogonal complement to X_0. Define Y, W_0, W_1 and W as in the proof of Theorem 1. We have the analogous seesaw dual pairs $(\mathrm{O}(X), \mathrm{Sl}(2))$ and $(\mathrm{O}(X_0) \times \mathrm{O}(X_1), \mathrm{Sl}(2) \times \mathrm{Sl}(2))$ in $\mathrm{Sp}(W)$, and the same seesaw diagram. We also have the auxiliary dual pairs $(\mathrm{O}(X_0), \mathrm{Sl}(2))$ and $(\mathrm{O}(X_1), \mathrm{Sl}(2))$ in $\mathrm{Sp}(W_0)$ and $\mathrm{Sp}(W_1)$, respectively. For the same reasons as before, the inverse images of $\mathrm{O}(X(\mathbb{A}))$, $\mathrm{Sl}(2, \mathbb{A})$ and $\mathrm{O}(X_0(\mathbb{A})) \times \mathrm{O}(X_1(\mathbb{A}))$ are split; since the dimensions of X_0 and X_1 are even, it also follows that the inverse images of $\mathrm{O}(X_0(\mathbb{A}))$ and $\mathrm{Sl}(2, \mathbb{A})$ in $\mathrm{Sp}(W_0(\mathbb{A}))$ and of $\mathrm{O}(X_1(\mathbb{A}))$ and $\mathrm{Sl}(2, \mathbb{A})$ in $\mathrm{Sp}(W_1(\mathbb{A}))$ are split. This implies that the inverse image of $\mathrm{Sl}(2, \mathbb{A}) \times \mathrm{Sl}(2, \mathbb{A})$ in $\mathrm{Sp}(W(\mathbb{A}))$ is split.

We will use the same notation as in the proof of Theorem 1 for the Weil representations and their restrictions. However, now ω is a representation of $(\mathrm{Sl}(2, \mathbb{A}) \times \mathrm{Sl}(2, \mathbb{A})) \times (\mathrm{O}(X_0(\mathbb{A})) \times \mathrm{O}(X_1(\mathbb{A})))$, and

$$\omega((g_0, g_1), (h_0, h_1))\varphi(x_0 \oplus x_1) = \omega_0(g_0, h_0)\varphi_0(x_0)\omega_1(g_1, h_1)\varphi_1(x_1)$$

for $\varphi = \varphi_0 \otimes \varphi_1 \in L^2(X_0(\mathbb{A})) \otimes_{\mathbb{C}} L^2(X_1(\mathbb{A}))$, $(g_0, g_1) \in \mathrm{Sl}(2, \mathbb{A}) \times \mathrm{Sl}(2, \mathbb{A})$ and $(h_0, h_1) \in \mathrm{O}(X_0(\mathbb{A})) \times \mathrm{O}(X_1(\mathbb{A}))$.

In fact, for the proof we will need the similitude version of the seesaw. The similitude seesaw and identity require some more notation and observations. First, we claim that there is a quaternion algebra basis $1, \mathbf{i}, \mathbf{j}, \mathbf{k} = \mathbf{ij}$ for $D(k)$ such that $X_0(k) = E = k + k\cdot\mathbf{i}$ and $X_1(k) = k\cdot\mathbf{j} + k\cdot\mathbf{k} = E\cdot\mathbf{j}$. To see this, let $E = k + k\cdot\mathbf{i}$, where $\mathbf{i}^2 \in k^\times$. Since the canonical involution $*$ of D generates $\mathrm{Gal}(E/k)$, $\mathbf{i}^* = -\mathbf{i}$, and $(1, \mathbf{i}) = 0$. Let $x \in X_1(k)$ be nonzero. Consider the set E' of elements of $D(k)$ that commute with x. As $x \notin k$, this is $k + k \cdot x$. Now as a k algebra, E' is either a quadratic extension of k or isomorphic to $k \times k$. Moreover, the restriction of $*$ generates $\mathrm{Gal}(E'/k)$. Hence, there exists $\mathbf{j} \in E'$ such that $E' = k + k \cdot \mathbf{j}$, $\mathbf{j}^2 \in k^\times$ and $(1, \mathbf{j}) = 0$. Let $\mathbf{k} = \mathbf{ij}$. Since $x \in X_1(k)$, we have $(\mathbf{i}, x) = 0$, so that $(\mathbf{i}, \mathbf{j}) = 0$, i.e., $\mathbf{ij} = -\mathbf{ji}$. We conclude that any two distinct elements among $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ are orthogonal, and hence $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ is a basis. Since $(X_0(k), k \cdot \mathbf{j} + k \cdot \mathbf{k}) = 0$, we have $X_1(k) = k \cdot \mathbf{j} + k \cdot \mathbf{k}$. We note that since $X_0(k) = E$ and $X_1(k) = E\mathbf{j}$, the symmetric bilinear form on $X_1(k)$ is equivalent to $\mathrm{N}(\mathbf{j})$ times the symmetric bilinear form on $X_0(k)$.

Using the last observation, we can identify $\mathrm{GSO}(X_0(\mathbb{A}))$ and $\mathrm{GSO}(X_1(\mathbb{A}))$. For $a \in \mathbb{A}_E^\times$, let $m(a)$ denote both the element of $\mathrm{GSO}(X_0(\mathbb{A}))$ and of $\mathrm{GSO}(X_1(\mathbb{A}))$ defined by left multiplication by a. It is well known that the maps from $\mathbb{A}_E^\times$ to $\mathrm{GSO}(X_0(\mathbb{A}))$ and $\mathrm{GSO}(X_1(\mathbb{A}))$ which send a to $m(a)$ are isomorphisms. Clearly, the similitude factor $\lambda(m(a))$ of $m(a)$ for $a \in \mathbb{A}_E^\times$ is $\mathrm{N}_k^E(a)$. It follows that $\lambda(\mathrm{GSO}(X_0(\mathbb{A}))) = \lambda(\mathrm{GSO}(X_1(\mathbb{A}))) = \mathrm{N}_k^E(\mathbb{A}_E^\times)$.

The seesaw that we will use now is:

$$\begin{array}{ccc} [\mathrm{Gl}(2,\mathbb{A})^+ \times \mathrm{Gl}(2,\mathbb{A})^+] & & \mathrm{GSO}(X(\mathbb{A})) \\ \uparrow & \times & \uparrow \\ \mathrm{Gl}(2,\mathbb{A})^+ & & H(\mathbb{A}) = [\mathrm{GSO}(X_1(\mathbb{A})) \times \mathrm{GSO}(X_2(\mathbb{A}))] \end{array}.$$

Here, $\mathrm{Gl}(2,\mathbb{A})^+$ is the set of $g \in \mathrm{Gl}(2,\mathbb{A})$ such that

$$\det(g) \in \lambda(\mathrm{GSO}(X_0(\mathbb{A}))) = \lambda(\mathrm{GSO}(X_1(\mathbb{A}))) = \mathrm{N}_k^E(\mathbb{A}_E^\times)$$

and $[\mathrm{Gl}(2,\mathbb{A})^+ \times \mathrm{Gl}(2,\mathbb{A})^+]$ is the subgroup of pairs (g, g') of $\mathrm{Gl}(2,\mathbb{A})^+ \times \mathrm{Gl}(2,\mathbb{A})^+$ such that $\det(g) = \det(g')$, and $H(\mathbb{A}) = [\mathrm{GSO}(X_1(\mathbb{A})) \times \mathrm{GSO}(X_2(\mathbb{A}))]$ is the subgroup of pairs $(h, h') \in \mathrm{GSO}(X_0(\mathbb{A})) \times \mathrm{GSO}(X_2(\mathbb{A}))$ such that $\lambda(h) = \lambda(h')$. At this point we may as well state an identification of the right hand side of the diagram, analogous to the one of Theorem 1. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{GSO}(X(\mathbb{A})) & \xleftarrow{\sim} & (D(\mathbb{A})^\times \times D(\mathbb{A})^\times)/\mathbb{A}^\times \\ \Big\uparrow & & \Big\uparrow \\ H(\mathbb{A}) & \xleftarrow{\sim} & (\mathbb{A}_E^\times \times \mathbb{A}_E^\times)/\mathbb{A}^\times \end{array}.$$

Here the top map ρ is as in Theorem 1, and the bottom map is defined by $\rho(x,y) = (m(xy^{-1}), m(x\gamma(y)^{-1}))$. The vertical maps are inclusion.

As in the proof of Theorem 1, to introduce similitudes into the theta correspondence, we will use the extended representation of [HK1]. Let $R(\mathbb{A})$ be as in the proof of Theorem 1, and define $R_0(\mathbb{A})$ and $R_1(\mathbb{A})$ analogously. Then ω_0 and ω_1 extend to representations of $R_0(\mathbb{A})$ and $R_1(\mathbb{A})$, respectively, just as does ω'. Moreover, we have

$$\omega'(g,(h_0,h_1))(\varphi)(x_0 \oplus x_1) = \omega_0(g,h_0)\varphi_0(x_0)\omega_1(g,h_1)\varphi_1(x_1)$$

for $\varphi = \varphi_0 \otimes \varphi_1 \in L^2(X_0(\mathbb{A})) \otimes_{\mathbb{C}} L^2(X_1(\mathbb{A}))$, $(g, h_0) \in R_0(\mathbb{A})$ and $(g, h_1) \in R_1(\mathbb{A})$.

Using the extended representations we define the theta lifts, and finally state the seesaw identity. We define $\theta(f', \varphi)$ on $\mathrm{GSO}(X(\mathbb{A}))$ for $f' \in \pi$ and $\varphi \in \mathcal{S}(X(\mathbb{A}))$ as in the proof of Theorem 1. Fix a right $\mathrm{SO}(X_0(\mathbb{A}))$ invariant measure on $\mathrm{SO}(X_0(k))\backslash \mathrm{SO}(X_0(\mathbb{A}))$. Using the isomorphism of $\mathrm{GSO}(X_0(\mathbb{A}))$ with $\mathbb{A}_E^\times$ from above, define F_0 on $\mathrm{GSO}(X_0(\mathbb{A}))$ by $F_0(m(x)) = \chi_0(x)$. For $\varphi_0 \in \mathcal{S}(X_0(\mathbb{A}))$ define $\theta(F_0, \varphi_0)$ on $\mathrm{Gl}(2, \mathbb{A})^+$ by

$$\theta(F_1, \varphi_1)(g) = \int_{\mathrm{SO}(X_0(k))\backslash \mathrm{SO}(X_0(\mathbb{A}))} \theta(g, h_1 h; \varphi_1) F_1(h_1 h)\, dh_1$$

where h in $\mathrm{GSO}(X_0(\mathbb{A}))$ is such that $\det(g) = \lambda(h)$. For $\varphi_0 \in \mathcal{S}(X_0(\mathbb{A}))$, extend $\theta(F_0, \varphi_0)$ to a $\mathrm{Gl}(2, k)$ invariant function on $\mathrm{Gl}(2, \mathbb{A})$ by setting $\theta(F_0, \varphi_0)(g_0 g) = \theta(F_0, \varphi_0)(g)$ for $g_0 \in \mathrm{Gl}(2, k)$ and $g \in \mathrm{Gl}(2, \mathbb{A})^+$ and letting $\theta(F_0, \varphi_0)$ be 0 off $\mathrm{Gl}(2, k)\, \mathrm{Gl}(2, \mathbb{A})^+$. Then the automorphic representation of $\mathrm{Gl}(2, \mathbb{A})$ generated by these functions is $\pi(\chi_0)$. Similar notation and comments apply to X_1 and χ_1. See [HK2], Sect. 13.

An argument as in [HK1], Proposition 7.1.4, now shows that there exist invariant measures on $\mathbb{A}^\times \mathrm{Gl}(2, k)^+ \backslash \mathrm{Gl}(2, \mathbb{A})^+$ and $\mathbb{A}^\times H(k) \backslash H(\mathbb{A})$ such that for $\varphi_0 \in \mathcal{S}(X_0(\mathbb{A}))$, $\varphi_1 \in \mathcal{S}(X_1(\mathbb{A}))$ and $f \in \pi$,

$$\langle \theta(F_0, \varphi_0)\theta(F_1, \varphi_1), f \rangle_{\mathrm{Gl}(2,\mathbb{A})^+} = \langle \theta(f, \varphi_0 \otimes \varphi_1), F_0 \otimes F_1 \rangle_{H(\mathbb{A})}.$$

Here, the first integral is over $\mathbb{A}^\times \mathrm{Gl}(2, k)^+ \backslash \mathrm{Gl}(2, \mathbb{A})^+$, and the second integral is over $\mathbb{A}^\times H(k) \backslash H(\mathbb{A})$.

The lemma follows easily from this identity. Suppose that the product of the integrals in the statement of the lemma is nonzero, i.e., there exist $f_1 \in \mathrm{JL}(\pi)$ and $f_2 \in \mathrm{JL}(\pi)^\vee$,

$$\int_{\mathbb{A}^\times E^\times \backslash \mathbb{A}_E^\times} f_1(x)\chi_0(x)\chi_1(x)\, dx \cdot \int_{\mathbb{A}^\times E^\times \backslash \mathbb{A}_E^\times} f_2(y)\chi_0(y^{-1})\chi_1(\gamma(y)^{-1})\, dx \neq 0\ .$$

As was pointed out in the proof of Theorem 1, $f_1 \otimes f_2$ is a linear combination of functions $\theta(f', \varphi) \circ \rho$ where $f' \in \pi$ and $\varphi \in \mathcal{S}(X(\mathbb{A}))$. Moreover, the vectors $\varphi_0 \otimes \varphi_1$ for $\varphi_0 \in \mathcal{S}(X_0(\mathbb{A}))$ and $\varphi_1 \in \mathcal{S}(X_1(\mathbb{A}))$ span $\mathcal{S}(X(\mathbb{A}))$. It follows that since the last product of integrals is nonzero, for some $f' \in \pi$, $\varphi_0 \in \mathcal{S}(X_0(\mathbb{A}))$ and $\varphi_1 \in \mathcal{S}(X_1(\mathbb{A}))$ we have

$$\langle \theta(f', \varphi_0 \otimes \varphi_1), F_0 \otimes F_1 \rangle_{H(\mathbb{A})} \neq 0.$$

By the seesaw identity,

$$\langle \theta(F_0,\varphi_0)\theta(F_1,\varphi_1), f' \rangle_{\mathrm{Gl}(2,\mathbb{A})^+} \neq 0.$$

This implies that $T(\pi(\chi_0)\otimes\pi(\chi_1)\otimes\pi) \neq 0$. $\square$

The next lemma proves the necessity of the condition described at the beginning of this section.

Lemma 2. *Let* π *be a cuspidal automorphic representation of* $\mathrm{Gl}(2,\mathbb{A})$ *with trivial central character. Let* E *be a quadratic extension of* k*, and suppose* χ *is a Hecke character of* $\mathbb{A}_E^\times$ *that does not factor through* N_k^E*. If*

$$T(\tau(\chi)\otimes\tau(\chi)^\vee\otimes\pi) \neq 0$$

then

$$L(1/2,\pi)L(1/2,\pi\otimes\omega_{E/k}) \neq 0.$$

Proof. Suppose that $T(\pi(\chi)\otimes\pi(\chi)^\vee\otimes\pi) \neq 0$. By Theorem 1 it will suffice show that $T(\pi(\chi)\otimes\pi(\chi)^\vee\otimes(\pi\otimes\omega_{E/k})) \neq 0$. Now by the characterization of $\pi(\chi)$ from Lemma 1, there exists $f_1 \in \pi(\chi)$ with support in $\mathrm{Gl}(2,k)\,\mathrm{Gl}(2,\mathbb{A})^+$, $f_2 \in \pi(\chi)^\vee$ and $f \in \pi$ such that $T(\pi(\chi)\otimes\pi(\chi)^\vee\otimes\pi)(f_1\otimes f_2\otimes f) \neq 0$. Hence,

$$\begin{aligned}
&\int_{\mathbb{A}^\times \mathrm{Gl}(2,k)\backslash \mathrm{Gl}(2,\mathbb{A})} f_1(g) f_2(g) f(g)\omega_{E/k}(\det(g))\,dg \\
&\qquad = \int_{\mathbb{A}^\times \mathrm{Gl}(2,k)\backslash \mathrm{Gl}(2,k)\,\mathrm{Gl}(2,\mathbb{A})^+} f_1(g)f_2(g)f(g)\,dg \\
&\qquad = \int_{\mathbb{A}^\times \mathrm{Gl}(2,k)\backslash \mathrm{Gl}(2,\mathbb{A})} f_1(g)f_2(g)f(g)\,dg \\
&\qquad \neq 0\,.
\end{aligned}$$

This completes the proof. $\square$

Now we prove the main result of this section.

Theorem 2. *Let* D *be a quaternion algebra defined over* k*, and let* ϱ *be an infinite dimensional cuspidal automorphic representation of* $D(\mathbb{A})^\times$ *with trivial central character. Let* $\pi = \mathrm{JL}(\varrho)$*. Suppose that there exists a quadratic extension* E *contained in* D *such that for all places* v *of* k*,* $\mathrm{Hom}_{E_v^\times}(\varrho_v,\mathbf{1}) \neq 0$*. Let* $\mathrm{Gal}(E/k)=\{1,\gamma\}$*. Then*

$$L(1/2,\pi)L(1/2,\pi\otimes_{\mathbb{C}}\omega_{E/k}) \neq 0$$

if and only if there exists a Hecke character χ *of* $\mathbb{A}_E^\times$ *that does not factor through* N_k^E *such that*

$$T(\pi(\chi)\otimes\pi(\chi)^\vee\otimes\pi) \neq 0.$$

Moreover, suppose S is a finite set of places of k which stay prime in E, and α_v for $v \in S$ are characters of $E_v^\times$ such that $\mathrm{Hom}_{E_v^\times}(\varrho_v, \alpha_v) \neq 0$ for all $v \in S$. Then we may assume that in the previous statement $(\chi_v \circ \gamma)/\chi_v = \alpha_v$ for $v \in S$.

It is clear that if the statement in Theorem 2 involving χ holds for χ, then it holds for $\chi(\beta \circ \mathrm{N}_k^E)$, where β is a Hecke character of $\mathbb{A}^\times$.

Proof. Assume that $L(1/2, \pi)L(1/2, \pi \otimes_{\mathbb{C}} \omega_{E/k}) \neq 0$. By [W3], Théorème 2, it follows that

$$\int_{\mathbb{A}^\times E^\times \backslash \mathbb{A}_E^\times} f(x)\, dx \neq 0$$

for some $f \in \varrho = \mathrm{JL}(\pi)$. On the other hand, by an argument as in Lemma 1.4.9 of [H], there exists a Hecke character χ of $\mathbb{A}_E^\times$ such that χ does not factor through N_k^E, $(\chi_v \circ \gamma)/\chi_v = \alpha_v$ for $v \in S$, and

$$\int_{\mathbb{A}^\times E^\times \backslash \mathbb{A}_E^\times} f'(x)(\chi(\gamma(x))\chi(x)^{-1})^{-1}\, dx \neq 0$$

for some $f' \in \varrho = \mathrm{JL}(\pi)$. By Lemma 1 with $\chi_0 = \chi^{-1}$ and $\chi_1 = \chi$, we have $T(\pi(\chi) \otimes \pi(\chi)^\vee \otimes \pi) \neq 0$.

The other implication of the theorem follows from Lemma 2. $\square$

3. Applications to modular forms

The main result of this section is Theorem 3, a version of Theorem 2 for new forms in $S_k(\Gamma_0(p))$, where k is an even integer such that $k/2$ is odd, and p is a prime such that $p \equiv 3 \pmod 4$. To obtain Theorem 3 as an application of Theorem 2, it is necessary to show locally that some trilinear forms do not vanish on certain pure tensors composed of a combination of new and old vectors. In particular, we need more information than is contained in [GP], where the case of a triple tensor product of unramified representations or a triple tensor product of special representations is treated. To obtain the required result, we need to generalize the description of the new vector in a Kirillov model from [GP] to the case when the central character is not trivial. The result on trilinear forms appears in Lemma 3, and the new vector in a Kirillov model is described in the discussion preceding the lemma. We begin the section by giving the straightforward application of one direction of Theorem 1 to modular forms.

Proposition 1. *Let N be a nonnegative integer, and let k be a positive even integer. Let $F \in S_k(\Gamma_0(N))$ be a new form. Let M be a nonnegative integer such that $N|M$, let χ be a Dirichlet character modulo M, and let*

F_1 in $S_{k/2}(\Gamma_0(M),\chi)$ be an eigenform for $T(p)$ for $p\nmid M$. If there exists a divisor d of M/N such that

$$\langle F|\begin{pmatrix} d & 0\\ 0 & 1\end{pmatrix}_k, F_1\cdot F_1|[W_M]_{k/2}\rangle_{\Gamma_0(M)\backslash\mathfrak{H}}\neq 0$$

then

$$L(\frac{k}{2},F)\neq 0.$$

Proof. Let $F_2=F_1|[W_M]_{k/2}$ and

$$F'=F|\begin{pmatrix} d & 0\\ 0 & 1\end{pmatrix}_k.$$

Let f_F, $f_{F'}$, f_{F_1} and f_{F_2} be the functions on $\mathrm{Gl}(2,\mathbb{A})$ corresponding to F, F', F_1 and F_2, respectively, as in [Ge], Sect. 3.A. Note that for $h\in\mathrm{Gl}(2,\mathbb{A})$, $f_{F'}(h)=f_F(hh_0)$, where

$$h_0=\prod_{p|d}\begin{pmatrix} d^{-1} & 0\\ 0 & 1\end{pmatrix}_p.$$

Let π, σ and $\tilde\sigma$ be the irreducible cuspidal automorphic representations generated by f_F, f_{F_1} and f_{F_2}, respectively. Then $\tilde\sigma=\sigma^\vee$. Now

$$\begin{aligned}&\int_{\mathbb{A}^\times\,\mathrm{Gl}(2,\mathbb{Q})\backslash\,\mathrm{Gl}(2,\mathbb{A})} f_F(hh_0)\overline{f_{F_1}(h)f_{F_2}(h)}\,dh\\ &=\langle F|\begin{pmatrix} d & 0\\ 0 & 1\end{pmatrix}_k, F_1\cdot F_1|[W_M]_{k/2}\rangle_{\Gamma_0(M)\backslash\mathfrak{H}}\\ &\neq 0.\end{aligned}$$

Since $\overline{f_{F_1}}$ is in $\overline{\sigma}=\sigma^\vee$ and $\overline{f_{F_2}}$ is in $\overline{\tilde\sigma}=\sigma$, the conclusion follows from Theorem 1 of Sect. 1 and Example 6.19 of [Ge]. $\square$

To prove Theorem 3 we need a lemma about new vectors and trilinear forms. Before stating the lemma we recall some definitions and results. Suppose for the moment that k is a nonarchimedean local field of characteristic zero, with ring of integers $\mathfrak{O}_k$. Let $\mathfrak{P}_k$ be the prime ideal of $\mathfrak{O}_k$, and let π_k be a uniformizing element, i.e., $\mathfrak{P}_k=\pi_k\mathfrak{O}_k$. Suppose $\sigma\in\mathrm{Irr}(\mathrm{Gl}(2,k))$ is infinite dimensional. For each nonnegative integer n, let $L(\sigma,n)$ be the space of $f\in\sigma$ such that $\sigma(k)f=\omega_\sigma(a)f$ for

$$k=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0(n)=\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{Gl}(2,\mathfrak{O}_k):c\equiv 0\pmod{\mathfrak{P}_k^n}\}.$$

It is well known that $L(\sigma,n)\neq 0$ for some n and that for the smallest such n, the conductor $c(\sigma)$ of σ, $\dim_{\mathbb{C}}L(\sigma,n)=1$. We call any nonzero vector in

$L(\sigma, c(\sigma))$ a new vector of σ. It is easy to see that if $f \in L(\sigma, n)$ is nonzero, then there exists a nonzero $f^\vee \in L(\sigma^\vee, n)$ such that $\langle f, f^\vee\rangle \neq 0$, where $\langle\ ,\rangle$ is the canonical pairing between σ and $\sigma^\vee$. In particular, $c(\sigma) = c(\sigma^\vee)$. New vectors can be explicitly described in the Kirillov model $K(\sigma, \psi)$ of σ with respect to a nontrivial additive character ψ of k. Assume that the conductor of ψ is $\mathfrak{O}_k$. If σ is supercuspidal, then by [S2], $f_1 = \omega_\sigma \chi_{\mathfrak{O}_k^\times}$ is a new vector in $K(\sigma, \psi)$. If σ is the irreducible principal series representation $\pi(\mu_1, \mu_2)$, then a new vector f_1 is given by the following formulas:

$$f_1(x) = \begin{cases} \chi_{\mathfrak{O}_k}(x)|x|^{1/2}\mu_1(x)\mu_2(x) \displaystyle\sum_{n=0}^{\mathrm{val}(x)} \mu_1(\pi_k)^{n-\mathrm{val}(x)}\mu_2(\pi_k)^{-n} \\ \qquad\qquad \text{if } \mathrm{c}(\mu_1) = \mathrm{c}(\mu_2) = 0, \\ \mu_2(x)\chi_{\mathfrak{O}_k}(x)|x|^{1/2} \text{ if } \mathrm{c}(\mu_1) = 0, \mathrm{c}(\mu_2) > 0, \\ \mu_1(x)\chi_{\mathfrak{O}_k}(x)|x|^{1/2} \text{ if } \mathrm{c}(\mu_1) > 0, \mathrm{c}(\mu_2) = 0, \\ \mu_1(x)\mu_2(x)\chi_{\mathfrak{O}_k^\times}(x) \text{ if } \mathrm{c}(\mu_1) > 0, \mathrm{c}(\mu_2) > 0. \end{cases}$$

These formulas can be obtained using the Weil representation model for $\pi(\mu_1, \mu_2)$, for example. Using a trick, we can also describe a new vector in the model for $\sigma^\vee$ which has as underlying space $K(\sigma, \psi)$ and action defined by $\sigma'(g) = \omega_\sigma(\det(g))^{-1}\sigma(g)$. Let

$$g_0 = \begin{pmatrix} 0 & \pi_k^{-c(\sigma)} \\ 1 & 0 \end{pmatrix}.$$

Then a computation shows that $f_2 = \sigma(g_0)f_1$ is a new vector for $(\sigma', K(\sigma, \psi))$. Recall that by [Go], p. 1.22, the map $(\sigma', K(\sigma, \psi)) \to (\sigma^\vee, K(\sigma^\vee, \psi))$ that sends f to $\omega_\sigma^{-1} f$ is an isomorphism of $\mathrm{Gl}(2, k)$ representations. It follows that $\omega_\sigma^{-1} f_2$ is a new vector in $(\sigma^\vee, K(\sigma^\vee, \psi))$. Using our explicit description, we find that there is a nonzero constant $c \in \mathbb{C}^\times$ such that if σ is supercuspidal representation then $f_2 = c\chi_{\mathfrak{O}_k^\times}$, and if σ is the irreducible principal series representation $\pi(\mu_1, \mu_2)$, then

$$f_2(x) = c \cdot \begin{cases} \chi_{\mathfrak{O}_k}(x)|x|^{1/2} \displaystyle\sum_{n=0}^{\mathrm{val}(x)} \mu_1(\pi_k)^{-n+\mathrm{val}(x)}\mu_2(\pi_k)^{n} \\ \qquad\qquad \text{if } \mathrm{c}(\mu_1) = \mathrm{c}(\mu_2) = 0, \\ \mu_1(x)\chi_{\mathfrak{O}_k}(x)|x|^{1/2} \text{ if } \mathrm{c}(\mu_1) = 0, \mathrm{c}(\mu_2) > 0, \\ \mu_2(x)\chi_{\mathfrak{O}_k}(x)|x|^{1/2} \text{ if } \mathrm{c}(\mu_1) > 0, \mathrm{c}(\mu_2) = 0, \\ \chi_{\mathfrak{O}_k^\times}(x) \text{ if } \mathrm{c}(\mu_1) > 0, \mathrm{c}(\mu_2) > 0. \end{cases}$$

For information about trilinear forms, see [P]. The following result should be compared to Propositions 6.1 and 6.3 of [GP].

Lemma 3. *Let k be a nonarchimedean local field. Let $\sigma, \pi \in \mathrm{Irr}(\mathrm{Gl}(2,k))$, with $\omega_\pi = 1$. Let $f_1 \in \sigma$ and $f_2 \in \sigma^\vee$ be new vectors. If σ is either supercuspidal or an element of the continuous series, and there exists an unramified character μ of $k^\times$ such that $\pi = \pi(\mu, \mu^{-1})$, then there exist $T \in \mathrm{Hom}_{\mathrm{Gl}(2,k)}(\sigma \otimes \sigma^\vee \otimes \pi, \mathbf{1})$ and $f \in \pi$ fixed under $\Gamma_0(c(\sigma))$ such that*

$$T(f_1 \otimes f_2 \otimes f) \neq 0.$$

Proof. By our remark concerning pairing of vectors in $L(\pi, n)$ and $L(\pi^\vee, n)$ it suffices to construct an element of $\mathrm{Hom}_{\mathrm{Gl}(2,k)}(\sigma \otimes \sigma^\vee, \pi^\vee)$ that is nonzero on $f_1 \otimes f_2$. By Frobenius reciprocity,

$$\mathrm{Hom}_{\mathrm{Gl}(2,k)}(\sigma \otimes \sigma^\vee, \rho(\mu, \mu^{-1})) = \mathrm{Hom}_B(\sigma \otimes \sigma^\vee, \alpha),$$

where α is the quasi-character of the Borel subgroup B defined by

$$\alpha \begin{pmatrix} t_1 & x \\ 0 & t_2 \end{pmatrix} = \mu(t_1)\mu(t_2)^{-1}|t_1/t_2|^{1/2}.$$

To prove the lemma it suffices to produce an element L of $\mathrm{Hom}_B(\sigma \otimes_{\mathbb{C}} \sigma^\vee, \alpha)$ such that $L(f_1 \otimes f_2) \neq 0$. We will use the Kirillov model of σ and the model $(\sigma', K(\sigma, \psi))$ for $\sigma^\vee$ from the paragraph preceding the lemma. Define $L : \sigma \otimes \sigma^\vee \to \alpha$ by

$$L(f \otimes f') = \int_{k^\times} f(x) f'(-x) \omega_\sigma(x)^{-1} \mu(x)^{-1} |x|^{-1/2} \, d^\times x.$$

It is easy to check that this integral always converges, and defines a B map. Using the descriptions of f_1 and f_2 from above, a computation shows that $L(f_1 \otimes f_2) \neq 0$. $\square$

To state Theorem 3, we need some notation. Let E be an imaginary quadratic extension of $\mathbb{Q}$, and let χ be a Hecke character of $\mathbb{A}_E^\times$ that does not factor through $\mathrm{N}_{\mathbb{Q}}^E$. We let $M(\chi)$ be the conductor of $\pi(\chi)$. We say that χ is of modular form type if $\pi(\chi)_\infty$ is of modular form type. An element $\pi \in \mathrm{Irr}(\mathrm{Gl}(2, \mathbb{R}))$ is of modular form type if and only if there exists a positive integer l such that if $l = 1$ then $\pi = \pi(1, \mathrm{sign})$, and if $l > 1$, then

$$\pi = \begin{cases} \sigma(|\,|^{(l-1)/2}, |\,|^{-(l-1)/2}) & \text{if } l \text{ is even}, \\ \sigma(|\,|^{(l-1)/2}, |\,|^{-(l-1)/2}\mathrm{sign}) & \text{if } l \text{ is odd}. \end{cases}$$

Here, the notation is as in [Ge]. The terminology is motivated by the following facts. If for some positive l, nonnegative integer N, and Dirichlet character ψ modulo N, $F \in S_l(\Gamma_0(N), \psi)$ is a nonzero new form, and $\pi = \otimes_v \pi_v$ is the irreducible cuspidal automorphic representation associated to F, then π_∞ is as above. Conversely, if $\pi = \otimes_v \pi_v$ is an irreducible cuspidal automorphic representation of $\mathrm{Gl}(2,\mathbb{A})$, and π_∞ is of modular form type and as described as above, then π canonically induces a new form in $S_l(\Gamma_0(N),\psi)$, where N is the conductor of π, and ψ is related to the central character of π. For more, see Lemma 5.16 and the discussion in Sect. C of [Ge]. Let $\chi_\infty(z)=z^m\overline{z}^n(z\overline{z})^r$, where m and n are nonnegative integers, with at most one nonzero, and $r\in\mathbb{C}$. Then χ is of modular form type if and only if there exists a positive integer l such that

$$m+n=l-1, \qquad r=-\frac{(l-1)}{2}.$$

See [Ge], Remark 7.7.

Theorem 3. *Let p be a prime such that $p\equiv 3 \pmod 4$, and let k be an even positive integer such that $k/2$ is odd. Let $F\in S_k(\Gamma_0(p))$ be a new form. Let S be a finite set of primes of $\mathbb{Q}$ not including p and ∞ that do not split in $E=\mathbb{Q}(\sqrt{-p})$, and let α_q, $q\in S$, be a collection of characters of $E_q^\times$ that are trivial on $\mathbb{Q}_q^\times$. Let $\mathrm{Gal}(E/\mathbb{Q})=\{1,\gamma\}$. Then*

$$L(\frac{k}{2},F)L(\frac{k}{2},F,\left(\tfrac{\cdot}{p}\right))\neq 0,$$

if and only if there exists a Hecke character χ of $\mathbb{A}_E^\times$ of modular form type that does not factor through $\mathrm{N}_{\mathbb{Q}}^E$ and a positive integer d such that p divides $M=M(\chi)$ exactly, χ_p is unramified, $(\chi_q\circ\gamma)/\chi_q=\alpha_q$ for $q\in S$, $\chi_\infty(z)=z^{k/2-1}(z\overline{z})^{(1-k/2)/2}$, $d|(M/p)$ and

$$\langle F|\begin{pmatrix} d & 0\\ 0 & 1\end{pmatrix}_k, F_1\cdot F_1|[W_M]_{k/2}\rangle_{\Gamma_0(M)\backslash\mathfrak{H}}\neq 0,$$

where F_1 is the new form of weight $k/2$ and level M associated to χ, as in the preceding discussion.

Proof. Let π be the irreducible cuspidal automorphic representation of $\mathrm{Gl}(2,\mathbb{A})$ associated to F as in [Ge], Proposition 5.21. Then the product from the statement of the theorem does not vanish if and only if $L(1/2,\pi)L(1/2,\pi\otimes\omega_{E/\mathbb{Q}})\neq 0$.

Assume that $L(1/2,\pi)L(1/2,\pi\otimes\omega_{E/\mathbb{Q}})\neq 0$. Since $L(1/2,\pi)\neq 0$, it follows that $\epsilon(1/2,\pi)=1$. By Theorem 6.15 and Theorem 6.16 of [Ge], it follows that $\epsilon(1/2,\pi_p)=-1$; here, and in the following, the ϵ-factor at the place v of $\mathbb{Q}$ is defined with respect to the standard additive character of $\mathbb{Q}_v$. Now $\pi_p=\mathrm{Sp}\otimes\eta$ where where η is an unramified character such that $\eta^2=1$; see [GP], Lemma 4.1. Since $\epsilon(1/2,\pi_p)=-1$ it follows that $\eta=1$. Now let D be the quaternion algebra over $\mathbb{Q}$ ramified at exactly p and ∞. By [V], E is contained in D. For an explicit description of D, see [S2]. Since π_p

and π_∞ are in the discrete series, it follows that $\mathrm{JL}(\pi) \neq 0$; let $\varrho = \mathrm{JL}(\pi)$. Since $\varrho_p = \mathbf{1}$, we have $\mathrm{Hom}_{E_p^\times}(\varrho_p, \mathbf{1}) \neq 0$. By [W1], $\mathrm{Hom}_{E_q^\times}(\varrho_q, \mathbf{1}) \neq 0$ for all $q < \infty$, $q \neq p$. Since k is even, $\pi_\infty = \sigma(|\ |^{(k-1)/2}, |\ |^{-(k-1)/2})$, in the notation of [Ge], p.58. Hence, by p. 142 of [Ge], $\varrho_\infty = \mathrm{N}^{(2-k)/2}\,\rho_{k-2}$. Identifying E_∞ with $\mathbb{C}$, it is easy to see that

$$\varrho_\infty|_{E_\infty^\times} = \bigoplus_{i=0}^{k-2} z^{i+(2-k)/2}\overline{z}^{-i-(2-k)/2}.$$

It follows that $\mathrm{Hom}_{E_\infty^\times}(\varrho_\infty, \mathbf{1}) \neq 0$.

Now we apply Theorem 2. In addition to the α_q for $q \in S$, let $\alpha_p = \mathbf{1}$ and $\alpha_\infty(z) = z^{-(k/2-1)}\overline{z}^{k/2-1}$. Then by $\varrho_p = \mathbf{1}$, the characterization of ϱ_∞, and for example, Theorem 1.4.4 of [H],

$$\mathrm{Hom}_{E_v^\times}(\varrho_v, \alpha_v) \neq 0$$

for $v \in S \cup \{p, \infty\}$. By Theorem 2, it now follows that there is a Hecke character χ of $\mathbb{A}_E^\times$ that does not factor through $\mathrm{N}_{\mathbb{Q}}^E$ such that $(\chi_v \circ \gamma)/\chi_v = \alpha_v$ for $v \in S \cup \{p, \infty\}$ and $T(\sigma \otimes \sigma^\vee \otimes \pi) \neq 0$, where $\sigma = \pi(\chi)$. Let m and n be nonnegative integers with at most one nonzero and let $r \in \mathbb{C}$ be such that $\chi_\infty(z) = z^m \overline{z}^n (z\overline{z})^r$ for $z \in \mathbb{C}^\times$. Since $(\chi_\infty \circ \gamma)/\chi_\infty = \alpha_\infty$, it follows that $m = k/2 - 1$ and $n = 0$. Also, since $\alpha_p = \mathbf{1}$, χ_p factors through $\mathrm{N}_{\mathbb{Q}_p}^{E_p}$. By the comment after Theorem 2, we may replace χ by $\chi(\beta \circ \mathrm{N}_{\mathbb{Q}}^E)$ for any Hecke character β of $\mathbb{A}_{\mathbb{Q}}^\times$. It follows that we may assume that $\chi_p = \mu \circ \mathrm{N}_{\mathbb{Q}_p}^{E_p}$ for some unramified character μ of $\mathbb{Q}_p^\times$, and $\chi_\infty(z) = z^{k/2-1}(z\overline{z})^{(1-k/2)/2}$.

To show how the nonvanishing of the trilinear form gives the nonvanishing of the inner product from the statement of the theorem we begin with some notation and observations. We have that $\sigma_\infty = \sigma_\infty^\vee = \sigma(|\ |^{(k/2-1)/2}, |\ |^{-(k/2-1)/2}\mathrm{sign})$ if $k > 1$ and $\sigma_\infty = \sigma_\infty^\vee = \pi(1, \mathrm{sign})$ if $k = 1$. For each finite prime q of $\mathbb{Q}$, let $f_{1,q} \in \sigma_q$ and $f_{2,q} \in \sigma_q^\vee$ be new vectors, and let $f_{1,\infty} \in \sigma_\infty$ and $f_{2,\infty} \in \sigma_\infty^\vee$ be nonzero vectors of weight $k/2$. Let $f_1 = \otimes_v f_{1,v}$ and $f_2 = \otimes_v f_{2,v}$. Then $\overline{f_1} \in \sigma^\vee$ and $\overline{f_2} \in \sigma$. Moreover, $\overline{f_1}$ is a nonzero multiple of $\otimes_q f_{2,q} \otimes f'_{2,\infty}$ and $\overline{f_2}$ is a nonzero multiple of $\otimes_q f_{1,q} \otimes f'_{1,\infty}$, where $f'_{1,\infty}$ and $f'_{2,\infty}$ are nonzero vectors of weight $-k/2$ in σ_∞ and $\sigma_\infty^\vee$, respectively. Let M be the conductor of σ, i.e., $M = \prod_q q^{\mathrm{c}(\sigma_q)}$. Note that since $\sigma_p = \pi(\mu, \mu\omega_{E_p/\mathbb{Q}_p})$, $\mathrm{c}(\sigma_p) = 1$, and p divides M exactly. Define a Dirichlet character $\alpha : (\mathbb{Z}/M\mathbb{Z})^\times \to \mathbb{C}^\times$ by $\alpha(a) = \prod_{q|M} \omega_{\sigma,q}(a)$. Let $F_1 \in S_{k/2}(\Gamma_0(M), \alpha)$ correspond to f_1, i.e., define F_1 by $F_1(g \cdot i) = f_1(g_\infty) j(g,i)^{k/2}$. Let $F' = F_1|[W_M]_{k/2}$. Consider $f_{F'}$. As in the proof of Proposition 1, the space generated by $f_{F'}$ is $\sigma^\vee$, and $f_{F'}$ is a nonzero multiple of f_2. It follows that if $F_2 \in S_{k/2}(\Gamma_0(M), \alpha^{-1})$ corresponds to f_2, then F' is a nonzero multiple of F_2.

We now claim that there exists $f' = \otimes_v f'_v \in \pi$ such that for all finite primes q of $\mathbb{Q}$, $f'_q \in L(\pi_q, \mathrm{c}(\sigma_q))$, f'_∞ is a vector of weight k, and $T(\overline{f_2} \otimes \overline{f_1} \otimes f') \neq 0$. By [P], for all places v of $\mathbb{Q}$ we have $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{Gl}(2,\mathbb{Q}_v)}(\sigma_v \otimes \sigma_v^\vee \otimes \pi_v, \mathbf{1}) \leq 1$. Thus, to prove our claim it suffices to show that for each finite place q there exist $T_q \in \mathrm{Hom}_{\mathrm{Gl}(2,\mathbb{Q}_q)}(\sigma_q \otimes \sigma_q^\vee \otimes \pi_q, \mathbf{1})$ and $f'_q \in L(\pi_q, \mathrm{c}(\sigma_q))$ such that $T_q(f_{1,q} \otimes f_{2,q} \otimes f'_q) \neq 0$, and there exist $T_\infty \in \mathrm{Hom}_{\mathrm{Gl}(2,\mathbb{Q}_\infty)}(\sigma_\infty \otimes \sigma_\infty^\vee \otimes \pi, \mathbf{1})$ and $f'' \in \pi_\infty$ of weight k such that $T_\infty(f'_{1,\infty} \otimes f'_{2,\infty} \otimes f'') \neq 0$. For $q \neq p$ this follows from Lemma 3. For p and ∞ we argue as follows. Since $F_1 \cdot F_1|[W_M]_{k/2} \in S_k(\Gamma_0(M))$ is nonzero, there exists $G \in S_k(\Gamma_0(M))$ that is an eigenform for the Hecke operators $T(q)$ for $q \nmid M$ and

$$\langle G, F_1 \cdot F_1|[W_M]_{k/2} \rangle \neq 0.$$

The cuspidal automorphic representation π' generated by f_G is irreducible, $\pi'_\infty = \pi_\infty$, and since π' has trivial central character and p divides M exactly, $\pi'_p = \pi_p = \mathrm{Sp}$. Moreover, the nonvanishing of the last inner product implies the nonvanishing of

$$\int_{\mathbb{A}^\times \mathrm{Gl}(2,\mathbb{Q}) \backslash \mathrm{Gl}(2,\mathbb{A})} \overline{f_2}(g) \overline{f_1}(g) f_G(g)\, dg,$$

which implies the conditions for p and ∞.

Next, we define d. For each finite prime q of $\mathbb{Q}$ let f_q be a new vector for π_q, and let f_∞ be a nonzero vector of weight k in π_∞. Let $f = \otimes_v f_v$. We may assume that $f_F = f$. Let q be a finite prime of $\mathbb{Q}$. It is well known that $L(\pi_q, \mathrm{c}(\sigma_q))$ is spanned by the vectors

$$f_q, \pi_q \begin{pmatrix} q^{-1} & 0 \\ 0 & 1 \end{pmatrix} f_q, \ldots, \pi_q \begin{pmatrix} q^{-\mathrm{c}(\sigma_q)+\mathrm{c}(\pi_q)} & 0 \\ 0 & 1 \end{pmatrix} f_q.$$

By writing each f'_q as linear combination of these vectors, it follows that we may assume that each f'_q is of the form

$$f'_q = \pi_q \begin{pmatrix} q^{-j_q} & 0 \\ 0 & 1 \end{pmatrix} f_q,$$

where $0 \leq j_q \leq \mathrm{c}(\sigma_q) - \mathrm{c}(\pi_q)$. Thus, we may assume that $f' = \pi(h_0) f$, where

$$h_0 = \prod_{q|d} \begin{pmatrix} d^{-1} & 0 \\ 0 & 1 \end{pmatrix}_q,$$

and $d = \prod_q q^{j_q}$. As $\mathrm{c}(\pi_p) = \mathrm{c}(\mathrm{Sp}) = 1$, $p \nmid d$, and $d|(M/p)$.

The nonvanishing of $T(\overline{f_2} \otimes \overline{f_1} \otimes f')$ now implies the nonvanishing of the inner product from the theorem.

Finally Lemma 2, combined with an argument as in the proof of Proposition 1, shows that if χ as in the theorem exists, then the product of L-values from the theorem does not vanish. $\square$

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