NONVANISHING OF GLOBAL THETA LIFTS FROM ORTHOGONAL GROUPS

Brooks Roberts

University of Idaho

Abstract. Let X be an even dimensional symmetric bilinear space defined over a totally real number field F with adeles $\mathbb{A}$, and let $\sigma = \otimes_v \sigma_v$ be an irreducible tempered cuspidal automorphic representation of $\mathrm{O}(X, \mathbb{A})$. We give a sufficient condition for the nonvanishing of the theta lift $\Theta_n(\sigma)$ of σ to the symplectic group $\mathrm{Sp}(n, \mathbb{A})$ ($2n$ by $2n$ matrices) for $2n \geq \dim X$ for a large class of X. As a corollary, we show that if $2n = \dim X$ and all the local theta lifts $\Theta_n(\sigma_v)$ are nonzero, then $\Theta_n(\sigma)$ is nonzero if the standard L-function $L^S(s, \sigma)$ is nonzero at 1, and $\Theta_{n-1}(\sigma)$ is nonzero if $L^S(s, \sigma)$ has a pole at 1. The proof uses only essential structural features of the theta correspondence, along with a new result in the theory of doubling zeta integrals.

Let H and G be reductive linear algebraic groups defined over a number field F with ring of adeles $\mathbb{A}$. A fundamental problem in the theory of automorphic forms is to investigate the existence of liftings of irreducible automorphic representations of $H(\mathbb{A})$ to irreducible automorphic representations of $G(\mathbb{A})$. Here, by a lifting we mean a map from a subset of the set of irreducible automorphic representations of $H(\mathbb{A})$ to the set of irreducible automorphic representations of $G(\mathbb{A})$ such that if σ maps to π, then the local unramified components of σ and π are related by the functorality principle of Langlands, or by a natural extension of this principle in the nonconnected case. The existences of such liftings have important consequences in number theory.

Several programs exist for constructing and investigating liftings, and in this paper we shall be concerned with one such method, the theta correspondence. The theta correspondence provides liftings in the case that H and G form a reductive dual pair. While the theta correspondence thus applies only to a limited number of pairs H and G, liftings constructed via the theta correspondence are part of a rich structure with roots in the classical theory of automorphic forms. Examples of structures related to theta lifts include the Siegel-Weil formula and its consequences, period integrals, Fourier coefficients, and the behavior of L-functions at special points. In this work we consider the case when H is the orthogonal group of an even dimensional quadratic space and G is a symplectic group. If

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σ is an irreducible cuspidal automorphic representation of $H(\mathbb{A})$, then the theta lift $\Theta(\sigma)$ of σ is defined and is either zero or an automorphic representation of $G(\mathbb{A})$. In this paper we provide conditions under which $\Theta(\sigma) \neq 0$.

To state the main theorem we require some notation. Let X be an even dimensional nondegenerate symmetric bilinear space defined over a number field F, with orthogonal group $\mathrm{O}(X)$. For n a nonnegative integer, let $\mathrm{Sp}(n)$ be the symplectic group of rank n ($2n$ by $2n$ matrices). Fix a nontrivial additive character ψ of $\mathbb{A}/F$, and let ω be the corresponding Weil representation of $\mathrm{Sp}(n,\mathbb{A}) \times \mathrm{O}(X,\mathbb{A})$ realized on the Schrödinger model $L^2(X(\mathbb{A})^n)$, and let $\mathcal{S}(X(\mathbb{A})^n) \subset L^2(X(\mathbb{A})^n)$ be the subspace of functions defined in the notation section. Let $\chi_X = (\cdot, \operatorname{disc} X(F))_F$ be the character of $\mathbb{A}^\times/F^\times$ associated to $X(F)$, where $(\cdot,\cdot)_F$ is the Hilbert symbol of F. If $\varphi \in \mathcal{S}(X(\mathbb{A})^n)$, we set

$$\theta(g,h;\varphi) = \sum_{x \in X(F)^n} \omega(g,h)\varphi(x)$$

for $g \in \mathrm{Sp}(n,\mathbb{A})$ and $h \in \mathrm{O}(X,\mathbb{A})$. This series converges absolutely. If σ is an irreducible cuspidal automorphic representation of $\mathrm{O}(X,\mathbb{A})$ and V_σ is a realization of σ in the space of cusp forms, let

$$\theta(f,\varphi)(g) = \int_{\mathrm{O}(X,F)\backslash \mathrm{O}(X(\mathbb{A}))} \theta(g,h;\varphi) f(h)\, dh$$

for $f \in V_\sigma$, $\varphi \in \mathcal{S}(X(\mathbb{A})^n)$ and $g \in \mathrm{Sp}(n,\mathbb{A})$. This integral converges absolutely, and each $\theta(f,\varphi)$ is an automorphic form on $\mathrm{Sp}(n,\mathbb{A})$. The $\mathbb{C}$ vector space of all the functions $\theta(f,\varphi)$ will be denoted by $\Theta_n(V_\sigma)$, and is either zero or an automorphic representation of $\mathrm{Sp}(n,\mathbb{A})$; note that $\Theta_n(V_\sigma)$ depends on the realization V_σ. For further exposition of these global definitions see [HPS]. Also, for S a finite set of places including the archimedean primes and places where σ_v is ramified, let $L^S(s,\sigma)$ be the partial standard L-function of σ (see [KR1], section 2). Locally, suppose that v is a place of F, ω_v is the Weil representation of $\mathrm{Sp}(n,F_v) \times \mathrm{O}(X,F_v)$ on $L^2(X(F_v)^n)$ corresponding to ψ_v, and $\sigma \in \mathrm{Irr}(\mathrm{O}(X,F_v))$. Again, we have a character $\chi_{X(F_v)} = (\cdot, \operatorname{disc} X(F_v))_v$ of $F_v^\times$ associated to $X(F_v)$, where $(\cdot,\cdot)_v$ is the Hilbert symbol of F_v. We say that σ occurs in the theta correspondence for $\mathrm{O}(X,F_v)$ and $\mathrm{Sp}(n,F_v)$ if there exists $\pi \in \mathrm{Irr}(\mathrm{Sp}(n,F_v))$ such that

$$\mathrm{Hom}_{\mathrm{Sp}(n,F_v)\times \mathrm{O}(X,F_v)}((\omega_v, \mathcal{S}(X(F_v)^n)), \pi \otimes \sigma^\vee) \neq 0.$$

The contragredient is introduced for compatibility with the global lifting [KR2], and we say that π and σ correspond. Here, $S(X(F_v)^n)$ is again as defined in the notation section, and if v is infinite, then $(\omega_v, \mathcal{S}(X(F_v)^n))$, π and σ are actually Harish-Chandra modules and the homomorphism space consists of homomorphisms of Harish-Chandra modules. If v does not lie over 2, then the local Howe duality conjecture is known to hold [H], [Wal], such a π is uniquely determined, and we write $\pi = \theta_n(\sigma)$. Let k be the smallest nonnegative integer such that σ occurs in the theta correspondence with $\mathrm{Sp}(k,F_v)$; then we call any corresponding $\pi \in \mathrm{Irr}(\mathrm{Sp}(k, F_v))$ a first occurrence of σ, and it is known that $k \leq 2\dim X$. Again, if v does not lie over 2, then the first occurrence of σ is uniquely determined. Now suppose additionally that v is finite. We say that σ is tempered if the irreducible components of the restriction of σ to the connected group $\mathrm{SO}(X, F_v)$ are tempered. Suppose σ and a first occurrence of σ are tempered; by [R3], Theorem 4.4, if $\pi \in \mathrm{Irr}(\mathrm{Sp}(n, F_v))$ corresponds to σ and $2n \geq \dim X$, then π is uniquely determined, and we may also write $\pi = \theta_n(\sigma)$ even if v lies over 2. Throughout this paper, we *do not* assume the local Howe duality conjecture for primes lying over 2. Finally, we define

$$s_X(n) = \frac{2n - \dim X}{2}.$$

Theorem 1.2 (Main Theorem). *Let F be a totally real number field, and let X be an even dimensional nondegenerate symmetric bilinear space defined over F. Suppose that at each infinite place of F the signature of X is of the form $(2p, 2q)$ with $p \geq 0$, $q \geq 0$ and $p - q$ even (the signature is allowed to be different for different infinite places). Let σ be an irreducible cuspidal automorphic representation of $\mathrm{O}(X, \mathbb{A})$ with $\sigma \cong \otimes_v \sigma_v$, and let V_σ be a realization of σ in the space of cusp forms. Let n be a positive integer such that $2n \geq \dim X$. Assume:*

1. *σ_v occurs in the theta correspondence for $\mathrm{O}(X, F_v)$ and $\mathrm{Sp}(n, F_v)$ for all v;*
2. *for all v, σ_v is tempered and if σ_v first occurs in the theta correspondence with $\mathrm{Sp}(n', F_v)$ with $2n' > \dim X$, then the first occurrence of σ_v is tempered;*
3. *in the case $2n = \dim X$, $L^S(s, \sigma)$ does not vanish at $s_X(n + 1) = 1$ (a pole is permitted).*

Then $\Theta_n(V_\sigma) \neq 0$.

We note that the nonvanishing of $\Theta_n(V_\sigma)$ is independent of the realization V_σ of σ.

The proof of this theorem was inspired by the work of Böcherer and Schulze-Pillot in the case $\dim X = 4$ in the context of classical modular forms. The works [BSP1] and [BSP2] solved the Yoshida nonvanishing problem.

We have the following simple corollary.

Corollary 1.3. *Let F and X be as in the main theorem, and let $2n = \dim X$. Let σ be an irreducible cuspidal automorphic representation of $\mathrm{O}(X, \mathbb{A})$ with $\sigma \cong \otimes_v \sigma_v$, and let V_σ be a realization of σ in the space of cusp forms. Assume σ_v is tempered and occurs in the theta correspondence for $\mathrm{O}(X, F_v)$ and $\mathrm{Sp}(n, F_v)$ for all places v.*

1. *If $L^S(s, \sigma)$ does not vanish at $s_X(n + 1) = 1$ then $\Theta_n(V_\sigma) \neq 0$;*
2. *If $L^S(s, \sigma)$ has a pole at $s_X(n + 1) = 1$ then $\Theta_{n-1}(V_\sigma) \neq 0$.*

This corollary applies to a large class of nonhyperbolic spaces X and nongeneric σ, and guarantees that a given σ has a nonzero theta lift under manageable and expected conditions. Since tempered cuspidal automorphic representations occur naturally in number theory (any automorphic representation associated to a Galois representation will be tempered, for example), we expect the corollary to have some interesting number theoretic applications. Indeed, in a subsequent work we will use the corollary of the main theorem along with other results ([R1], [R2]) to construct the L-packets of representations of $\mathrm{GSp}(2, \mathbb{A})$ associated to many of the imprimitive four dimensional representations of $\mathrm{Gal}(\overline{F}/F)$ into ${}^L\,\mathrm{GSp}(2) = \mathrm{GSp}(2, \mathbb{C}) \subset \mathrm{Gl}(4, \mathbb{C})$ which do not factor through a proper parabolic subgroup of ${}^L\,\mathrm{GSp}(2)$, and show that those representations in the L-packets which are predicted to be cuspidal automorphic by the conjectural multiplicity formula of Arthur [A] are in fact cuspidal automorphic.

Besides [BSP1] and [BSP2], some recent works about the nonvanishing of global theta lifts include [GRS] and [M2]. The paper [GRS] is, in part, concerned with theta lifts of generic representations from even dimensional special orthogonal groups. It contains results for generic representations which are quite analogous to Corollary 1.3. If one assumes the weak Ramanujan conjecture that generic cuspidal automorphic representations are tempered, then Corollary 1.3 might be regarded as a generalization of the corresponding results of [GRS]. In [M2], the existence of a twist (by a one dimensional character) having a nonzero theta lift is related to poles of certain Eisenstein series.

The main theorem and its corollary are consistent with the expected general result for the nonvanishing of theta lifts from $\mathrm{O}(X, \mathbb{A})$ to $\mathrm{Sp}(n, \mathbb{A})$ when $2n \geq \dim X$ as proposed by Rallis. If the notation is as in our introductory remarks, the Rallis inner product formula asserts that for n in the stable range $2n \geq 2\dim X$, the inner product of the theta lift $\theta(f, \varphi)$ with itself is the nonzero number $L^S(s_X(n+1), \sigma)$ times the product over the places in S of certain local factors [Ra2], [Ra3]; here, S is a finite set of places including the archimedean places and the places where σ is ramified, and $L^S(s, \sigma)$ is the standard partial L-function of σ. By [L], these local factors for $v \in S$ are related to the local theta correspondence at v, and are nonzero. Thus, for $2n \geq 2\dim X$, the theta lift $\Theta_n(\sigma)$ is nonzero. Because the classical Siegel-Weil formula is for the stable range $2n \geq 2\dim X$, the Rallis inner product formula does not extend apriori into the range $2\dim X > 2n \geq \dim X$, though one might investigate the existence of a regularized inner product formula, as has been done for lifts from the symplectic group to even orthogonal groups. However, even for this case a nonvanishing result still needs to be deduced, and in particular the local factors of the formula still need to be related to the local theta correspondences ([KR2], section 8).

Still, one might expect a result similar to the one deduced from the Rallis inner product formula in the stable range. That is, if $L^S(s, \sigma)$ does not vanish at certain points (a global condition) and the local components of σ occur in the local theta correspondences with $\mathrm{Sp}(n)$ (a local condition), then the theta lift $\Theta_n(\sigma)$ is nonzero. Our criterion for nonvanishing includes such local and global conditions. Assumption (1) of the main theorem is the expected local condition, while (3) and the assumption in (2) that σ_v be tempered at the finite places together give the global condition. Because of the temperedness assumption, $L^S(s, \sigma)$ does not vanish at $s_X(k)$ for $k > n + 1$; thus, by (2) and (3), $L^S(s, \sigma)$ does not vanish at $s_X(k)$ for $k \geq n + 1$. This is used in the proof of the main theorem. Of course, for the purposes of our proof, in (2) we are assuming more than that $L^S(s, \sigma)$ does not vanish at $s_X(k)$ for $k > n+1$.

As we have mentioned, the proof of the main theorem is based on a clever idea from [BSP1], and the main ingredients are the local and global behavior of theta lifts in Witt towers, the local and global theory of zeta integrals for the symplectic group introduced by Piatetski-Shapiro and Rallis, and a deep result of Kudla and Rallis about the simplicity of the poles of certain Eisenstein series. A detailed description of the proof appears below. The key point is that, locally, after a first occurrence of σ_v in the theta correspondence, the theta lifts of σ_v further up the Witt tower of symplectic groups have Langlands data obtained from that of the previous theta lift one step down the tower by the addition of a known one dimensional quasi-character. Now if $\Theta_n(\sigma) = 0$, then for some $l > n$, $\Theta_l(\sigma)$ is nonzero and cuspidal. The key point, along with a new result in the theory of zeta integrals, then implies that order of vanishing at $s_X(l)$ of the twisted partial standard L-function of $\Theta_l(\sigma)$ exceeds the order allowed by functorality of the theta correspondence at the unramified places. The proof is thus based on essential structural features of the theta correspondence, along with the theory of zeta integrals. It does not use the Siegel-Weil formula nor an inner product identity.

Here is a sketch of the proof. As our basic setup we regard the orthogonal group from the main theorem as being fixed, and consider the various theta lifts $\Theta_k(\sigma)$ as we go up the Witt tower of symplectic groups $\mathrm{Sp}(k, \mathbb{A})$ for $2k \geq \dim X$. As general background, we know by [Ra1] that in the stable range $2k \geq 2\dim X$ we have $\Theta_k(\sigma) \neq 0$, and for all k, if $\Theta_k(\sigma) \neq 0$, then the theta lifts further up the Witt tower will also be nonzero, i.e., $\Theta_{k'}(\sigma) \neq 0$ for $k' > k$. Now our goal is to show that if n is as in the statement of the main theorem, then $\Theta_n(\sigma) \neq 0$. As we have just mentioned, initially all we know is that some theta lift, possibly far up the Witt tower, is nonzero; let $\Theta_{k+1}(\sigma) \neq 0$ be such a nonzero theta lift. The main step of the proof is to show that the theta lift one step down the Witt tower is also nonzero, i.e., $\Theta_k(\sigma) \neq 0$, provided that $k \geq n$. The theorem then follows by induction. To prove the implication $\Theta_{k+1}(\sigma) \neq 0 \implies \Theta_k(\sigma) \neq 0$ we argue by contradiction: suppose $\Theta_k(\sigma) = 0$. Let π be an irreducible constituent of the cuspidal representation $\Theta_{k+1}(\sigma)$ (note that $\Theta_{k+1}(\sigma)$ is cuspidal by [Ra1]); we obtain a contradiction by computing the order of vanishing of $L^S(s, \pi, \chi)$ at $s_X(k+1)$ in two different ways and obtaining conflicting answers (here $\chi = \chi_X$). On the one hand, $L^S(s, \pi, \chi)$ can be written in terms of $L^S(s, \sigma)$ and certain abelian factors using the result of Rallis [KR2] about the theta correspondence at the unramified places. Using that $L^S(s, \sigma)$ does not vanish at $s_X(k+1)$, we find easily that $L^S(s, \pi, \chi)$ has a zero of order at most $|S| - 2$ at $s_X(k+1)$. This uses only information at the unramified places.

On the other hand, we can also bring the ramified places into the argument. Thanks to the work of Piatetski-Shapiro and Rallis, we can also write $L^S(s, \pi, \chi)$ as a certain global χ-zeta integral associated to π, divided by a product of certain local χ_v-zeta integrals associated to the local components π_v, for $v \in S$. The key now is to show that each of the local χ_v-zeta integrals for $v \in S$ can be chosen to have a pole at $s_X(k+1)$; assume this for the moment. Then since the global χ-zeta integral can have at most a simple pole at $s_X(k+1)$ by an important theorem of Rallis and Kudla, we find that $L^S(s,\pi,\chi)$ has a zero at $s_X(k+1)$ of order at least $|S|-1$. This is a contradiction, and the main theorem is thus proven under the assumption of the claim about the local χ_v-zeta integrals.

This claim is proven in Lemma 1.1, the main lemma, which can be described as follows. Fix a place $v \in S$. We are given $\sigma_v \in \mathrm{Irr}(\mathrm{O}(X,F_v))$, and know that σ_v occurs in the theta correspondence with $\mathrm{Sp}(n,F_v)$. Further up the Witt tower, we also know that $\pi_v \in \mathrm{Irr}(\mathrm{Sp}(k+1,F_v))$ corresponds to σ_v, and the assertion of the main lemma is that there exists a χ_v-zeta integral for π_v (see the main lemma for a precise statement) which has a pole at $s_X(k+1)$. The proof of the main lemma has two components. The first is the description of the structure of π_v as a Langlands quotient. This was described in [R3] for nonarchimedean v and in [M1] for real v. Though these two references assert more about the Langlands data of π_v, the essential point is that the "first"element of the Langlands data for π_v is the quasi-character $\chi_v|\cdot|^{s_X(k+1)}$. The second result used in the proof of the main lemma is the main technical lemma, Lemma 1.4. The main technical lemma shows that for any element of $\mathrm{Irr}(\mathrm{Sp}(n,F_v))$ whose Langlands data has as first element a quasi-character $\delta=\chi_v|\cdot|^{s_0}$ for some $s_0 \in \mathbb{C}$, there exists a χ_v-zeta integral for π_v which has a pole at s_0. The main lemma clearly follows from these two results.

The proof of the main theorem is thus reduced to the proof of the main technical lemma, and the majority of this paper is devoted to proving this lemma. We will limit our comments about the main technical lemma in this introduction since at various points in the paper we give guiding comments; see in particular Section 4, which describes the idea of the proof of the main technical lemma. One point which we will describe here is the place of the main technical lemma in the conjectural basic theory of zeta integrals for $\mathrm{Sp}(n)$ analogous to theory of Godement-Jacquet zeta integrals for $\mathrm{Gl}(n)$ [GJ]. In summary, it appears that the main technical lemma is asserting a bit more than would follow from the conjectural basic theory. To be precise, let us use the notation of the main technical lemma, and assume F is nonarchimedean. Let $Z_\chi(\pi)$ be the $\mathbb{C}$ vector space generated by the $Z(s,f,\Phi)$ for f a coefficient of π and Φ a good χ-section as mentioned at the end of Section 2; define $Z_\chi(\delta_1),\dots,Z_\chi(\delta_t),Z_\chi(\tau)$ similarly. By analogy to the case of $\mathrm{Gl}(n)$, one might conjecture that

$$Z_\chi(\delta_1)\cdots Z_\chi(\delta_t)Z_\chi(\tau)=Z_\chi(\pi).$$

What does this conjectural equality say about the main technical lemma? It is not too difficult to see that $L(s+1/2,\chi\delta_1^{-1}) \in Z_\chi(\delta_1)$; if the above quality holds, then $L(s+1/2,\chi\delta_1^{-1}) \in Z_\chi(\pi)$. From this one can deduce that there exists a coefficient f of π and a good χ-section Φ for G such that $Z(s-1/2,f,\Phi)$ has a pole at s_0 if $\delta_1=\chi|\cdot|^{s_0}$. However, it is not evident that one can arrange Φ to be standard, as claimed by the main technical lemma. Thus, the main technical lemma appears to be asserting more than what would follow from the conjectural general theory.

While the main technical lemma does not follow from the conjectural theory of zeta integrals for $\mathrm{Sp}(n)$, our proof of the main technical lemma does provide some elements which may be useful for developing the general theory. Examples include the explicit description in Section 5 of the groups and embeddings needed to relate zeta integrals of parabolically induced representations to the inducing data, as first mentioned in [PSR1]; the introduction of auxiliary zeta integrals in the real case and the important estimate of Lemma 6.6, which builds on a result of [KR1]; the identification of a product of zeta integrals of Langlands quotient data as a zeta integral of the quotient under certain conditions in Theorems 6.7 and 6.10, in analogy to results from [GJ]; and the construction of χ-sections in section 7. Also, at various points in the sections 2–8 we make some comments on the development of the general theory. One point that is perhaps worth repeating here is that the conjectural theory of zeta integrals for $\mathrm{Sp}(n)$ over nonarchimedean local fields may require further progress in representation theory. If F is nonarchimedean, then for $\mathrm{Gl}(n)$ the equality analogous to the last displayed equality above is proven by an indirect argument using the classification of the tempered dual of $\mathrm{Gl}(n, F)$; the argument for $\mathrm{Sp}(n)$ may also require knowledge of the tempered dual of $\mathrm{Sp}(n, F)$, which has yet to appear.

Finally, it is likely that the main theorem can be generalized, and in particular some of its hypotheses may be unnecessary. The main theorem makes two assumptions on the signature of X at each of the infinite places v. The first is that signature is of the form $(2p, 2q)$; the second is that $p-q$ is even, i.e., $\chi_v = \chi_{X(F_v)} = 1$. The first assumption is required to apply the results of [M1], and we are not sure if they may be omitted. However, at the finite places, the analogous results of [R3] apply to any even dimensional symmetric bilinear space, so the assumption that the signature be of the form $(2p, 2q)$ at the real places might be superfluous. We use $\chi_v = 1$ in the proof of the main technical lemma in the real case. This assumption is needed to apply the results of [KR1], stated in Lemma 6.5 and Theorem 6.8 below. Given the conjectural general theory of zeta integrals, and since the analogous assumption is not needed at the finite places, is likely that the assumption $\chi_v = 1$ can be dropped when v is real. This would require that the results of [KR1] be generalized.

Perhaps the most interesting assumption of the main theorem is (2). As we have mentioned, the temperedness of σ_v at the finite places implies that $L^S(s, \sigma)$ does not vanish at $s_X(k)$ for $k > n+1$, which is used in the proof of the main theorem. Instead of assuming that σ_v is tempered at the finite places, one could assume that $L^S(s, \sigma)$ does not vanish at these points. However, the temperedness assumptions are also required for the application of [R3] and [M1]. One might wonder when these assumptions can be dropped. That is, suppose $\sigma_v \in \mathrm{Irr}(\mathrm{O}(X, F_v))$ occurs in the theta correspondence with $\mathrm{Sp}(k, F_v)$ for some $2k \geq \dim X$. Let $\pi_v \in \mathrm{Irr}(\mathrm{Sp}(k+1, F_v))$ correspond to σ_v in the theta correspondence for $\mathrm{O}(X, F_v)$ and $\mathrm{Sp}(k+1, F_v)$. Is it true that the first element in the Langlands data for π_v is the quasi-character $\chi_v |\cdot|^{s_X(k+1)}$? One could also ask if a weaker, but still adequate, result holds: Is it true that there exist a standard section Φ_v and a matrix coefficient f_v for π_v such that $Z(s-1/2, f_v, \Phi_v)$ has a pole at $s_X(k+1)$? It would be very interesting to know when these statements hold.

We also expect that there is a result analogous to the main theorem and its corollary if one begins instead with an irreducible cuspidal automorphic representation of the symplectic group, and lifts to an even dimensional orthogonal group. For this, a generalization of [R3] to this case would be required, along with a strengthening of [M1]. The arguments of this paper probably generalize in a straightforward way to this case. Note in particular that the results of [KR1] would also apply without generalization.

The paper is organized in the following way. In Section 1 we prove the main theorem, assuming the main technical lemma. In the remainder of the paper we prove the main technical lemma. We begin by recalling in Sections 2 and 3 some definitions and results about zeta integrals of symplectic and general linear groups. Given this background, in Section 4 we outline the proof of the main technical lemma, which has two steps. Before carrying out these two steps we first define some required groups and embeddings in Section 5. Then in Sections 6 and 7 we carry out the two steps required for the proof of the technical lemma. Finally, in Section 8 we sum up and prove the main technical lemma.

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Notation. Suppose F is a number field; besides the notation introduced above, we will use the following definitions. Let $\mathbb{A}_f$ be the finite adeles of F, and set $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$. The space $\mathcal{S}(X(\mathbb{A})^n)$ used in the definition of global theta lifts is the restricted direct product of the spaces $\mathcal{S}(X(F_v)^n)$ for v a place of F. If v is finite, then $\mathcal{S}(X(F_v)^n)$ is the space of locally constant, compactly supported functions on $X(F_v)^n$. If v is infinite, then $\mathcal{S}(X(F_v)^n)$ is the space of $\tilde{K}$ finite vectors for the Weil representation (Schrödinger model) of the metaplectic group $\mathrm{Mp}(X\otimes Y, F_v)$ on $L^2(X(F_v)^n)$; here, Y is the symplectic F space of dimension $2n$ and $\tilde{K}$ is the usual maximal compact subgroup of the metaplectic group. Let G be a semi-simple linear algebraic group defined over F, let $\mathfrak{g}$ be the Lie algebra of $G(F_\infty)$, and let K be a maximal compact subgroup of $G(F_\infty)$. For the purposes of this paper, an automorphic representation will be a $(\mathfrak{g}, K)\times G(\mathbb{A}_f)$ submodule of the space of automorphic forms on $G(F)\backslash G(\mathbb{A})$.

Next, suppose that F is a local field of characteristic zero, and G is a reductive linear algebraic group defined over F, with $F = \mathbb{R}$ if F is archimedean. Suppose first F is nonarchimedean with integers $\mathfrak{O}$, prime ideal $\mathfrak{p} = \pi_F\mathfrak{O} \subset \mathfrak{O}$, Hilbert symbol $(\cdot,\cdot)_F$, and valuation $|\cdot|$ such that if μ is an additive Haar measure on F, then $\mu(xA) = |x|\mu(A)$ for $x \in F$ and $A \subset F$. Let $q = |\mathfrak{O}/\mathfrak{p}|$. The compactly supported, smooth (i.e., locally constant) functions on $G(F)$ will be denoted by $C_c^\infty(G(F))$ or $\mathcal{S}(G(F))$. Let $\mathrm{Irr}(G(F))$ be the set of equivalence classes of smooth admissible irreducible representations of $G(F)$. Let π be a smooth representation of $G(F)$. The smooth contragredient representation of π is $\pi^\vee$, and if π admits a central character, we denote it by ω_π. A coefficient of π is a finite $\mathbb{C}$ linear combination of matrix coefficients of π. In the case G is connected, we say that π is tempered if ω_π is unitary and every matrix coefficient of π lies in $\mathrm{L}^{2+\epsilon}(G(F)/\mathrm{Z}(G(F)))$ for all $\epsilon > 0$. If $\pi \in \mathrm{Irr}(\mathrm{Gl}(n,F))$, then $e(\pi)$ is the unique real number such that the central character of $\pi\otimes|\det|^{-e(\pi)}$ is unitary, and π is essentially tempered if $\pi\otimes|\det|^{-e(\pi)}$ is tempered. Suppose M and N are closed subgroups of $G(F)$ such that M normalizes N, $M\cap N = 1$, $P = MN$ is closed in $G(F)$, N is unimodular and $P\backslash G(F)$ is compact. Fix a

Haar measure dn on N, and for $m \in M$, let $\delta_P(m)$ be the positive number such that all $f \in \mathcal{S}(N)$,

$$\int_N f(m^{-1}nm)\,dn = \delta_P(m) \int_N f(n)\,dn.$$

Suppose that σ is a smooth representation of M. Then $\mathrm{Ind}_P^{G(F)} \sigma$ is the representation of $G(F)$ by right translation on the $\mathbb{C}$ vector space of smooth functions f on $G(F)$ with values in σ such that $f(mng) = \delta(m)^{1/2}\sigma(m)f(g)$ for $m \in M$, $n \in N$ and $g \in G(F)$. If χ is a quasi-character of $F^\times$, then $c(\chi)$ is the conductor of χ, i.e., $c(\chi) = 0$ if χ is unramified and otherwise $c(\chi)$ is the smallest positive integer such that $\chi(1 + \mathfrak{P}^{c(\chi)}) = 1$.

Suppose $F = \mathbb{R}$. We will use definitions and results from [W]. Let K be a maximal compact subgroup of $G(\mathbb{R})$, and let $\mathfrak{g}$ be the Lie algebra of the real Lie group $G(\mathbb{R})$. A Hilbert representation ϱ of $G(\mathbb{R})$ is a continuous representation of G on a separable Hilbert space such that the restriction $\varrho|_K$ of ϱ to K is unitary. Let $\mathrm{Irr}(G(\mathbb{R}))$ be the set of equivalence classes of irreducible $(\mathfrak{g}, K)$-modules (which we will also call Harish-Chandra modules). If ϱ is a Hilbert representation of $G(\mathbb{R})$, then the space ϱ_K of smooth, K-finite vectors in ϱ is a Harish-Chandra module. Let $\pi \in \mathrm{Irr}(G(\mathbb{R}))$. The contragredient of π will be denoted by $\pi^\vee$. A matrix coefficient of π is a function from $G(\mathbb{R})$ to $\mathbb{C}$ defined by $g \mapsto \langle \varrho(g)v, w\rangle$ for some $v, w \in \pi$ and ϱ an irreducible Hilbert representation of $G(\mathbb{R})$ with inner product $\langle \cdot, \cdot \rangle$ such that $\varrho_K = \pi$ (such a ϱ exists). The $\mathbb{C}$ vector space generated by the matrix coefficients of π is independent of the choice of ϱ. A coefficient of π is a finite $\mathbb{C}$ linear combination of matrix coefficients of π. There is the concept of π being tempered ([W], 5.1.1). If π is tempered than there exists a unique irreducible unitary representation ϱ of $G(\mathbb{R})$ such that $\varrho_K = \pi$. We say that an irreducible unitary representation ϱ of $G(\mathbb{R})$ is tempered if $\varrho_K \in \mathrm{Irr}(G(\mathbb{R}))$ is tempered. If ϱ is an irreducible Hilbert representation of $\mathrm{Gl}(n, \mathbb{R})$ we define $e(\varrho)$, etc., as in the nonarchimedean case. Let M, N, P and δ_P be as in the nonarchimedean case. Assume additionally $G(\mathbb{R}) = PK$. Let σ be a Hilbert representation of M with respect to $K \cap M$. There is the concept of the Hilbert representation of $G(\mathbb{R})$ unitarily induced from σ, with a definition similar to the nonarchimedean case ([W], 1.5.2); the $(\mathfrak{g}, K)$ module of K-finite, smooth vectors in this representation will be denoted by $\mathrm{Ind}_P^{G(\mathbb{R})} \sigma$. The compactly supported, smooth functions on $G(\mathbb{R})$ will be denoted by $C_c^\infty(G(\mathbb{R}))$.

Finally, we shall need the Langlands classification for $\mathrm{Sp}(n, F)$. This can be stated as follows. Let $n = n_1 + \cdots + n_t + n_0$, where $n_1, \ldots, n_t, n_0$ are nonnegative integers, with $n_1, \ldots, n_t$ positive if $t > 0$. For $1 \le i \le t$ let δ_i be essentially tempered representations of $Gl(n_i, F)$ such that $e(\delta_1) > \cdots > e(\delta_t) > 0$, and let τ be a tempered representation of $\mathrm{Sp}(n_0, F)$. Let $P_{n_1,\ldots,n_t}$ be the standard parabolic subgroup of $\mathrm{Sp}(n, F)$ (as defined in [R3], for example) with Levi subgroup isomorphic to $\mathrm{Gl}(n_1, F) \times \cdots \times \mathrm{Gl}(n_t, F) \times \mathrm{Sp}(n_0, F)$. Then $\mathrm{Ind}_{P_{n_1,\ldots n_t}}^{\mathrm{Sp}(n,F)} (\delta_1 \otimes \cdots \otimes \delta_t \otimes \tau)$ has a unique nonzero quotient $L(\delta_1 \otimes \cdots \otimes \delta_t \otimes \tau)$ in $\mathrm{Irr}(\mathrm{Sp}(n, F))$, which we call the Langlands quotient of the Langlands data $(\delta_1, \ldots, \delta_t, \tau)$. Moreover, every element of $\mathrm{Irr}(\mathrm{Sp}(n, F))$ is of the form $L(\delta_1 \otimes \cdots \otimes \delta_t \otimes \tau)$ for some unique

Langlands data $(\delta_1, \dots, \delta_t, \tau)$. For information about Langlands quotients see for example [W], [BW] and [T].

1. Proof of the main theorem

In this section we prove the main theorem and its corollary. For a sketch of the proof, see the introduction to this paper. We begin by proving the main lemma, which in turn depends on [R3], [M1] and the main technical lemma, Lemma 1.4. In the following lemma we write $\theta_{k+1}(\sigma)$ for the element of $\mathrm{Irr}(\mathrm{Sp}(k+1, F_v))$ corresponding to σ. As we mentioned before the statement of the main theorem, this is justified even if F is nonarchimedean of even residual characteristic by Theorem 4.4 of [R3]. The notation and definitions for zeta integrals appear in Section 2. The proof of the main theorem can be understood without a detailed knowledge of zeta integrals outside the statement of the main lemma.

Lemma 1.1 (Main Lemma). *Let F be a local field of characteristic zero, with $F = \mathbb{R}$ if F is archimedean. Let X be an even dimensional nondegenerate symmetric bilinear space defined over F, and let k be a positive integer such that $2k \geq \dim X$. Let $\sigma \in \mathrm{Irr}(\mathrm{O}(X, F))$ and assume σ occurs in the theta correspondence with $\mathrm{Sp}(k, F)$. Assume σ is tempered and if σ first occurs in the theta correspondence with $\mathrm{Sp}(n', F)$ with $2n' > \dim X$, then the first occurrence of σ is tempered. If $F = \mathbb{R}$, assume the signature of X is of the form $(2p, 2q)$ with $p \geq 0$, $q \geq 0$ and $p - q$ even. Then there exists a standard K'-finite χ_X-section Φ for $\mathrm{Sp}(k+1, F)$ and a matrix coefficient f of $\theta_{k+1}(\sigma) \in \mathrm{Irr}(\mathrm{Sp}(k+1, F))$ such that $Z(s - 1/2, f, \Phi)$ has a pole at $s_X(k+1)$.*

Proof. We claim first that $\theta_{k+1}(\sigma)$ has Langlands quotient form (see the notation section)

$$\theta_{k+1}(\sigma) = L(\chi_X |\cdot|^{s_X(k+1)} \otimes \delta_2 \otimes \cdots \otimes \delta_t \otimes \tau)$$

where $\chi_X|\cdot|^{s_X(k+1)}$ is the quasi-character of $Gl(1, F) = F^\times$ such that $x \mapsto \chi_X(x)|x|^{s_X(k+1)}$. If F is nonarchimedean this follows from (1) of Theorem 4.2 and Theorem 4.4 of [R3]. Suppose $F = \mathbb{R}$. Assume σ first occurs in the theta correspondence with $\mathrm{Sp}(n', F)$ with $2n' \leq \dim X$. Then in the terminology of [M1], by Théorème IV.3 of [M1], σ satisfies $(\dagger)'$ of [M1]. As σ is tempered, by (ii) of Théorème III.13 of [M1], $\theta_{k+1}(\sigma)$ has the claimed form. Next, assume σ first occurs in the theta correspondence with $\mathrm{Sp}(n', F)$ with $2n' > \dim X$. Then $\theta_{n'}(\sigma)$ is tempered by assumption; by Corollaire IV.5 (ii) of [M], $\theta_{k+1}(\sigma)$ again has the claimed form. Since $p - q$ is even if $F = \mathbb{R}$, $\chi_X = \mathrm{sign}^{p-q} = 1$ if $F = \mathbb{R}$. The lemma now follows from the Lemma 1.4, the main technical lemma. □

We can now give the proof of the main theorem.

Proof of Theorem 1.2, the main theorem. To begin, we observe that the hypotheses of the theorem hold if n is replaced by k a positive integer, for $k \geq n$. The only hypothesis that needs to be checked is (1); this hypothesis holds for n replaced by k, $k \geq n$, by the persistence property of local theta lifts, which asserts that if an element of $\mathrm{Irr}(\mathrm{O}(X, F_v))$

occurs in the theta correspondence with $\mathrm{Sp}(t, F_v)$, then it occurs in the theta correspondence with $\mathrm{Sp}(t+1, F_v)$; if v is finite see, for example, a) of the Remarque on p. 67 of [MVW] (note that if v is finite, even if v lies over 2, our definition of a representation occurring in the theta correspondence from the introduction is equivalent to the representation being a quotient of the Weil representation by 2) a) of Théorème principal, p. 69 of [MVW]). Thus, if the theorem is true, then $\Theta_k(V_\sigma) \neq 0$ for $k \geq n$; this is consistent with the persistence property of global theta lifts [Ra1]. Now it is well known that if k is large enough, then $\Theta_k(V_\sigma) \neq 0$; in fact, by [Ra1], $\Theta_k(V_\sigma) \neq 0$ for k in the stable range $k \geq \dim X$. We may thus assume that $\dim X > n$. We will prove the theorem by showing inductively that $\Theta_{\dim X}(V_\sigma) \neq 0, \Theta_{\dim X-1}(V_\sigma) \neq 0, \ldots, \Theta_{n+1}(V_\sigma) \neq 0, \Theta_n(V_\sigma) \neq 0$. We already noted that $\Theta_{\dim X}(V_\sigma) \neq 0$. Assume $\Theta_{k+1}(V_\sigma) \neq 0$ for some $\dim X > k \geq n$; we must show $\Theta_k(V_\sigma) \neq 0$. Assume $\Theta_k(V_\sigma) = 0$; we will obtain a contradiction.

Now $\Theta_{k+1}(V_\sigma)$ is cuspidal since $\Theta_k(V_\sigma) = 0$; see Remark I.2.1, p. 351 of [Ra1] or the remark on p. 78 after Corollary 5.4 of [HPS]. By [M3], $\pi = \Theta_{k+1}(V_\sigma)$ is irreducible. Let $\pi \cong \otimes_v \pi_v$. Then $\pi_v = \theta_{k+1}(\sigma_v)$ for all v by the proof of Theorem I.2.2 on p. 355 of [Ra1]. Let S be the union of the set of archimedean places and the set of places where σ, π and $\chi = \chi_X$ are ramified; then S is a finite set. We will derive a contradiction by computing the order of vanishing of $L^S(s, \pi, \chi)$ at $s(k+1) = s_X(k+1)$ in two ways and obtaining conflicting results. Here, $L^S(s, \pi, \chi)$ is the partial standard L-function of π twisted by χ; see Section 7 of [KR2] or section 2 of [KR1].

First, by the functorality of the theta correspondence at the unramified places, i.e., those places not in S, we have

$$\begin{aligned} L^S(s,\pi,\chi) = {} & \zeta^S(s) L^S(s,|\cdot|^{s(k+1)}) L^S(s,|\cdot|^{-s(k+1)}) L^S(s,|\cdot|^{s(k+1)-1}) L^S(s,|\cdot|^{-(s(k+1)-1)}) \\ & \cdots L^S(s,|\cdot|) L^S(s,|\cdot|^{-1}) L^S(s,\sigma). \end{aligned}$$

See Corollary 7.1.4 of [KR2]. Here, $\zeta(s)$ is the zeta function of F, and the superscript S means the product of all Euler factors whose corresponding places are not in S. Since σ is tempered at the finite places, $L^S(s, \sigma)$ does not vanish at $s(k+1)$ if $s(k+1) > 1$, i.e., $2k > \dim X$; if $2k = \dim X$, then $L^S(s, \sigma)$ does not vanish at $s(k+1)$ by hypothesis. Let us consider the behavior of the other factors at $s(k+1)$. Suppose first $s(k+1) > 1$. Then (see for example [E] 450D), $L^S(s, |\cdot|^{-s(k+1)})$ has a zero of order $|S|-1$ at $s(k+1)$, $L^S(s, |\cdot|^{-(s(k+1)-1)})$ has a simple pole at $s(k+1)$, and the remaining factors other than $L^S(s, \sigma)$ have neither a pole nor a zero at $s(k+1)$. Thus, if $s(k+1) > 1$, $L^S(s, \pi, \chi)$ has a zero of order at most $|S|-2$ at $s(k+1)$. Suppose $s(k+1) = 1$. Then $L^S(s, \pi, \chi) = \zeta^S(s)L^S(s, |\cdot|)L^S(s, |\cdot|^{-1})L^S(s, \sigma)$, $L^S(s, |\cdot|^{-1})$ has a zero of order $|S|-1$ at $s(k+1) = 1$, $\zeta^S(s)$ has a simple pole at $s(k+1) = 1$, and the remaining factor $L^S(s, |\cdot|)$ besides $L^S(s, \sigma)$ has neither a pole nor a zero at $s(k+1)$. Thus, in the case $s(k+1) = 1$ we again find that $L^S(s, \pi, \chi)$ has a zero of order at most $|S|-2$ at $s(k+1)$. Thus, altogether, $L^S(s, \pi, \chi)$ has a zero of order at most $|S|-2$ at $s(k+1)$.

On the other hand, we can bring the places in S into the argument. We may assume that each π_v is endowed with an inner product such that if $w = \otimes_v w_v$, $w' = \otimes_v w'_v \in \pi$, then

$$\langle w, w' \rangle = \prod_v \langle w_v, w'_v \rangle,$$

where $\langle w, w' \rangle$ is the Petersson inner product with respect to our fixed invariant measure on $\mathrm{Sp}(k+1, F) \backslash \mathrm{Sp}(k+1, \mathbb{A})$. For $v \notin S$, let w_v be the K_v-invariant vector in π_v used to define the restricted tensor product $\otimes_v \pi_v$; we may assume that $\langle w_v, w_v \rangle = 1$ for $v \notin S$. Also, for $v \notin S$ let Φ_v be the unramified χ_v-section for $\mathrm{Sp}(k+1, F_v)$. If $v \notin S$ and f_v is the matrix coefficient of π_v defined by $f_v(g) = \langle \pi(g) w_v, w_v \rangle$, then it is known that (see for example [KR2])

$$Z(s - 1/2, f_v, \Phi_v) = L(s, \pi_v, \chi_v).$$

Let $v \in S$. By the main lemma, there exist $w_v, w'_v \in \pi_v$ and a K'_v-finite standard χ_v-section Φ_v for $\mathrm{Sp}(k+1, F_v)$ such that if f_v is the matrix coefficient of π_v defined by $f_v(g) = \langle \pi_v(g) w_v, w'_v \rangle$, then $Z(s - 1/2, f_v, \Phi_v)$ has a pole at $s(k+1)$. Let

$$w = \otimes_v w_v, \qquad w' = \otimes_{v \notin S} w_v \otimes \otimes_{v \in S} w'_v$$

and $\Phi = \prod \Phi_v$. Then by, for example, (7.2.8) of [KR2],

$$L^S(s, \pi, \chi) = \frac{Z^*(s - 1/2, w, w', \Phi)}{\prod_{v \in S} Z(s - 1/2, f_v, \Phi_v)}$$

where $Z^*(s, w, w', \Phi) = b^S_{2(k+1)}(s, \chi) Z(s, w, w', \Phi)$, and $b^S_{2(k+1)}(s, \chi)$ and $Z(s, w, w', \Phi)$ are defined as in [KR2]; $b^S_{2(k+1)}(s, \chi)$ is a certain normalizing factor and $Z(s, w, w', \Phi)$ is a global zeta integral. By Theorem 1.1 of [KR2] (see the comment before Corollary 7.2.3 of [KR2]), $Z^*(s - 1/2, w, w', \Phi)$ has at most a simple pole at $s(k+1)$. It follows that $L^S(s, \pi, \chi)$ has a zero at $s(k+1)$ of order at least $|S| - 1$. This is a contradiction. □

Corollary 1.3 follows from the main theorem and some other results.

Proof of Corollary 1.3. As in the statement of the corollary, let $2n = \dim X$. Assume that σ_v is tempered for all finite places v, σ_v occurs in the theta correspondence for $\mathrm{O}(X, F_v)$ and $\mathrm{Sp}(n, F_v)$ for all places v, and $L^S(s, \sigma)$ does not vanish at 1. Let v be a finite place. Since σ_v occurs in the theta correspondence with $\mathrm{Sp}(n, F_v)$, $2n = \dim X$, and σ_v is tempered, by Theorem 4.2 of [R3], the first occurrence of σ_v is tempered. The main theorem now implies that $\Theta_n(V_\sigma) \neq 0$. Now assume additionally that $L^S(s, \sigma)$ has a pole at 1. Suppose $\Theta_{n-1}(V_\sigma) = 0$. Then, as in the proof of the main theorem, $\pi = \Theta_n(V_\sigma)$ is irreducible and cuspidal. By corollary 7.1.4 of [KR2],

$$L^S(s, \pi, \chi) = \zeta^S(s) L^S(s, \sigma).$$

Since $\zeta^S(s)$ and $L^S(s, \sigma)$ have poles at 1, $L^S(s, \pi, \chi)$ has at least a double pole at $s = 1$. This contradicts Theorem 7.2.5 of [KR2] which asserts that $L^S(s, \pi, \chi)$ has at most a simple pole at $s = 1$. □

Lemma 1.4 (Main Technical Lemma). *Let n be a positive integer, and let F be a local field of characteristic zero, with $F = \mathbb{R}$ if F is archimedean. Let χ be a quasi-character of $F^\times$, with $\chi = 1$ if $F = \mathbb{R}$. Let $\pi \in \mathrm{Irr}(\mathrm{Sp}(n, F))$, and write π as a Langlands quotient (see the notation section) $\pi = L(\delta_1 \otimes \cdots \otimes \delta_t \otimes \tau)$, where $n = n_1 + \cdots + n_t + n_0$ is an ordered partition of n with $n_1, \ldots, n_t$ positive if $t > 0$, δ_i are essentially tempered irreducible representations of $\mathrm{Gl}(n_i, F)$ for $1 \le i \le t$ with $e(\delta_1) > \cdots > e(\delta_t) > 0$, and τ is a tempered irreducible representation of $\mathrm{Sp}(n_0, F)$. Assume that $t > 0$ and $n_1 = 1$, so that δ_1 is a quasi-character, and suppose $\delta_1 = \chi|\cdot|^{s_0}$ for some $s_0 \in \mathbb{C}$. Then there exists a coefficient f of π and a standard K'-finite χ-section Φ for G such that $Z(s - 1/2, f, \Phi)$ has a pole at s_0.*

After preparations in sections 2–7, the proof of the main technical lemma appears in Section 8.

Remark 1.5. In the nonarchimedean case we will actually show the stronger statement that for any quasi-character δ_1, not just those of the form $\chi|\cdot|^{s_0}$, there exist a coefficient f of π, a standard K'-finite χ-section Φ for G and constants A and B, $A \neq 0$, such that

$$Z(s - 1/2, f, \Phi) = Aq^{Bs} L(s, \chi\delta_1^{-1}).$$

2. Zeta integrals for the symplectic group

The remainder of this paper is devoted to the proof of the main technical lemma. In this section and the next, we give definitions and recall results concerning zeta integrals. In Section 4 we then describe the idea of the proof of the main technical lemma.

We follow [PSR1,2,3]. The following notation will be fixed for the remainder of the paper. Let F be a local field of characteristic zero, with $F = \mathbb{R}$ if F is archimedean. Let $(V, \langle\cdot,\cdot\rangle)$ be a nondegenerate symplectic bilinear space of dimension $2n$ over F. Let $G = \mathrm{Sp}(V)$. Define a nondegenerate symplectic bilinear space $(V', \langle\cdot,\cdot\rangle)$ over F by letting $V' = V \times V$ and $\langle(v_1, v_2), (v_1', v_2')\rangle = \langle v_1, v_1'\rangle - \langle v_2, v_2'\rangle$. Let $H = \mathrm{Sp}(V')$. We have an embedding

$$i : G \times G \hookrightarrow H$$

defined by $i(g, g')(v, v') = (gv, g'v')$. Let V^d be the subspace of pairs (v, v) for $v \in V$. Then V^d is a Lagrangian of V'. Let P' be the parabolic subgroup of H stabilizing V^d. Note that G embedded in H on the diagonal is contained in P'. Also, the set $P'i(G \times 1)$ is dense in H. Fix a symplectic basis $\mathcal{B} = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ of V, and identify G with $\mathrm{Sp}(n, F)$ via this basis. Let K be the usual maximal compact subgroup of G, so that $K = \mathrm{Sp}(n, \mathfrak{O})$ if F is nonarchimedean, and K is the stabilizer of the point i in the Siegel upper half space of degree n if $F = \mathbb{R}$. The elements

$$(x_1, 0), \ldots, (x_n, 0), (0, y_1), \ldots, (0, y_n), (y_1, 0), \ldots, (y_n, 0), (0, x_1), \ldots, (0, x_n)$$

form a symplectic basis for V'. Identifying H with $\mathrm{Sp}(2n, F)$ via this basis, we let K' be the usual maximal compact subgroup of H. Since with this choice of basis

$$i(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}) = \begin{bmatrix} a & 0 & b & 0 \\ 0 & d' & 0 & c' \\ c & 0 & d & 0 \\ 0 & b' & 0 & a' \end{bmatrix},$$

it follows that $i(K \times K) \subset K'$. This choice of basis for V' has the advantage that i takes on a simple form; however, the elements of P' are not so easily described. We will instead use the symplectic basis $\mathcal{B}' = \{e_1, \dots, e_n, f_1, \dots, f_n, e_1', \dots e_n', f_1', \dots, f_n'\}$ for V', where $e_i = (x_i, x_i)$, $f_i = (y_i, y_i)$, $e_i' = (y_i, 0)$ and $f_i' = (0, x_i)$ for $1 \leq i \leq n$. This choice of basis will be fixed for the remainder of the paper. With respect to $\mathcal{B}'$, the elements of the Siegel parabolic P' have the usual form, and

$$i(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, 1) = \begin{bmatrix} a & b & b & 0 \\ 0 & 1 & 0 & 0 \\ c & d-1 & d & 0 \\ 1-a & -b & -b & 1 \end{bmatrix}, \quad i(1, \begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ c & d & 0 & c \\ -c & 1-d & 1 & -c \\ a-1 & b & 0 & a \end{bmatrix}.$$

Note that $K' = \mathrm{Sp}(2n, \mathfrak{O})$ if F is nonarchimedean, and K' is conjugate to the usual maximal compact subgroup of $H = \mathrm{Sp}(2n, \mathbb{R})$ if $F = \mathbb{R}$.

Next, we define the test functions for the zeta integrals. Suppose first that F is nonarchimedean. Fix a quasi-character χ of $F^\times$. For $s \in \mathbb{C}$, let $\mathrm{I}_H(s, \chi)$ be the $\mathbb{C}$ vector space of functions $\Phi : H \to \mathbb{C}$ that are right invariant under a compact open subgroup of H and satisfy $\Phi(ph) = \alpha_\chi(p, s)\Phi(h)$ for $h \in H$ and $p \in P'$; here,

$$p = \begin{bmatrix} a & b \\ 0 & {}^t a^{-1} \end{bmatrix}, \quad \alpha_\chi(p, s) = \chi(\det a)|\det a|^{s+\rho_{2n}},$$

where $\rho_{2n} = (2n+1)/2$. Under right translation, $\mathrm{I}_H(s, \chi)$ is a smooth representation of H. A χ-**section for** G, or simply a χ-section, is a continuous function $\Phi : H \times \mathbb{C} \to \mathbb{C}$ such that $\Phi(\cdot, s) \in \mathrm{I}_H(s, \chi)$ for all $s \in \mathbb{C}$, $\Phi(h, \cdot)$ is holomorphic for all $h \in H$, and there exists a compact open subgroup L of H such that $\Phi(hk, s) = \Phi(h, s)$ for all $h \in H$, $k \in L$ and $s \in \mathbb{C}$. Let $\mathrm{I}_H(\chi)$ be the $\mathbb{C}$ vector space of all χ-sections. Then $\mathrm{I}_H(\chi)$ is a smooth representation of H under right translation. A χ-section Φ is **standard** if the restriction of Φ to K is independent of s, i.e., $\Phi(k, s) = \Phi(k, s')$ for $k \in K'$ and $s, s' \in \mathbb{C}$. Let $\mathrm{I}_H^{\mathrm{Stan}}(\chi)$ be the $\mathbb{C}$ subspace of $\mathrm{I}_H(\chi)$ consisting of all standard χ-sections. Clearly, $\mathrm{I}_H^{\mathrm{Stan}}(\chi)$ is closed under the right translation action of K', and restriction to K' defines a K' isomorphism from $\mathrm{I}_H^{\mathrm{Stan}}(\chi)$ to $\mathrm{Ind}_{P' \cap K'}^{K'} \chi$. If χ is unramified, we define the **unramified section** $\Phi \in \mathrm{I}_H(\chi)$ by $\Phi(pk, s) = \alpha_\chi(p, s)$ for $p \in P'$, $k \in K'$ and $s \in \mathbb{C}$. Clearly, this is well-defined.

Now suppose that $F = \mathbb{R}$. Fix a quasi-character χ of $\mathbb{R}^\times$. For now χ will be arbitrary; beginning in Section 6 we will assume $\chi = 1$. For $s \in \mathbb{R}$, let $\mathrm{I}_H(s, \chi)$ be the $\mathbb{C}$ vector space of functions $\Phi : H \to \mathbb{C}$ such that Φ is smooth and $\Phi(ph) = \alpha_\chi(p,s)\Phi(h)$ for $p \in P'$ and $h \in H$; here α_χ is defined as in the nonarchimedean case. Let $\mathcal{D}(H)$ be the $\mathbb{C}$ vector space of differential operators on H. For $D \in \mathcal{D}(H)$ and C a compact subset of H, define a semi-norm $\|\cdot\|_{D,C}$ on $\mathrm{I}_H(s,\chi)$ by

$$\|\Phi\|_{D,C} = \sup\{|(D\Phi)(h)| : h \in C\}.$$

With the topology defined by this family of semi-norms, $\mathrm{I}_H(s,\chi)$ is a Frechet space. Moreover, $\mathrm{I}_H(s,\chi)$ is a differentiable representation of H under right translation in the sense of [War], p. 259. A **χ-section for** G is a smooth function $\Phi : H \times \mathbb{C} \to \mathbb{C}$ such that $\Phi(\cdot, s) \in \mathrm{I}_H(s,\chi)$ for all $s \in \mathbb{C}$ and $\Phi(h,\cdot)$ is holomorphic for all $h \in H$. Let $\mathrm{I}_H(\chi)$ be the $\mathbb{C}$ vector space of all χ-sections. If $D \in \mathcal{D}(H)$, C is a compact subset of H, and C' is a compact subset of $\mathbb{C}$, define a semi-norm $\|\cdot\|_{D,C,C'}$ on $\mathrm{I}_H(\chi)$ by

$$\|\Phi\|_{D,C,C'} = \sup\{|(D\Phi)(h,s)| : h \in C, s \in C'\}.$$

With the topology defined by this family of semi-norms, $\mathrm{I}_H(\chi)$ is a Frechet space, and $\mathrm{I}_H(\chi)$ is a differentiable representation of H under right translation. As in the nonarchimedean case, we have the concept of a standard section.

We can now define the zeta integrals. Fix a Haar measure on G. Let $\pi \in \mathrm{Irr}(G)$. It is known that there exists a real number σ_0 such that for all $\Phi \in \mathrm{I}_H(\chi)$, coefficients f of π, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \sigma_0$,

$$Z(s,f,\Phi) = \int_G \Phi(i(g,1),s) f(g)\, dg$$

converges absolutely. Let $\Phi \in \mathrm{I}_H(\chi)$ and let f be a coefficient of π. It is known that $Z(s,f,\Phi)$ has an analytic continuation to a meromorphic function on $\mathbb{C}$, and if F is nonarchimedean, then there is a rational function $p(X) \in \mathbb{C}(X)$ such that $Z(s,f,\Phi) = p(q^{-s})$.

In addition, there is the concept of a good χ-section. See [PSR3], p. 110, and [HKS], p. 970. Good sections consist of the standard sections, the image of the standard sections under a certain operator, and, if χ is unramified, translates of a normalization of the unramified section. Since we shall only work with standard sections we omit the precise definition.

3. Zeta integrals for the general linear group

There is a similar development for the general linear group. Again, F is a local field of characteristic zero, with $F = \mathbb{R}$ if F is archimedean. Let X be a vector space over F of dimension k. In Section 5, X will be a subspace of the space V from the last section, and the groups we define will be subgroups of H. Let $G_1 = \mathrm{Gl}(X)$, $X' = X \times X$, and $H_1 = \mathrm{Gl}(X')$. We again have an embedding

$$i_1 : G_1 \times G_1 \hookrightarrow H_1$$

given by $i_1(g,g')(x,x') = (gx, g'x')$. Let X^d be the subspace of pairs (x,x) for $x \in X$. Let P_1' be the maximal parabolic subgroup of G_1 stabilizing X^d; again, G_1 embedded on the diagonal is contained in P_1'. Let $\mathcal{B}_1 = \{x_1, \dots, x_k\}$ be an ordered basis for X, and identify G_1 with $\mathrm{Gl}(k,F)$ via this basis. Let K_1 be the usual compact subgroup of G_1, so that $K_1 = \mathrm{Gl}(k,\mathfrak{O})$ if F is nonarchimedean, and $K_1 = \mathrm{O}(k,\mathbb{R})$ if $F = \mathbb{R}$. The elements

$$(x_1,0),\dots,(x_k,0),(0,x_1),\dots,(0,x_k)$$

form a basis for X_1'. Identifying H_1 with $\mathrm{Gl}(2k,F)$ via this basis, we let K_1' be the usual maximal compact subgroup of H_1'. Since with this choice of basis,

$$i_1(h,h') = \begin{bmatrix} h & 0 \\ 0 & h' \end{bmatrix},$$

we have $i_1(K_1 \times K_1) \subset K_1'$. However, again we will instead use a different ordered basis for X', namely $\mathcal{B}_1' = \{e_1,\dots,e_k,f_1',\dots,f_k'\}$, where $e_i = (x_i,x_i)$ and $f_i' = (0,x_i)$ for $1 \le i \le k$. Then the elements of P_1' have the usual upper triangular form of the $k+k$ standard parabolic of $\mathrm{Gl}(2k,F)$, and

$$i_1(h,1) = \begin{bmatrix} h & 0 \\ 1-h & 1 \end{bmatrix}, \quad i_1(1,h) = \begin{bmatrix} 1 & 0 \\ h-1 & h \end{bmatrix}.$$

Note that $K_1' = \mathrm{Gl}(2k,\mathfrak{O})$ if F is nonarchimedean, and K_1' is conjugate to the usual maximal compact subgroup of $H_1' = \mathrm{Gl}(2k,\mathbb{R})$ if $F = \mathbb{R}$.

Test functions are defined analogously to the case of $\mathrm{Sp}(n)$. Fix quasi-characters μ_1 and μ_2 of $F^\times$. For now, μ_1 and μ_2 will be arbitrary, but just before Proposition 5.3 we will make a fixed choice. For $s \in \mathbb{C}$, we define the inducing quasi-character $\alpha_{\mu_1,\mu_2}(\cdot,s)$ of P_1' by

$$\alpha_{\mu_1,\mu_2}(p,s) = |\det a/\det c|^{s+k/2}\mu_1(\det a)\mu_2(\det c), \quad p = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in P_1'.$$

Again, we have the spaces $\mathrm{I}_{H_1}(\mu_1,\mu_2,s)$, $\mathrm{I}_{H_1}(\mu_1,\mu_2)$ and $\mathrm{I}_{H_1}^{\mathrm{Stan}}(\mu_1,\mu_2)$. The zeta integrals associated to an element of $\mathrm{Irr}(G_1)$ are defined analogously and the same results hold.

4. Idea of the proof

Having introduced zeta integrals, we now discuss the idea of the proof of the main technical lemma. Let the notation and assumptions be as in the main technical lemma. Let $\delta = \delta_1$, let π' be the Langlands quotient of the Langlands data $(\delta_2,\dots,\delta_t,\tau)$, and let P be the standard maximal parabolic of $G = \mathrm{Sp}(n,F)$ with Levi factor $M \cong G_1 \times G_2 = \mathrm{Gl}(1,F) \times \mathrm{Sp}(n-1,F)$ as defined in Section 5. The basic idea of the proof, which has two steps, is as follows: In the first step, we investigate when a product of zeta integrals for δ and π' can be identified with a zeta integral for π. Precisely, we show that if f_1 and f_2 are coefficients for δ and π', respectively, Φ_1 is a μ_1, μ_2-section for G_1, Φ_2 is a χ-section for G_2, and Φ is a $K \times K$-finite χ-section for G such that

$$\Phi_1(i_1(g_1, 1), s)\Phi_2(i_2(g_2, 1), s) = \int_{U \times \overline{U}} \Phi(i(\bar{u}^{-1}um(g_1, g_2), 1), s)\, d(u\bar{u}) \tag{4.1}$$

for $g_1 \in G_1$, $g_2 \in G_2$ and s in some right half plane, then there exists a coefficient f of π such that

$$Z(s, f_1 \otimes \delta_P^{-1/2}, \Phi_1)Z(s, f_2, \Phi_2) = Z(s, f, \Phi). \tag{4.2}$$

Here, U is the unipotent radical of P, $\overline{U} = {}^tU$, and the factor $\delta_P^{-1/2}$, regarded as a quasi-character on G_1, naturally appears because we must take $\mu_1 = \chi\delta_P^{1/2}$ and $\mu_2 = \chi^{-1}\delta_P^{1/2}$; see the comment before Proposition 5.3. In the second step, we show that there exist f_1, f_2, Φ_1, Φ_2 and a χ-section Φ for G such that (4.1) holds, $Z(s-1/2, f_1 \otimes \delta_P^{-1/2}, \Phi_1)$ has a pole at s_0, and $Z(s, f_2, \Phi_2) = 1$. Steps one and two clearly imply the main technical lemma, if the χ-section Φ for G from the second step can also be shown to be $K \times K$-finite and standard.

In the nonarchimedean case, this is indeed exactly how we prove the main technical lemma: the Φ from the second step is $K \times K$-finite and standard. However, a complication arises in the real case: While one can always find f_1, f_2, Φ_1, Φ_2 and Φ as in the second step, in the real case Φ may not be $K \times K$-finite; hence, we cannot directly apply the first step. To get around this, we use a density argument. The key insight is to view the products $Z(s, f_1 \otimes \delta_P^{-1/2}, \Phi_1)Z(s, f_2, \Phi_2)$ for which there exists a (not necessarily $K \times K$-finite) Φ satisfying (4.1) as belonging to an intermediate class of auxiliary zeta integrals which behave like ordinary zeta integrals for π. Namely, suppose f_1, f_2, Φ_1, Φ_2 and Φ are given and (4.1) holds. Then

$$Z(s, f_1 \otimes \delta_P^{-1/2}, \Phi_1)Z(s, f_2, \Phi_2) = Z(s, f_1, f_2, \Phi) \tag{4.3}$$

where

$$Z(s, f_1, f_2, \Phi) = \int_{G_1 \times G_2 \times U \times \overline{U}} \Phi(i(\bar{u}^{-1}um, 1), s) f_1(g_1) f_2(g_2) \delta_P(m)^{-1/2}\, d(g_1 g_2 u\bar{u}).$$

Here, m is the element of M determined by g_1 and g_2. We thus consider the class of auxiliary zeta integrals $Z(s, f_1, f_2, \Phi)$ for f_1 and f_2 coefficients of δ and π', respectively, and Φ a χ-section for G. Now [KR1] establishes two fundamental properties for ordinary zeta integrals. First, if f is a coefficient for π, then for every nonnegative integer M, there exist differential operators $D_1, \dots, D_k$ on H such that

$$|Z(s, f, \Phi)| \leq \int_G |\Phi(i(g, 1), s) f(g)|\, dg \leq \|\Phi(\cdot, s)\|_{K', D_1} + \cdots + \|\Phi(\cdot, s)\|_{K', D_k}$$

for s in a half plane $\mathrm{Re}(s) > \sigma_0 + b - cM$ and Φ in the set $\mathrm{I}_H(\chi)_M$ of χ-sections which vanish to order M on the boundary of $P'\backslash H$. Here, σ_0 and c are constants depending only on G, and b is a constant depending on π. See the comment before Lemma 6.5 for more precise definitions. Second, for every nonnegative integer M, there exists an element $X_M(s)$ of $\mathbb{C}[s] \otimes Z(\mathfrak{g})$ which has a nonzero action on π and maps $\mathrm{I}_H(\chi)$ into $\mathrm{I}_H(\chi)_M$; here, $Z(\mathfrak{g})$ is the center of the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of $G \cong G \times 1 \subset H$. From these two properties, it follows that for every half plane $\mathrm{Re}(s) > \sigma_0 + b - cM$ there exists a polynomial $p(s)$ depending on π such that for $\Phi \in \mathrm{I}_H(\chi)$, $Z(s, f, X_M(s)\Phi)$ converges absolutely in $\mathrm{Re}(s) > \sigma_0 + b - cM$, and

$$p(s)Z(s, f, \Phi) = Z(s, f, X_M(s)\Phi)$$

for $\mathrm{Re}(s) > \sigma_0 + b$. Thus, $Z(s, f, \Phi)$ has an analytic continuation to $\mathrm{Re}(s) > \sigma_0 + b - cM$ with poles among the zeros of $p(s)$. We prove analogous properties for the $Z(s, f_1, f_2, \Phi)$. We estimate $|Z(s, f_1, f_2, \Phi)|$ by decomposing the domain of integration, and bounding $|Z(s, f_1, f_2, \Phi)|$ by a sum of two integrals: one integral reduces to the case of an ordinary zeta integral, while the other requires a new argument. Using this estimate, we next show that if Φ is $K \times K$-finite, then there is a coefficient f of π such that

$$Z(s, f_1, f_2, \Phi) = Z(s, f, \Phi).$$

In particular, this allows us to deduce that the analogue of the above second property also holds for auxiliary zeta integrals.

These results in the real case amount to an enhancement of the first step mentioned above. The second step in the real case is similar to the nonarchimedean situation. In the real case, we now prove the main technical lemma as follows. We pick f_1, f_2, Φ_1, Φ_2 and Φ such that $Z(s, f_1 \otimes \delta_P^{-1/2}, \Phi_1)$ and $Z(s, f_2, \Phi_2)$ are as desired and (4.1) holds; then (4.3) also holds. Then we choose a sequence (Φ_n) in $\mathrm{I}_H(\chi)$ of K'-finite elements converging to Φ. Using that $Z(s - 1/2, f_1, f_2, \Phi)$ has pole at s_0 by construction, an argument shows that some $Z(s - 1/2, f_1, f_2, \Phi_n)$ has a pole at s_0. Since $Z(s, f_1, f_2, \Phi_n) = Z(s, f, \Phi_n)$ for some coefficient f of π, and since every K'-finite χ-section is a finite linear combination of standard K'-finite χ-sections, the proof is complete.

5. An auxiliary parabolic

As we mentioned in the last section, to prove the main technical lemma we will need to relate zeta integrals for δ and π' to zeta integrals for π. To do so we need to introduce some more groups and embeddings. We will use the notation of Sections 2 and 3.

Though the main technical lemma only requires the standard maximal parabolic P with Levi isomorphic to $\mathrm{Gl}(1) \times \mathrm{Sp}(n-1)$, in this section we will consider the case of a general standard maximal parabolic. In the next section we will assume that P has Levi isomorphic to $\mathrm{Gl}(1) \times \mathrm{Sp}(n-1)$. Fix an integer k such that $1 \leq k \leq n$, and let X be the subspace of V with ordered basis $\{x_1, \ldots, x_k\}$, let Y be the subspace with ordered basis

$\{y_1, \ldots, y_k\}$, and let Z be the subspace with ordered basis $\{x_{k+1}, \ldots, x_n, y_{k+1}, \ldots, y_n\}$. As in section 3, let $G_1 = \mathrm{Gl}(X)$. Also, let $G_2 = \mathrm{Sp}(Z)$. The definitions of sections 2 and 3 apply to G_1 and G_2. Let P be the maximal parabolic subgroup of G stabilizing X. The elements of the Levi component M and unipotent radical U of P have the form

$$m(g_1, g_2) = \begin{bmatrix} g_1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & {}^t g_1^{-1} & 0 \\ 0 & c & 0 & d \end{bmatrix}, \quad u(x, y, z) = \begin{bmatrix} 1 & x & y & z \\ 0 & 1 & {}^t z & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -{}^t x & 1 \end{bmatrix},$$

respectively. Here, $g_1 \in G_1$, $g_2 \in G_2$ has form

$$g_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

$x \in \mathrm{M}_{k\times(n-k)}(F)$, $y \in \mathrm{M}_{k\times k}(F)$ and $z \in \mathrm{M}_{k\times(n-k)}(F)$ with ${}^t(y - x{}^t z) = y - x{}^t z$. We have $U = U_1 \rtimes U_2$, with U_1 the subgroup of elements of the form $u(0, y, z)$ and U_2 the subgroup of elements of the form $u(x, 0, 0)$. Clearly, $U_1 \cong \mathrm{Sym}_{k\times k}(F) \times \mathrm{M}_{k\times(n-k)}(F)$ and $U_2 \cong \mathrm{M}_{k\times(n-k)}(F)$. The groups U_1, U_2 and U are all unimodular. Using the formula for Haar measures on semi-direct products, if Haar measures on $\mathrm{Sym}_{k\times k}(F)$ and $\mathrm{M}_{k\times(n-k)}(F)$ are fixed, then a Haar measure on U is given by

$$\int_U F(u)\, du = \int_A F(u(x, y - z{}^t x, z))\, d(xyz)$$

for $F \in C_c(U)$, where $A = \mathrm{M}_{k\times(n-k)}(F) \times \mathrm{Sym}_{k\times k}(F) \times \mathrm{M}_{k\times(n-k)}(F)$, and we use the product measure on A. We will also need to use the parabolic subgroup $\overline{P}$ opposite to P. In matrices, this is the group of ${}^t p$ with $p \in P$. The Levi component of $\overline{P}$ is of course M, and we let $\overline{U} = {}^t U$ denote the unipotent radical of $\overline{P}$. Fixing Haar measures on $\mathrm{M}_{k\times(n-k)}(F)$ and $\mathrm{Sym}_{k\times k}(F)$ determines a Haar measure on $\overline{U}$.

Next, we introduce an auxiliary parabolic of H that will be the key to relating zeta integrals on G to zeta integrals on G_1 and G_2. In doing so, we were inspired by the comments in section 1 of [PSR1]. Let $P_{1,2}$ be the maximal parabolic of H stabilizing the totally isotropic subspace $X' = X \times X$ of V'. We have the decomposition $V' = X' \oplus Z' \oplus Y'$, where $X' = X \times X$, $Y' = Y \times Y$ and $Z' = Z \times Z$. Evidently, X' and Y' are totally isotropic and dually paired, and the orthogonal complement of $X' \oplus Y'$ is Z'. The Levi component $M_{1,2}$ of $P_{1,2}$ is isomorphic to $\mathrm{Gl}(X') \times \mathrm{Sp}(Z')$. Letting $H_1 = \mathrm{Gl}(X')$ and $H_2 = \mathrm{Sp}(Z')$, we obtain embeddings

$$i_1' : H_1 \hookrightarrow M_{1,2} \subset P_{1,2} \subset H, \quad i_2' : H_2 \hookrightarrow M_{1,2} \subset P_{1,2} \subset H.$$

Note that by sections 2 and 3 we also have embeddings

$$i_1 : G_1 \times G_1 \hookrightarrow H_1, \quad i_2 : G_2 \times G_2 \hookrightarrow H_2.$$

We use $\{e_1, \ldots, e_k, f'_1, \ldots, f'_k\}$ and $\{e_{k+1}, \ldots, e_n, f_{k+1}, \ldots, f_n, e'_{k+1}, \ldots, e'_n, f'_{k+1}, \ldots, f'_n\}$ as ordered bases for X' and Z', respectively. These choices of bases are consistent with the definitions of section 2 and 3 applied to G_1 and G_2. Explicitly, suppose $h_1 \in H_1$ and $h_2 \in H_2$ with

$$h_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad {}^t h_1^{-1} = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}, \quad h_2 = \begin{bmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{bmatrix}.$$

Then

$$i'_1(h_1) = \begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d' & 0 & -c' & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -b' & 0 & a' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ c & 0 & 0 & 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad i'_2(h_2) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & a_2 & 0 & b_1 & 0 & b_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_3 & 0 & a_4 & 0 & b_3 & 0 & b_4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & d_1 & 0 & d_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & c_3 & 0 & c_4 & 0 & d_3 & 0 & d_4 \end{bmatrix}.$$

One can verify that $i'_1(K'_1) \subset K'$ and $i'_2(K'_2) \subset K'$. Moreover, the embeddings i'_1 and i'_2 are compatible with i_1 and i_2 in the sense of the following lemma. We will use this lemma in the second step (Section 7) of the proof of the main technical lemma.

Lemma 5.1. *If $g_1 \in G_1$ and $g_2 \in G_2$, then $i(m(g_1, g_2), 1) = i'_1 i_1(g_1, 1) i'_2 i_2(g_2, 1)$.*

Proof. This follows by a direct computation. □

In the yet to be completed theory of zeta integrals for the symplectic group, an important role will be played by integrals of good χ-sections for G over $i(U \times 1)$. The operator defined by this integration should send the space of good χ-sections for H onto the tensor product of the corresponding spaces for H_1 and H_2. Such a result should suffice to show that the zeta integral ideal of a representation of G parabolically induced from P is the product of the zeta integral ideals of the inducing data. By an involved argument, we can prove this when F is nonarchimedean and $\chi = 1$. While interesting from the standpoint of the general theory, this result is tangental to our proof of the main technical lemma. We will thus prove only what we need, namely that formally the operator takes sections for H to the tensor product of sections for H_1 and H_2.

Lemma 5.2. *Let $u(x, y, z) \in U$, $p_1 \in P'_1$ and $p_2 \in P'_2$, with*

$$p_1 = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \quad p_2 = \begin{bmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ 0 & 0 & a'_1 & a'_2 \\ 0 & 0 & a'_3 & a'_4 \end{bmatrix}.$$

Then there exist $p' = \begin{bmatrix} a' & b' \\ 0 & {}^t a'^{-1} \end{bmatrix}, p'' = \begin{bmatrix} a'' & b'' \\ 0 & {}^t a''^{-1} \end{bmatrix}$ *in* P' *with* $\det a' = \det a'' = 1$ *and*

$$\begin{aligned} i(u(x,y,z),1)i_1'(p_1) &= i_1'(p_1)p'i(u(c^{-1}x, c^{-1}y^tc^{-1}, c^{-1}z),1), \\ i(u(x,y,z),1)i_2'(p_2) &= i_2'(p_2)p''i(u(x', y+z^tx+y'-z'^tx', z'),1) \end{aligned}$$

where $x' = xa_1 + za_3$, $z' = xa_2 + za_4$, *and* $y' = (xb_1+zb_3)^tx' + (xb_2+zb_4)^tz'$.

Proof. The lemma follows by a direct computation, with

$$a' = \begin{bmatrix} 1 & dx & dw^tc^{-1} & dz \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad b' = \begin{bmatrix} (a^{-1}+a^{-1}bc^{-1})y^td + dw^tc^{-1} & dz & 0 & 0 \\ {}^t(dz) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $d = a^{-1} - c^{-1} + a^{-1}bc^{-1}$ and $w = y + z^tx - x^tz$, and

$$a'' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & {}^t(x(b_1-a_2)+z(a_1'-a_4+b_3)) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & {}^t(xb_2+z(a_2'+b_4)) & 1 \end{bmatrix}, \quad b'' = \begin{bmatrix} 0 & e & 0 & f \\ {}^te & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ {}^tf & 0 & 0 & 0 \end{bmatrix},$$

where $e = x(b_1-a_2)+z(a_1'-a_4+b_3)$ and $f = xb_2 + z(a_2'+b_4)$. □

As the next proposition shows, in integrating a χ-section for G over U we obtain essentially a product of a μ_1, μ_2-section for G_1 with a χ-section for G_2. However, contrary to what one might first guess, we do not have $\mu_1 = \chi$ and $\mu_2 = \chi^{-1}$; instead, μ_1 and μ_2 are χ and χ^{-1}, respectively, twisted by the same factor $|\cdot|^{(2n-k+1)/2}$. Since $\delta_P(m(g_1,g_2)) = |\det g_1|^{2n-k+1}$ for $g_1 \in G_1$ and $g_2 \in G_2$, from now on $\delta_P(x)$ for $x \in F^\times$ shall mean $|x|^{2n-k+1}$. For the remainder of the paper, let

$$\mu_1 = \chi\delta_P^{1/2}, \qquad \mu_2 = \chi^{-1}\delta_P^{1/2}.$$

Proposition 5.3. *Fix Haar measures on* $\mathrm{Sym}_{k\times k}(F)$ *and* $\mathrm{M}_{k\times(n-k)}(F)$ *so that a Haar measure on* U *is determined. Fix* $s \in \mathbb{C}$, *and let* $\Phi : H \to \mathbb{C}$ *be such that* $\Phi(ph) = \alpha_\chi(p,s)\Phi(h)$ *for* $p \in P'$ *and* $h \in H$. *Assume that* $\Phi(\cdot h)|_U \in L^1(U)$ *for* $h \in H$, *and define* $\Phi_{1,2} : H_1 \times H_2 \to \mathbb{C}$ *by*

$$\Phi_{1,2}(h_1,h_2) = \int_U \Phi(i(u,1)i_1'(h_1)i_2'(h_2))\,du.$$

Then $\Phi_{1,2}(p_1h_1, p_2h_2) = \alpha_{\mu_1,\mu_2}(p_1,s)\alpha_\chi(p_2,s)\Phi_{1,2}(h_1,h_2)$ *for* $p_1 \in P_1', p_2 \in P_2', h_1 \in H_1$ *and* $h_2 \in H_2$.

Proof. The proof follows by a straightforward computation using Lemma 5.2. □

6. Step one: identification of zeta integrals

In this section we carry out the first step in the proof of the main technical lemma. For the remainder of this paper π will be as in the statement of the main technical lemma. Write

$$\delta = \delta_1 = \chi|\cdot|^{s_0}, \qquad \pi' = L(\delta_2 \otimes \cdots \otimes \delta_t \otimes \tau),$$

where π' is the Langlands quotient of the Langlands data $(\delta_2, \ldots, \delta_t, \tau)$. If $F = \mathbb{R}$, so that by hypothesis $\chi = 1$, we will write

$$a = e(\delta) = \mathrm{Re}(s_0).$$

In this case, when $F = \mathbb{R}$, $|\delta(x)| = |x|^{e(\delta)} = |x|^a$; by definition, a is positive. We will use the notation of Section 5 with $k = 1$, so that P from Section 5 is the parabolic subgroup with Levi factor $M = G_1 \times G_2 \cong \mathrm{Gl}(1, F) \times \mathrm{Sp}(n-1, F)$.

As we mentioned in Section 4, the first step of the proof of the main technical lemma is to show that if Φ_1 is a μ_1, μ_2-section for G_1, Φ_2 is a χ-section for G_2 and Φ is a $K \times K$-finite χ-section for G, and Φ_1, Φ_2 and Φ are related by equation (4.1), then the product of a zeta integral $Z(s, f_1 \otimes \delta_P^{-1/2}, \Phi_1)$ for δ with a zeta integral $Z(s, f_2, \Phi_2)$ for π' is a zeta integral $Z(s, f, \Phi)$ for π. When F is nonarchimedean we prove this in Theorem 6.10 at the end of this section. A similar result holds if $F = \mathbb{R}$ (see Theorem 6.7), but as we mentioned in Section 4, this is not quite sufficient for our purposes because for the Φ_1, Φ_2 and Φ that we find in Section 7 such that (4.1) holds and $Z(s, f_1 \otimes \delta_P^{-1/2}, \Phi_1)$ and $Z(s, f_2, \Phi_2)$ have the right form, Φ will not generally be $K \times K$-finite. However, the $K \times K$-finite vectors are dense in $\mathrm{I}_H(\chi)$, and so it is natural to try a limit argument. When $F = \mathbb{R}$, this necessitates that we view the products $Z(s, f_1 \otimes \delta_P^{-1/2}, \Phi_1)Z(s, f_2, \Phi_2)$ as belonging to a larger class of auxiliary zeta integrals $Z(s, f_1, f_2, \Phi)$ for which one has good estimates along with analytic continuation. The first, and most essential, result of this section is an estimate of these auxiliary integrals in terms of the semi-norms on $\mathrm{I}_H(\chi, s)$. After this, we show that if Φ is $K \times K$-finite, then an auxiliary zeta integral $Z(s, f_1, f_2, \Phi)$ is equal to a zeta integral $Z(s, f, \Phi)$ for π; using results of [KR1] about zeta integrals for π, this result allows us to analytically continue $Z(s, f_1, f_2, \Phi)$ for $K \times K$-finite Φ. Until the final result of this section, Theorem 6.10, we let $F = \mathbb{R}$; when $F = \mathbb{R}$, as in the statement of the main technical lemma, we assume $\chi = 1$. As a reference for the representation theory of real reductive groups we use [W].

We will need to measure the size of elements of $G = \mathrm{Sp}(n, \mathbb{R})$, and more generally, of elements of $\mathrm{M}_{k \times l}(\mathbb{R})$. For $x \in \mathrm{M}_{k \times l}(\mathbb{R})$, define

$$\|x\| = (\sum_{i,j} |x_{ij}|^2)^{1/2}.$$

If N is a positive integer, then $\|gh\| \leq \|g\|\|h\|$ for $g, h \in \mathrm{Gl}(N, \mathbb{R})$, and $\|gk\| = \|kg\| = \|g\|$ for $g \in \mathrm{Gl}(N, F)$ and $k \in \mathrm{O}(N, \mathbb{R})$. We begin with two technical results which will be used in the proof of Lemma 6.6.

Lemma 6.1. *For $\bar{u} \in \overline{U}$, let*

$$\bar{u} = u(\bar{u})m(\bar{u})k(\bar{u}) = u(\bar{u})m(g_1(\bar{u}), g_2(\bar{u}))k(\bar{u}),$$

where $u(\bar{u}) \in U$, $g_1(\bar{u}) \in G_1$, $g_2(\bar{u}) \in G_2$ and $k(\bar{u}) \in K$. Let $\bar{u} \in \overline{U}$, and write

$$\bar{u} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ {}^t x & 1 & 0 & 0 \\ y & z & 1 & -x \\ {}^t z & 0 & 0 & 1 \end{bmatrix}$$

with $x \in \mathrm{M}_{1\times(n-1)}(\mathbb{R})$, $y \in \mathbb{R}$ and $z \in \mathrm{M}_{1\times(n-1)}(\mathbb{R})$. Then

$$|g_1(\bar{u})| = \frac{1}{\sqrt{1 + \|x\|^2 + \|y\|^2 + \|z\|^2}}.$$

Proof. Let $k(\bar{u}) = (k_{ij})$ with $1 \le i, j \le 4$ and k_{ij} of the same size as the corresponding entry of $\bar{u}$. The equality $\bar{u} = u(\bar{u})m(g_1(\bar{u}), g_2(\bar{u}))k(\bar{u})$ implies that $y = g_1(\bar{u})^{-1}k_{31}, z = g_1(\bar{u})^{-1}k_{32}, 1 = g_1(\bar{u})^{-1}k_{33}, -x = g_1(\bar{u})k_{34}$. Hence,

$$1 + \|x\|^2 + \|y\|^2 + \|z\|^2 = |g_1(\bar{u})|^{-2}(\|k_{31}\|^2 + \|k_{32}\|^2 + \|k_{33}\|^2 + \|k_{34}\|^2).$$

Since $K \subset \mathrm{O}(2n, \mathbb{R})$, $\|k_{31}\|^2 + \|k_{32}\|^2 + \|k_{33}\|^2 + \|k_{34}\|^2 = 1$. □

Lemma 6.2. *For $g \in G$, let*

$$i(g, 1) = p'(g)k'(g) = \begin{bmatrix} a'(g) & b'(g) \\ 0 & {}^t a'(g)^{-1} \end{bmatrix} k'(g)$$

where $p'(g) \in P'$ and $k'(g) \in K'$. Then $|\det a'(g)| \le 1$ for all $g \in G$, and there exist a positive constant $E > 1$ such that

$$|\det a'(g)| \le E\|g\|^{-1}$$

for all $g \in G$.

Proof. This is as in [KR1], Lemma 3.1.2. Let $g \in G$. Write $g = kak'$ with $k, k' \in K$ and $a = \mathrm{diag}(a_1, \dots, a_n, a_1^{-1}, \dots, a_n^{-1})$ with $a_1 \ge \cdots \ge a_n \ge 1$. Then a computation shows that

$$|\det a'(g)| = \prod_{i=1}^{n} \frac{2}{a_i + a_i^{-1}}.$$

This implies $|\det a'(g)| \le 1$ and $2^n\sqrt{2n}\|g\|^{-1} \ge |\det a'(g)|$. □

Next, we describe two needed results about matrix coefficients. The first is the well known result that matrix coefficients are at most exponential in growth.

Lemma 6.3. *Let $\Pi \in \mathrm{Irr}(G)$. There exists a nonnegative integer r, depending only on Π, such that if f is any coefficient of π, then for some $C > 0$,*

$$|f(g)| \leq C\|g\|^r, \qquad g \in G.$$

Proof. See, for example, Lemma 2.A.2.2 of [W]. Note that in section 2.A.2 of [W] the norm $\|g\|_{\mathrm{op}}$ of an element g of $G = \mathrm{Sp}(n, \mathbb{R})$ is defined to be the operator norm of

$$\begin{bmatrix} g & 0 \\ 0 & {}^t g^{-1} \end{bmatrix}.$$

For $g \in G$, we have $(1/\sqrt{2n})\|g\| \leq \|g\|_{\mathrm{op}} \leq \|g\|$. $\square$

The next result about matrix coefficients is more technical and essentially amounts to the convergence of the integral defining the intertwining operator used to define Langlands quotients. In the following lemma, the exponent $e(\delta_2)$ is defined in the notation section.

Lemma 6.4. *Let f_2 be a coefficient of π'. Then there exist positive constants C and r and a continuous function $h : G_2 \to [0, \infty)$ such that*

$$|f_2(g_2(\bar{u})g_2)| \leq C\|g_2\|^r h(g_2(\bar{u}))$$

for $\bar{u} \in \overline{U}$ and $g_2 \in G_2$, and for $a' > e(\delta_2)$,

$$\int_{\overline{U}} |g_1(\overline{u})|^{a'} \delta_P(m(\bar{u}))^{1/2} h(g_2(\bar{u}))\, d\bar{u} < \infty.$$

Here, $g_1(\bar{u})$, $g_2(\bar{u})$ and $m(\bar{u})$ are as in Lemma 6.1. If in the statement of the main technical lemma $t = 1$, then $a' > e(\delta_2)$ is taken to mean $a' > 0$.

Proof. Suppose first π' is tempered. Denote by ϱ the irreducible unitary tempered representation of G such that $\varrho_K = \pi'$ (see the notation section). Since π' is tempered, ϱ satisfies the weak inequality, i.e., there exists a nonnegative constant d such that for $v, w \in \pi'$ and $g \in G_2$,

$$|\langle \varrho(g)v, w\rangle| \leq s(v)(1 + \log\|g\|)^d \Xi(g),$$

where s is a continuous semi-norm on ϱ^∞ depending only on w, and Ξ is the zonal spherical function of Harish-Chandra; see 5.1.1 and 5.1.2 of [W]. We may assume that there exist $v, w \in \pi'$ such that $f_2(g) = \langle \varrho(g)v, w\rangle$ for $g \in G_2$. Then

$$|f_2(g_2(\bar{u})g)| \leq s(\varrho(g)v)(1 + \log\|g_2(\bar{u})\|)^d \Xi(g_2(\bar{u}))$$

for $\bar{u} \in \overline{U}$ and $g \in G_2$. Now there exist positive constants C and r such that

$$|s(\varrho(g)v)| \leq C\|g\|^r$$

for $g \in G_2$. We thus have

$$|f_2(g_2(\bar{u})g)| \le C\|g\|^r h(g_2(\bar{u}))$$

for $\bar{u} \in \overline{U}$ and $g \in G_2$, where $h : G_2 \to [0, \infty)$ is defined by $h(g) = (1 + \log\|g\|)^d \Xi(g)$. Clearly, h is continuous. As for the convergence of the integral from the statement of the lemma, this essentially amounts to the convergence of the integral defining the intertwining operator used to write π as a Langlands quotient. See the proof of Lemma 5.3.1 of [W].

The proof when π' is not tempered is similar, though more complicated since we are taking an intermediate Langlands quotient. We omit the argument; besides Lemma 5.3.1 of [W], three ingredients are the proof of Theorem 4.5.6, 5.2.7 and the proof of Lemma 5.2.8, all of [W]. □

Finally, we need a fundamental result from [KR1] which estimates ordinary zeta integrals in terms of the semi-norms on $\mathrm{I}_H(s, \chi)$ defined in Section 2. To state this result, we need a concept from [KR1], section 3.1. Let $\Phi \in \mathrm{I}_H(\chi)$. Let $s \in \mathbb{C}$. Since $\chi = 1$, we can regard $\Phi(\cdot, s)$ as a function on $(K' \cap P')\backslash K'$. Now $(K' \cap P')\backslash K'$ is diffeomorphic to the smooth manifold Ω of all Lagrangians in the symplectic space V' from Section 2, which is also diffeomorphic to $P'\backslash H$ (we let H act on Ω by $h \cdot L = h^{-1}(L)$ for $h \in H$ and $L \in \Omega$). The $G \times G$ orbit Ω_0 of the Lagrangian V^d from Section 2 is open. Now let M be a nonnegative integer, and let $\mathrm{I}_H(\chi)_M$ be the $\mathbb{C}$ subspace of all $\Phi \in \mathrm{I}_H(\chi)$ such that for all $s \in \mathbb{C}$, $\Phi(\cdot, s)$ and all its partial derivatives of order less than Mn vanish on the negligible set $\Omega - \Omega_0$. Also, as in Section 2, let $\mathcal{D}(H)$ be the $\mathbb{C}$ vector space of all differential operators on H.

Lemma 6.5 (Kudla-Rallis). *Let $b \ge 0$. There exist real constants $\sigma_0 > 0$ and $c > 0$ depending only on G such that if M is a nonnegative integer, then there exist $D_1, \dots, D_k \in \mathcal{D}(H)$ such that*

$$\int_G |\Phi(i(g, 1), s)|\|g\|^b \, dg \le \|\Phi(\cdot, s)\|_{D_1, K'} + \cdots + \|\Phi(\cdot, s)\|_{D_k, K'}.$$

for $\Phi \in \mathrm{I}_H(\chi)_M$ and $\mathrm{Re}(s) > \sigma_0 + b - cM$.

Proof. This is essentially proven in section 3.1 of [KR1]. Note that [KR1] considers the case $b = 0$. By Lemma 6.2, $\|g\|^b \le E^b|\det a'(g)|^{-b}$. This translates the right half plane of convergence $\mathrm{Re}(s) > \sigma_0 - cM$ in [KR1] to $\mathrm{Re}(s) > \sigma_0 + b - cM$. □

The next lemma, which is the key technical result of this section, gives a version of the above estimate of [KR1] for auxiliary zeta integrals. The proof uses the result of [KR1], along with a new idea. In the following lemma, $\|\Phi(\cdot, s)\|_{K'}$ for $\Phi \in \mathrm{I}_H(\chi)$ and $s \in \mathbb{C}$ means the supremum of $|\Phi(\cdot, s)|$ on K'.

Lemma 6.6. *Fix coefficients f_1 and f_2 of δ and π', respectively. For $\Phi \in \mathrm{I}_H(\chi)$, let*

$$Z(s, f_1, f_2, \Phi) = \int_{G_1 \times G_2 \times U \times \overline{U}} \Phi(i(\bar{u}^{-1}um, 1), s) f_1(g_1) f_2(g_2) \delta_P(m)^{-1/2} \, d(g_1 g_2 u \bar{u})$$

and let $|Z|(s, f_1, f_2, \Phi)$ *be the integral over* $G_1 \times G_2 \times U \times \overline{U}$ *of the absolute value of the integrand of* $Z(s, f_1, f_2, \Phi)$*. Here,* $m = m(g_1, g_2)$*. There exist* $C > 0$ *and* $\sigma_0' > 0$ *depending only on* f_1*,* f_2 *and* π *such that if* M *is a nonnegative integer, then there exist* $D_1, \ldots, D_k \in \mathcal{D}(H)$ *so that*

$$|Z|(s, f_1, f_2, \Phi) \leq C(\|\Phi(\cdot, s)\|_{K'} + \|\Phi(\cdot, s)\|_{D_1,K'} + \cdots + \|\Phi(\cdot, s)\|_{D_k,K'}).$$

for $\Phi \in \mathrm{I}_H(\chi)_M$*,* $\mathrm{Re}(s) \geq -\rho_{2n}$ *and* $\mathrm{Re}(s) > \sigma_0 + \sigma_0' - cM$*.*

Proof. The proof is based on the following idea. To start, let $\Phi \in \mathrm{I}_H(\chi)$ and $s \in \mathbb{C}$; then we want to estimate

$$\int_{G_1 \times G_2 \times U \times \overline{U}} |\Phi(i(\bar{u}^{-1}um, 1), s) f_1(g_1) f_2(g_2)| \delta_P(m)^{-1/2}\, d(g_1 g_2 u \bar{u}).$$

Given the form of this integral, one natural approach is to use the Gelfand-Naimark integration formula, which in one form asserts that if $\phi \in C_c(G)$, (writing $m = m(g_1, g_2)$),

$$\int_{G_1 \times G_2 \times U \times \overline{U}} \phi(\bar{u}^{-1}um)\, d(g_1 g_2 u \bar{u}) = \int_G \phi(g)\, dg.$$

See [W], Lemma 2.4.5. To use this, we would bound $|f_1(g_1) f_2(g_2)| \delta_P(m)^{-1/2}$ by a positive power b of $\|\bar{u}^{-1}um\|$; then by the Gelfand-Naimark formula our integral would be less than

$$\int_G |\Phi(i(g, 1), s)| \|g\|^b\, dg,$$

and finally by Lemma 6.5 we would obtain the desired estimate. This approach has a flaw: unless the exponent $a = e(\delta)$ is large in comparison to the exponent in the bound for $|f_2(g_2)|$ obtained via Lemma 6.3, $|f_1(g_1) f_2(g_2)| \delta_P(m)^{-1/2}$ cannot be bounded by a positive power of $\|\bar{u}^{-1}um\|$.

However, this idea can be rescued. The first step is to rewrite our integral in what will turn out to be a more manageable form. As in the proof of Lemma 6.1, for $\bar{u} \in \overline{U}$, let

$$\bar{u} = u(\bar{u}) m(\bar{u}) k(\bar{u}) = u(\bar{u}) m(g_1(\bar{u}), g_2(\bar{u})) k(\bar{u}),$$

where $u(\bar{u}) \in U$, $g_1(\bar{u}) \in G_1$, $g_2(\bar{u}) \in G_2$ and $k(\bar{u}) \in K$. Let$|f_1| = C_1 |\cdot|^a$ for some $C_1 \geq 0$. We have

$$\begin{aligned}
&\int_{G_1 \times G_2 \times U \times \overline{U}} |\Phi(i(\bar{u}^{-1}um, 1), s) f_1(g_1) f_2(g_2)| \delta_P(m)^{-1/2}\, d(g_1 g_2 u \bar{u}) \\
&= \int_{G_1 \times G_2 \times U \times \overline{U}} |\Phi(i(k(\bar{u})^{-1} m(\bar{u})^{-1} u(\bar{u})^{-1} um, 1), s) f_1(g_1) f_2(g_2)| \delta_P(m)^{-1/2}\, d(g_1 g_2 u \bar{u})
\end{aligned}$$

$$= \int_{G_1\times G_2\times U\times\overline{U}} |\Phi(i(k(\bar u)^{-1}m(\bar u)^{-1}um,1),s)f_1(g_1)f_2(g_2)|\delta_P(m)^{-1/2}\,d(g_1g_2u\bar u)$$
$$= \int_{G_1\times G_2\times U\times\overline{U}} |\Phi(i(k(\bar u)^{-1}um(\bar u)^{-1}m,1),s)f_1(g_1)f_2(g_2)|\delta_P(m)^{-1/2}\delta_P(m(\bar u))\,d(g_1g_2u\bar u)$$
$$= C_1\int_{G_1\times G_2\times U\times\overline{U}} |\Phi(i(k(\bar u)^{-1}um,1),s)||g_1(\bar u)g_1|^{a-n}|f_2(g_2(\bar u)g_2)|\delta_P(m(\bar u))\,d(g_1g_2u\bar u)$$

Here we have used $\delta_P(m)=|g_1|^{2n}$. Again, there is a Gelfand-Naimark type integration formula which might be applied to this integral for an argument as proposed above. This formula asserts that

$$\int_{G_1\times G_2\times U\times\overline{U}} \phi(k(\bar u)^{-1}um)\delta_P(m(\bar u))\,d(g_1g_2u\bar u)=\int_G\phi(g)\,dg$$

for $\phi\in C_c(G)$; this follows from the second identity of Lemma 2.4.5 of [W] combined with the integration formula corresponding to $G=KP$. Again, though, in general $|g_1(\bar u)g_1|^{a-n}|f_2(g_2(\bar u)g_2)|$ cannot be bounded by a positive power of $\|k(\bar u)^{-1}um\|=\|um\|$. We come now to the main insight of the proof which saves this approach: Split the domain of integration into two parts so that on the good part $|g_1(\bar u)g_1|^{a-n}|f_2(g_2(\bar u)g_2)|$ is bounded by a positive power of $\|k(\bar u)^{-1}um\|$, and on the bad part one can still estimate the resulting integral. As it turns out, to bound $|g_1(\bar u)g_1|^{a-n}|f_2(g_2(\bar u)g_2)|$ by a positive power of $\|k(\bar u)^{-1}um\|$ it suffices to bound $|g_1(\bar u)g_1|^{-1}$ by a positive power of $\|k(\bar u)^{-1}um\|$. We thus now let B be the set of bad points, i.e., we let l be a positive constant, to be chosen more precisely later, and we let B be the set of $(g_1,g_2,u,\bar u)\in G_1\times G_2\times U\times\overline{U}$ such that

$$\|k(\bar u)^{-1}um(g_1,g_2)\|^l=\|um(g_1,g_2)\|^l\le|g_1(\bar u)g_1|^{-1}.$$

We let B^c be the set of good points, i.e., the complement of B in $G_1\times G_2\times U\times\overline{U}$. Let $|Z|_B(s,f_1,f_2,\Phi)$ be the integrand of $|Z|(s,f_1,f_2\Phi)$ integrated over B, and define $|Z|_{B^c}(s,f_1,f_2,\Phi)$ similarly. Then

$$|Z|(s,f_1,f_2,\Phi)=|Z|_B(s,f_1,f_2,\Phi)+|Z|_{B^c}(s,f_1,f_2,\Phi).$$

We first investigate $|Z|_B(s,f_1,f_2,\Phi)$; this will result in a fixed choice of l depending only on f_1, f_2 and π. To estimate this term we will assume, as we need to, that $\mathrm{Re}(s)\ge-\rho_{2n}$. The key ingredient will be Lemma 6.4. Now using the notation of Lemma 6.2 we have

$$\begin{aligned}&|Z|_B(s,f_1,f_2,\Phi)\\&=C_1\int_B|\alpha(p'(k(\bar u)^{-1}um),s)||\Phi(k'(k(\bar u)^{-1}um),s)|\\&\qquad\cdot|g_1(\bar u)g_1|^{a-n}|f_2(g_2(\bar u)g_2)|\delta_P(m(\bar u))\,d(g_1g_2u\bar u)\end{aligned}$$

$$\leq C_1\|\Phi(\cdot,s)\|_{K'}\int_B |\alpha(p'(k(\bar u)^{-1}um),s)||g_1(\bar u)g_1|^{a-n}|f_2(g_2(\bar u)g_2)|\delta_P(m(\bar u))\,d(g_1g_2u\bar u)$$
$$\leq C_1\|\Phi(\cdot,s)\|_{K'}\int_B |g_1(\bar u)g_1|^{a-n}|f_2(g_2(\bar u)g_2)|\delta_P(m(\bar u))\,d(g_1g_2u\bar u),$$

since $|\alpha(p'(k(\bar u)^{-1}um),s)|\leq 1$ by Lemma 6.2; recall that we are assuming $\mathrm{Re}(s)+\rho_{2n}\geq 0$.

Rewriting $|g_1(\bar u)g_1|^{a-n}\delta_P(m(\bar u))$ and applying the first part of Lemma 6.4, we find that the last integral is less than

$$C_1C_2\int_B |g_1|^{a-n}\delta_P(m(\bar u))^{1/2}|g_1(\bar u)|^a\|g_2\|^r h(g_2(\bar u))\,d(g_1g_2u\bar u),$$

where C_2 and r depend on f_2. Let $(g_1,g_2,u,\bar u)\in B$. Write

$$u=\begin{bmatrix}1&x&y&z\\0&1&{}^tz&0\\0&0&1&0\\0&0&-{}^tx&1\end{bmatrix}.$$

Then

$$|g_1|,\ |g_1|^{-1},\ \|g_2\|,\ \|g_1^{-1}x\|,\ \|g_1^{-1}y\|,\ \|g_1^{-1}z\|\leq\|um\|,$$

where as usual $m=m(g_1,g_2)$. The defining condition $\|um\|^l\leq|g_1(\bar u)g_1|^{-1}$ of the set B implies that

$$\begin{gathered}|g_1(\bar u)|^{1/(l-1)}\leq|g_1|\leq|g_1(\bar u)|^{-1/(l+1)},\\ \|g_2\|\leq|g_1(\bar u)|^{-1/l}|g_1|^{-1/l},\\ \|x\|,\ \|y\|,\ \|z\|\leq|g_1(\bar u)|^{-1/l}|g_1|^{1-1/l}.\end{gathered}$$

For the first inequality we need $l>1$, which we now assume. Let B' be the set of $(g_1,g_2,u,\bar u)\in G_1\times G_2\times U\times\overline{U}$ that satisfy these three lines of inequalities. Then $B\subset B'$, so that the last integral is less than

$$\begin{aligned}&\int_{B'}|g_1|^{a-n}\delta_P(m(\bar u))^{1/2}|g_1(\bar u)|^a\|g_2\|^r h(g_2(\bar u))\,d(g_1g_2u\bar u)\\ &=\int_{\overline{U}}\Big(\int_{|g_1(\bar u)|^{1/(l-1)}\leq|g_1|\leq|g_1(\bar u)|^{-1/(l+1)}}\Big(\int_{\|g_2\|\leq|g_1(\bar u)|^{-1/l}|g_1|^{-1/l}}\|g_2\|^r\,dg_2\Big)\\ &\cdot\Big(\int_{\|x\|,\|y\|,\|z\|\leq|g_1(\bar u)|^{-1/l}|g_1|^{1-1/l}}du\Big)|g_1|^{a-n}\,dg_1\Big)\,|g_1(\bar u)|^a\delta_P(m(\bar u))^{1/2}h(g_2(\bar u))\,d\bar u.\end{aligned}$$

Now there exist positive constants r' and C' depending on r, and hence on f_2, such that for $T \geq 0$,

$$\int_{\|g_2\| \leq T} \|g_2\|^r \, dg_2 \leq C' T^{r'}.$$

Hence, the last multiple integral is less than

$$C_3 \int_{\overline{U}} \Big(\int_{|g_1(\bar{u})|^{1/(l-1)} \leq |g_1| \leq |g_1(\bar{u})|^{-1/(l+1)}} |g_1|^{a-n-r'/l+(1-1/l)(2n-1)} \, dg_1 \Big) \cdot |g_1(\bar{u})|^{a-r'/l-(2n-1)/l} \delta_P(m(\bar{u}))^{1/2} h(g_2(\bar{u})) d\bar{u},$$

where C_3 depends on f_2. Let $\alpha = a - n - r'/l + (1-1/l)(2n-1)$. Then $\alpha \to a + n - 1 > 0$ as $l \to \infty$. Let l be large enough so that $l > 1$ and $\alpha > 0$; l and α depend on f_1, f_2 and π. Since in general

$$\int_{0 < b_1 \leq x \leq b_2} x^\beta \, d^\times x \leq (1/\beta) b_2^\beta$$

for $\beta > 0$, where $d^\times x$ is multiplicative Haar measure on $\mathbb{R}^\times$, the last multiple integral is less than

$$(1/\alpha) \int_{\overline{U}} |g_1(\bar{u})|^{a-r'/l-(2n-1)/l-\alpha/(l+1)} \delta_P(m(\bar{u}))^{1/2} h(g_2(\bar{u})) \, d\bar{u}$$

Now $a' = a - r'/l - (2n-1)/l - \alpha/(l+1) \to a$ as $l \to \infty$. Let l be large enough so that the previous conditions on l are satisfied and such that $a' > e(\delta_2)$ (recall that by the definition of Langlands data for the symplectic group, $a = e(\delta_1) > e(\delta_2)$); if in the statement of the main technical lemma $t = 1$, so that π' is tempered, then we require $a' > 0$. Then l and a' depend on f_1, f_2 and π. The last integral is

$$C_4 = \int_{\overline{U}} |g_1(\bar{u})|^{a'} \delta_P(m(\bar{u}))^{1/2} h(g_2(\bar{u})) \, d\bar{u},$$

which is finite by Lemma 6.4. Altogether, we have

$$|Z|_B(s, f_1, f_2, \Phi) \leq C_1 C_2 C_3 C_4 (1/\alpha) \|\Phi(\cdot, s)\|_{K'}.$$

Hence, there exists a choice of $l > 0$ and $C_5 > 0$, both depending on f_1, f_2, and π, such that

$$|Z|_B(s, f_1, f_2, \Phi) \leq C_5 \|\Phi(\cdot, s)\|_{K'}$$

for $\mathrm{Re}(s) \geq -\rho_{2n}$.

Having now fixed our choice of l, we consider $|Z|_{B^c}(s, f_1, f_2, \Phi)$. We again assume $s \in \mathbb{C}$ is arbitrary. As we mentioned above, for the good points B^c we can carry out the original idea for the proof. The first step is to bound $|g_1(\bar{u})g_1|^{a-n}|f_2(g_2(\bar{u})g_2)|$ by a positive power of $\|k(\bar{u})^{-1}um\|$. Let $(g_1, g_2, u, \bar{u}) \in B^c$. As above, let $m = m(g_1, g_2)$. First we consider $|g_1(\bar{u})g_1|^{a-n}$. Suppose $a - n \geq 0$. Since $|g_1(\bar{u})| \leq 1$ by Lemma 6.1, and $|g_1| \leq \|um\| = \|k(\bar{u})^{-1}um\|$, we have $|g_1(\bar{u})g_1|^{a-n} \leq |g_1|^{a-n} \leq \|k(\bar{u})^{-1}um\|^{a-n}$. Suppose $a - n < 0$. Since $(g_1, g_2, u, \bar{u}) \in B^c$, $|g_1(\bar{u})g_1|^{-1} \leq \|k(\bar{u})um\|^l$. Hence, $|g_1(\bar{u})g_1|^{a-n} \leq \|k(\bar{u})um\|^{l(n-a)}$. Thus, in both cases there exists $c_1 > 0$ depending on l and π such that $|g_1(\bar{u})g_1|^{a-n} \leq \|k(\bar{u})um\|^{c_1}$. Next, by Lemma 6.3, there exist $C_6 > 0$ depending on f_2 and c_2 depending on π' such that $|f_2(g_2(\bar{u})g_2)| \leq C_6\|g_2(\bar{u})g_2\|^{c_2}$. Hence, $|f_2(g_2(\bar{u})g_2)| \leq C_6\|g_2(\bar{u})\|^{c_2}\|g_2\|^{c_2}$. Now $\|g_2\| \leq \|k(\bar{u})^{-1}um\|$. As for $\|g_2(\bar{u})\|$, we have $\bar{u} = u(\bar{u})m(\bar{u})k(\bar{u})$, so that $\|g_2(\bar{u})\| \leq \|u(\bar{u})m(\bar{u})\| = \|\bar{u}\|$. By Lemma 6.1, $\|\bar{u}\| \leq \sqrt{2n}|g_1(\bar{u})|^{-1}$. Also, as $(g_1, g_2, u, \bar{u}) \in B^c$, $|g_1(\bar{u})|^{-1} \leq \|k(\bar{u})^{-1}um\|^l|g_1| \leq \|k(\bar{u})^{-1}um\|^{l+1}$. Hence, $|f_2(g_2(\bar{u})g_2)| \leq C_6(\sqrt{2n})^{c_2}\|k(\bar{u})^{-1}um\|^{c_2(l+2)}$. Summing up, there exist $C_7 > 0$ and $b > 0$ depending on f_1, f_2 and π such that

$$|g_1(\bar{u})g_1|^{a-n}|f_2(g_2(\bar{u})g_2)| \leq C_7\|k(\bar{u})^{-1}um\|^b$$

for $(g_1, g_2, u, \bar{u}) \in B^c$. Now using the second form of the Gelfand-Naimark integration formula mentioned at the beginning of the proof, we have

$$\begin{aligned}
&|Z|_{B^c}(s, f_1, f_2, \Phi) \\
&\leq C_1C_7 \int_{B^c} |\Phi(i(k(\bar{u})^{-1}um, 1), s)|\|k(\bar{u})^{-1}um\|^b \delta_P(m(\bar{u}))\, d(g_1g_2u\bar{u}) \\
&\leq C_1C_7 \int_{G_1\times G_2\times U\times\overline{U}} |\Phi(i(k(\bar{u})^{-1}um, 1), s)|\|k(\bar{u})^{-1}um\|^b \delta_P(m(\bar{u}))\, d(g_1g_2u\bar{u}) \\
&= C_1C_7 \int_G |\Phi(i(g, 1), s)|\|g\|^b\, dg.
\end{aligned}$$

To now complete the proof, let $\sigma_0' = b$ and $C = \max(C_1C_7, C_5)$. Then σ_0' and C depend only on f_1, f_2 and π. Let M be a nonnegative integer. By Lemma 6.5, there exist $D_1, \ldots, D_k \in \mathcal{D}(H)$ such that

$$\int_G |\Phi(i(g, 1), s)|\|g\|^b\, dg \leq \|\Phi(\cdot, s)\|_{D_1,K'} + \cdots + \|\Phi(\cdot, s)\|_{D_k,K'}$$

for $\Phi \in \mathrm{I}_H(\chi)_M$ and $\mathrm{Re}(s) > \sigma_0 + \sigma_0' - cM$. Thus, by the last paragraph,

$$|Z|_{B^c}(s, f_1, f_2, \Phi) \leq C(\|\Phi(\cdot, s)\|_{D_1,K'} + \cdots + \|\Phi(\cdot, s)\|_{D_k,K'})$$

for $\Phi \in \mathrm{I}_H(\chi)_M$ and $\mathrm{Re}(s) > \sigma_0 + \sigma_0' - cM$. From above,

$$|Z|_B(s, f_1, f_2, \Phi) \leq C\|\Phi(\cdot, s)\|_{K'}$$

for $\mathrm{Re}(s) \geq -\rho_{2n}$ and $\Phi \in \mathrm{I}_H(\chi)$. Adding, we get

$$|Z|(s, f_1, f_2, \Phi) \leq C(\|\Phi(\cdot, s)\|_{K'} + \|\Phi(\cdot, s)\|_{D_1,K'} + \cdots + \|\Phi(\cdot, s)\|_{D_k,K'})$$

for $\Phi \in \mathrm{I}_H(\chi)_M$, $\mathrm{Re}(s) > \sigma_0 + \sigma_0' - cM$, and $\mathrm{Re}(s) \geq -\rho_{2n}$. $\square$

Using Lemma 6.6 and the method of argument from Theorem 3.4 of [GJ], we can now prove step one for auxiliary zeta integrals. For the proof we shall need the following fact. Suppose $H : G \times G \to \mathbb{C}$ is a function such that $H(umg, \bar{u}mg) = H(g, g')$ for $u \in U$, $\bar{u} \in \overline{U}$, $m \in M$, and $g, g' \in G$; for $k, k' \in K$, the function on M defined by $m \mapsto H(mk, k')$ is a coefficient of $\delta \otimes \pi'$; and H is smooth and $K \times K$ finite on the right. Then the integral

$$f(g) = \int_{K \times \overline{U}} H(\bar{u}kg, k)\, d(k\bar{u})$$

is absolutely convergent for $g \in G$, and the function f is a coefficient of π. See (5.5) of [GJ].

Theorem 6.7. *Let f_1 and f_2 be coefficients of δ and π', respectively, and let $\Phi \in \mathrm{I}_H(\chi)$ be $K \times K$-finite. Then there exists a coefficient f of π such that for* $\mathrm{Re}(s) > \max(\sigma_0 + \sigma_0', -\rho_{2n})$

$$Z(s, f_1, f_2, \Phi) = Z(s, f, \Phi).$$

Proof. Let $\mathrm{Re}(s) > \max(\sigma_0 + \sigma_0', -\rho_{2n})$, so that by Lemma 6.6 $Z(s, f_1, f_2, \Phi)$ converges absolutely. To simplify the notation, we will write $F(g_1, g_2) = f_1(g_1) f_2(g_2) \delta_P(m(g_1, g_2))^{-1/2}$ for $g_1 \in G_1$ and $g_2 \in G_2$, and $\Phi(g, 1) = \Phi(i(g, 1), s)$ for $g \in G$.

To begin, we use the $K \times K$-finiteness of Φ to introduce an extra integration over $K \times K$. Since Φ is $K \times K$-finite, there exist irreducible unitary representations $\tau_1, \ldots, \tau_l$ of $K \times K$ such that if $\eta : K \times K \to \mathbb{C}$ is defined by $\eta = \dim(\tau_1)\bar{\chi}_{\tau_1} + \cdots + \dim(\tau_l)\bar{\chi}_{\tau_l}$, then

$$\Phi(h) = \int\limits_{K \times K} \Phi(h(k', k)) \eta(k, k')\, d(kk')$$

for $h \in H$. Here, χ_{τ_i} is the character of τ_i. Hence, $Z(s, f_1, f_2, \Phi)$ is

$$\int\limits_{G_1 \times G_2 \times U \times \overline{U}} [\int\limits_{K \times K} \Phi((\bar{u}^{-1} u m(g_1, g_2), 1)(k', k)) \eta(k, k')\, d(kk')] F(g_1, g_2)\, d(g_1 g_2 u \bar{u}).$$

Now by Lemma 6.6,

$$\int\limits_{K \times K} [\int\limits_{G_1 \times G_2 \times U \times \overline{U}} |\Phi((\bar{u}^{-1} u m(g_1, g_2), 1)(k', k)) F(g_1, g_2)|\, d(g_1 g_2 u \bar{u})] |\eta(k, k')|\, d(kk')$$

$$\leq C \int\limits_{K \times K} (\|\Phi(\cdot(k', k))\|_{K'} + \cdots + \|\Phi(\cdot(k', k))\|_{K, D_k}) |\eta(k, k')|\, d(kk'),$$

which is finite. Hence, we can apply Fubini's theorem and conclude that $Z(s, f_1, f_2, \Phi)$ is

$$\int_{K\times K} [\int_{G_1\times G_2\times U\times\overline{U}} \Phi((\bar{u}k)^{-1}um(g_1, g_2)k', 1)F(g_1, g_2)\, d(g_1g_2u\bar{u})]\eta(k, k')\, d(kk'),$$

where the inner integral converges absolutely for all $k, k' \in K$.

Next, we decompose the outer integral to obtain a function H as mentioned in the remark before the statement of the theorem. Let $L = K \cap \overline{P} = K \cap P = m(K_1 \times K_2)$. There exists a unique right K invariant quotient measure on $L\backslash K$ of total measure 1 such that $Z(s, f_1, f_2, \Phi)$ is

$$\int_{L\backslash K\times L\backslash K} [\int_{L\times L} [\int_{G_1\times G_2\times U\times\overline{U}} \Phi((\bar{u}hk)^{-1}um(g_1, g_2)h'k', 1)F(g_1, g_2)\, d(g_1g_2u\bar{u})]$$
$$\eta(hk, h'k')\, d(hh')]\, d(kk').$$

We consider the inner integral, which converges absolutely for all k, k', h and h'. Fix $k, k' \in K$ and $h, h' \in L$. Write $h = m(h_1, h_2)$ and $h' = m(h_1', h_2')$ for $h_1, h_1' \in K_1$ and $h_2, h_2' \in K_2$. Then

$$\int_{G_1\times G_2\times U\times\overline{U}} \Phi((\bar{u}hk)^{-1}um(g_1, g_2)h'k', 1)F(g_1, g_2)\, d(g_1g_2u\bar{u})$$
$$= \int_{G_1\times G_2\times U\times\overline{U}} \delta_P(m(g_1, g_2))\Phi((\bar{u}k)^{-1}m(g_1, g_2)uk', 1)F(h_1g_1{h_1'}^{-1}, h_2g_2{h_2'}^{-1})\, d(g_1g_2u\bar{u}),$$

where both integrals converge absolutely. Thus,

$$Z(s, f_1, f_2, \Phi) = \int_{L\backslash K\times L\backslash K} [\int_{L\times L} [\int_{G_1\times G_2\times U\times\overline{U}} \delta_P(m(g_1, g_2))$$
$$\Phi((\bar{u}k)^{-1}m(g_1, g_2)uk', 1)F(h_1g_1{h_1'}^{-1}, h_2g_2{h_2'}^{-1})\, d(g_1g_2u\bar{u})]\eta(hk, h'k')\, d(hh')]\, d(kk').$$

Since for any $k, k' \in K$ the inner double integral converges absolutely, $Z(s, f_1, f_2, \Phi)$ is

$$\int_{L\backslash K\times L\backslash K} [\int_{G_1\times G_2\times U\times\overline{U}} H_1(m(g_1, g_2), k, k')\Phi((\bar{u}k)^{-1}m(g_1, g_2)uk', 1)\, d(g_1g_2u\bar{u})]\, d(kk'),$$

where

$$H_1(m(g_1, g_2), k, k') = \quad \delta_P(m(g_1, g_2)) \int_{L\times L'} F(h_1g_1{h_1'}^{-1}, h_2g_2{h_2'}^{-1})\eta(hk, h'k')\, d(hh').$$

Define $H : G \times G \to \mathbb{C}$ by $H(umk, \bar{u}m'k') = H_1(m'^{-1}m, k', k)$ for $u \in U$, $\bar{u} \in \overline{U}$, $m, m' \in M$, and $k, k' \in K$. Then computations show that H is well-defined and H satisfies the conditions in the remark preceding the statement of the theorem. This is similar to (5.5.9) of [GJ]. We will apply the conclusion of the remark at the end of the proof.

Now $Z(s, f_1, f_2, \Phi)$ is

$$\begin{aligned}
&\int\limits_{L\backslash K\times L\backslash K} [\int\limits_{G_1\times G_2\times U\times\overline{U}} H(m(g_1,g_2)k',k)\Phi((\bar{u}k)^{-1}m(g_1,g_2)uk',1)\,d(g_1g_2u\bar{u})]\,d(kk') \\
&= \int\limits_{K\times K} [\int\limits_{G_1\times G_2\times U\times\overline{U}} H(m(g_1,g_2)k',k)\Phi((\bar{u}k)^{-1}m(g_1,g_2)uk',1)\,d(g_1g_2u\bar{u})]\,d(kk') \\
&= \int\limits_{K\times K\times\overline{U}} [\int\limits_{G_1\times G_2\times U} H(m(g_1,g_2)k',k)\Phi((\bar{u}k)^{-1}m(g_1,g_2)uk',1)\,d(g_1g_2u)]\,d(kk'\bar{u}) \\
&= \int\limits_{K\times K\times\overline{U}} [\int\limits_{G_1\times G_2} [\int\limits_{U} \delta_P(m(g_1,g_2))^{-1}H(um(g_1,g_2)k',k) \\
&\qquad\qquad \Phi((\bar{u}k)^{-1}um(g_1,g_2)k',1)\,du]\,d(g_1g_2)]\,d(kk'\bar{u}) \\
&= \int\limits_{K\times K\times\overline{U}} [\int\limits_{P} H(pk',k)\Phi((\bar{u}k)^{-1}pk',1)\,d_lp]\,d(kk'\bar{u}),
\end{aligned}$$

where d_lp is the left Haar measure on P determined by the Haar measures on M and U. By the same computations with absolute values, we have

$$\int\limits_{K\times K\times\overline{U}} [\int\limits_{P} |H(pk',k)\Phi((\bar{u}k)^{-1}pk',1)|\,d_lp]\,d(kk'\bar{u}) < \infty.$$

Hence,

$$\begin{aligned}
Z(s,f_1,f_2,\Phi) &= \int\limits_{K\times K\times\overline{U}\times P} H(pk',k)\Phi((\bar{u}k)^{-1}pk',1)\,d(kk'\bar{u}p) \\
&= \int\limits_{K\times\overline{U}} [\int\limits_{K} [\int\limits_{P} H(pk',k)\Phi((\bar{u}k)^{-1}pk',1)\,d_lp]\,dk']\,d(k\bar{u}) \\
&= \int\limits_{K\times\overline{U}} [\int\limits_{G} H(g,k)\Phi((\bar{u}k)^{-1}g,1)\,dg]\,d(k\bar{u}) \\
&= \int\limits_{K\times\overline{U}} [\int\limits_{G} H(\bar{u}kg,k)\Phi(g,1)\,dg]\,d(k\bar{u}),
\end{aligned}$$

where dg is the Haar measure on G determined by the Haar measures on P and K via the Iwasawa decomposition $G = PK$. Again, by the same computations with absolute values, we obtain

$$\int_{K\times\overline{U}} [\int_G |H(\bar{u}kg,k)\Phi(g,1)|\,dg]\,d(k\bar{u}) < \infty.$$

So,

$$\begin{aligned} Z(s,f_1,f_2,\Phi) &= \int_{K\times\overline{U}\times G} H(\bar{u}kg,k)\Phi(g,1)\,d(k\bar{u}g) \\ &= \int_G \Phi(g,1)[\int_{K\times\overline{U}} H(\bar{u}kg,k)\,d(k\bar{u})]\,dg. \end{aligned}$$

The theorem now follows from the remark preceding the statement of the theorem. □

As we mentioned at the beginning of this section and in Section 4, to prove the main technical lemma when $F = \mathbb{R}$ we will need to use a density argument. To do so we will need the following result of [KR1], which has as corollary the analytic continuation of the auxiliary zeta integrals.

Theorem 6.8 (Kudla-Rallis). *Let $\mathfrak{g}$ be the Lie algebra of $G \cong G\times 1 \subset H$, let $Z(\mathfrak{g})$ be the center of the universal enveloping algebra of $\mathfrak{g}$, and let M a nonnegative integer. Then there exists $X(s) \in Z(\mathfrak{g})\otimes\mathbb{C}[s]$ such that the polynomial $p(s)\in\mathbb{C}[s]$ by which $X(s)$ acts on π is nonzero, and $X(s)\,\mathrm{I}_H(\chi) \subset \mathrm{I}_H(\chi)_M$ for all $s \in \mathbb{C}$.*

Proof. See Proposition 3.2.1 of [KR1]. □

Corollary 6.9. *Let f_1 and f_2 be coefficients of δ and π', respectively, let σ_0 and σ_0' be as in Lemma 6.6, let M be a nonnegative integer, and let $X(s)$ and $p(s)$ be as in Theorem 6.8. Then for $K\times K$-finite $\Phi \in \mathrm{I}_H(\chi)$ and $\mathrm{Re}(s) > \max(\sigma_0+\sigma_0', -\rho_{2n})$,*

$$p(s)Z(s,f_1,f_2,\Phi) = Z(s,f_1,f_2,X(s)\Phi).$$

Hence, $p(s)Z(s,f_1,f_2,\Phi)$ is holomorphic in $\mathrm{Re}(s) > \max(\sigma_0+\sigma_0'-cM, -\rho_{2n})$.

Proof. Let $\Phi\in\mathrm{I}_H(\chi)$ be $K\times K$-finite. Let $\mathrm{Re}(s) > \max(\sigma_0+\sigma_0', -\rho_{2n})$. By Theorem 6.7, there exists a coefficient f of π such that $Z(s,f_1,f_2,\Phi) = Z(s,f,\Phi)$. Since the action of $X(s)$ on $\mathrm{I}_H(\chi)$ commutes with the action of $K\times K$, $X(s)\Phi$ has the same $K\times K$-type as Φ. It follows from the proof of Theorem 6.7 that $Z(s,f_1,f_2,X(s)\Phi) = Z(s,f,X(s)\Phi)$. By [KR1], $p(s)Z(s,f,\Phi) = Z(s,f,X(s)\Phi)$, so that $p(s)Z(s,f_1,f_2,\Phi) = Z(s,f_1,f_2,X(s)\Phi)$. Since by Lemma 6.6 $Z(s,f_1,f_2,X(s)\Phi)$ is holomorphic in $\mathrm{Re}(s) > \max(\sigma_0+\sigma_0'-cM, -\rho_{2n})$, so is $p(s)Z(s,f_1,f_2,\Phi)$. □

To end this section, we consider the nonarchimedean version of Theorem 6.7. As we remarked in the summary of Section 4, the nonarchimedean case of the main technical lemma is relatively straightforward, and auxiliary zeta integrals are unnecessary. Since the argument is very similar to the proof of Theorem 6.7, we will only state the result.

Theorem 6.10. *Suppose F is nonarchimedean. Let f_1 and f_2 be coefficients for δ and π', respectively, and let $\Phi_1 \in \mathrm{I}_{H_1}(\mu_1, \mu_2)$ and $\Phi_2 \in \mathrm{I}_{H_2}(\chi)$. Assume that $\Phi \in \mathrm{I}_H(\chi)$ is such that*

$$\Phi_1(i_1(g_1, 1), s)\Phi_2(i_2(g_2, 1), s) = \int_{U \times \overline{U}} \Phi(i(\bar{u}^{-1}um(g_1, g_2), 1), s)\, d(u\bar{u}),$$

for $g_1 \in G_1$, $g_2 \in G_2$ and $\mathrm{Re}(s)$ large. Assume also that

$$\int_{G_1 \times G_2 \times U \times \overline{U}} \Phi(i(\bar{u}^{-1}um(g_1, g_2), 1), s) f_1(g_1) f_2(g_2) \delta_P(m)^{-1/2}\, d(g_1 g_2 u\bar{u})$$

converges absolutely for $\mathrm{Re}(s)$ large. Then there exists a coefficient f of π such that

$$Z(s, f_1 \otimes \delta_P^{-1/2}, \Phi_1) Z(s, f_2, \Phi_2) = Z(s, f, \Phi).$$

7. Step two: sections giving poles

This section provides the second ingredient for the proof of the main technical lemma. It constructs a μ_1, μ_2-section Φ_1 for G_1, a χ-section Φ_2 for G_2, a χ-section Φ for G, and matrix coefficients f_1 and f_2 for δ and π', respectively, such that $Z(s - 1/2, f_1 \otimes \delta_P^{-1/2}, \Phi_1)$ has a pole at s_0, $Z(s - 1/2, f_2, \Phi_2) = 1$, and (4.1) holds. As usual, in this section F is a local field of characteristic zero, with $F = \mathbb{R}$ if F is archimedean, and we shall again use the notation of Lemma 1.4, the main technical lemma. In particular $\chi = 1$ if $F = \mathbb{R}$, though this is only used in Proposition 7.7.

The construction in this section goes as follows. To begin, we first recall the definition of a class of sections which are easy to manipulate, namely sections compactly supported on the open orbit of $G \times G$ in $P' \backslash H$; there are similar definitions for G_1 and G_2. We then show that for any pair of sections Φ_1 and Φ_2 for G_1 and G_2, respectively, compactly supported on the open orbits, there exists a χ-section Φ for G, also compactly supported on the open orbit, such that (4.1) holds. The key tool for this is the Gelfand-Naimark decomposition. Now for any χ-section Φ_2 for G_2 compactly supported on the open orbit and any coefficient f_2 for π', it is easy to see that $Z(s - 1/2, f_2, \Phi_2)$ is constant, and in fact Φ_2 and f_2 can be chosen so that $Z(s - 1/2, f_2, \Phi_2) = 1$, which we do. However, the same reasoning shows that $Z(s - 1/2, f_1 \otimes \delta_P^{-1/2}, \Phi_1)$ is also constant for any μ_1, μ_2-section Φ_1 for G_1 compactly supported in the open orbit and coefficient f_1 of δ. This is of course undesirable, but we can change Φ_1 to a section so that $Z(s - 1/2, f_1 \otimes \delta_P^{-1/2}, \Phi_1)$ has a pole at s_0, as follows. Keeping the choice of f_2 and Φ_2 we have already made, let Φ_1 be any μ_1, μ_2-section for G_1 compactly supported in the open orbit, and let Φ be a χ-section for G, also compactly supported in the open orbit, such that (4.1) holds. Then the idea is that one can rewrite (4.1) as

$$\Phi_1(i_1(g_1, 1), s)\Phi_2(i_2(g_2, 1), s) = \int_{U \times \overline{U}} \Phi(i(u, 1)i(m(g_1, g_2), 1)i(1, \bar{u}), s)\, d(u\bar{u}).$$

Using that

$$i(m(g_1,g_2),1) = i_1'i_1(g_1,1)i_2'i_2(g_2,1)$$

for $g_1 \in G_1$ and $g_2 \in G_2$, which follows from Lemma 5.1, we show that this new version of (4.1) actually holds on the largest possible set, i.e.,

$$\Phi_1(h_1,s)\Phi_2(h_2,s) = \int_{U\times\overline{U}} \Phi(i(u,1)i_1'(h_1)i_2'(h_2)i(\bar{u},1),s)\,d(u\bar{u})$$

for $h_1 \in H_1$ and $h_2 \in H_2$. To now alter Φ_1 and obtain a section with the right properties, it is natural to use the last equality and translate. However, the factor $i(1,\bar{u})$ appears to be an obstacle. But as it turns out, there is an essentially unique nontrivial element w of H_1 such that $i_1'(w)$ commutes with $i(1\times\overline{U})$, and this element suffices! That is, because $i_1'(w)$ commutes with $i(1\times\overline{U})$, (4.1) holds for $w\Phi_1$, Φ_2 and $i_1'(w)\Phi$; moreover if Φ_1 has compact support in a small enough neighborhood of the identity in the open orbit, then $Z(s-1/2, f_1\otimes\delta_P^{-1/2}, w\Phi_1)$ has a pole at s_0.

One might wonder whether or not things are really as complicated as they appear, and in particular whether or not in trying to solve (4.1) we are dealing with an intertwining operator. After all, one can also rewrite (4.1) as

$$\Phi_1(i_1(g_1,1),s)\Phi_2(i_2(g_2,1),s) = \int_{U\times\overline{U}} \Phi(i(u,1)i(1,\bar{u})i_1'i_1(g_1,1)i_2'i_2(g_2,1),s)\,d(u\bar{u}),$$

which naturally leads to the question: Is there a half-plane $\mathrm{Re}(s) > x_0$ such that for $\Phi \in \mathrm{I}_H(\chi)$, $h_1 \in H_1$ and $h_2 \in H_2$,

$$\Phi'(h_1,h_2,s) = \int_{U\times\overline{U}} \Phi(i(u,1)i(1,\bar{u})i_1'(h_1)i_2'(h_2),s)\,d(u\bar{u})$$

converges absolutely and defines an element an element of $\mathrm{I}_{H_1}(\mu_1,\mu_2,s)\otimes\mathrm{I}_{H_2}(\chi,s)$? The answer is no, as a simple example at the end of this section shows.

In contrast to the case of a fully induced representation which we discussed after Lemma 5.1, the complications of this section may seem surprising. However, an analogous difficulty arises for $\mathrm{Gl}(n)$ in [GJ]. In showing that the L-function of a Langlands quotient is the product of the L-functions of the Langlands data, Godement and Jacquet also encounter an equation like (4.1). See, for example, the proof of Theorem 3.4, [GJ]. This analogous equation also can be only solved in a special case, and to prove the equality of L-functions, Godement and Jacquet must resort to an indirect argument by contradiction. In particular, their argument uses the classification of the tempered dual of $\mathrm{Gl}(n)$. We do not know if proving the analogous equality of L-functions for the $\mathrm{Sp}(n)$–of which which we have proven a very small part–will require the classification of the tempered dual of $\mathrm{Sp}(n)$. In any case, though, it does seem that the final argument will not simply be a direct computation.

As we mentioned above, we begin by considering sections compactly supported on the open orbit. To define these sections precisely, recall that the set $Pi(G \times 1)$ is an open and dense subset of H, and that the map $P' \times G \to Pi(G \times 1)$ defined by $(p, g) \mapsto pi(g, 1)$ is a homeomorphism if F is nonarchimedean and a diffeomorphism if F is archimedean. If now $\phi \in C_c^\infty(G)$, define $\Phi_\phi : H \times \mathbb{C} \to \mathbb{C}$ by

$$\Phi_\phi(h, s) = \begin{cases} 0 \text{ if } h \notin P'i(G \times 1) \\ \alpha_\chi(p, s)\phi(g) \text{ if } h \in P'i(G \times 1),\ h = pi(g, 1). \end{cases}$$

Since $P' \cap i(G \times 1) = 1$, Φ_ϕ is well-defined. Clearly, $\Phi_\phi(ph, s) = \alpha_\chi(p, s)\Phi_\phi(h, s)$ for $p \in P'$, $h \in H$ and $s \in \mathbb{C}$.

Lemma 7.1. *Let $\phi \in C_c^\infty(G)$ and $s \in \mathbb{C}$. If $h \in H$ and $h \notin P'i(G \times 1)$, then $\Phi_\phi(h', s) = 0$ for h' in an open neighborhood of h.*

Proof. Assume that no such neighborhood of h exists. Then there exist sequences (g_n) in G and (p_n) in P' such that $i(g_n, 1)p_n \to h$ and $\phi(g_n) \neq 0$ for all n. Since ϕ has compact support, we may assume without loss of generality that $g_n \to g$ with g in the support of ϕ. Hence, $p_n \to p$ for some $p' \in P'$, and $i(g_n, 1)p_n \to i(g, 1)p = h$, contradicting $h \notin P'i(G \times 1)$. □

As the next result shows, the functions Φ_ϕ are χ-sections for G.

Lemma 7.2. *If $\phi \in C_c^\infty(G)$, then $\Phi_\phi \in \mathrm{I}_H(\chi)$. If F is nonarchimedean and $\phi \in C_c^\infty(G)$ has support in K, then $\Phi_\phi \in \mathrm{I}_H^{\mathrm{Stan}}(\chi)$.*

Proof. Suppose first that F is archimedean. To prove $\Phi_\phi \in \mathrm{I}_H(\chi)$ it will suffice to show that $\Phi_\phi(\cdot, s)$ is smooth for $s \in \mathbb{C}$. The map $P' \times G \to \mathbb{C}$ defined by $(p, g) \mapsto \alpha_\chi(p, s)\phi(g)$ is smooth; since the map $P' \times G \xrightarrow{\sim} P'i(G \times 1)$ has a smooth inverse it follows that the restriction of Φ_ϕ to $P'i(G \times 1)$ is smooth. Lemma 7.1 implies that Φ_ϕ is smooth at the points of H not contained in $P'i(G \times 1)$.

Now assume that F is nonarchimedean. To prove $\Phi_\phi \in \mathrm{I}_H(\chi)$ it will suffice to show that there exists a compact open subgroup L of G such that $\Phi_\phi(k'k, s) = \Phi_\phi(k', s)$ for $k' \in K'$, $k \in L$ and $s \in \mathbb{C}$. We will require some extra notation. If m is a positive integer, we let $K(m)$ be the m-th principal congruence subgroup of K consisting of the $k \in K$ such that k is congruent to 1 mod $\mathfrak{p}^m$, and we similarly define the m-th principal congruence subgroup $K'(m)$ of K'. Also, for m a positive integer, let $K_0'(m)$ be the subgroup of K' of elements whose lower left entries are congruent to 0 mod $\mathfrak{p}^m$. One can verify that $K_0'(m) = (K' \cap P')i(K(m) \times 1)$. Now using that ϕ is smooth and compactly supported, we may assume that for some positive $m \geq c(\chi)$, ϕ is the characteristic function of $K(m)$. The support of $\Phi_\phi(\cdot, s)|_{K'}$ is $(P' \cap K')i(K(m) \times 1) = K_0'(m)$, and if $k \in K_0'(m)$ then $\Phi_\phi(k, s) = \chi(\det a)$; here, a is the upper left entry of k. It follows that $\Phi_\phi(k'k, s) = \Phi_{\phi_m}(k', s)$ for $k' \in K'$, $k \in K'(m)$, and $s \in \mathbb{C}$.

Finally, an argument as in the last paragraph shows that if F is nonarchimedean and ϕ has support in K, then $\Phi_\phi \in \mathrm{I}_H^{\mathrm{Stan}}(\chi)$. $\square$

We call a section Φ_ϕ with $\phi \in C_c^\infty(G)$ a **section compactly supported on the open orbit**.

Next, we show that (4.1) is easy to solve for sections compactly supported in the open orbit. To prove this we will need the Gelfand-Naimark decomposition, which asserts that the set $\overline{U}P$ is open and dense in G, and that the map

$$j : U \times \overline{U} \times G_1 \times G_2 \to \overline{U}P$$

defined by $j(u, \bar{u}, g_1, g_2) = \bar{u}^{-1}um(g_1, g_2)$ is a homeomorphism if F is nonarchimedean, and a diffeomorphism if F is archimedean.

Lemma 7.3. *Let $\phi_1 \in C_c^\infty(G_1)$ and $\phi_2 \in C_c^\infty(G_2)$. Then there exists $\phi \in C_c^\infty(G)$ with support in $\overline{U}P$ such that*

$$\Phi_{\phi_1}(i_1(g_1, 1), s)\Phi_{\phi_2}(i_2(g_2, 1), s) = \int_{U\times\overline{U}} \Phi_\phi(i(\bar{u}^{-1}um(g_1, g_2), 1), s)\, d(u\bar{u})$$

for $g_1 \in G_1$, $g_2 \in G_2$ and $s \in \mathbb{C}$. If F is nonarchimedean and ϕ_1 and ϕ_2 have support in K_1 and K_2, respectively, then ϕ can be chosen to have support in K, so that Φ_ϕ is standard. Moreover, for any $\phi \in C_c^\infty(G)$ with support in $\overline{U}P$ such that the above identity holds,

$$\Phi_{\phi_1}(h_1, s)\Phi_{\phi_2}(h_2, s) = \int_{U\times\overline{U}} \Phi_\phi(i(u, 1)i_1'(h_1)i_2'(h_2)i(1, \bar{u}), s)\, d(u\bar{u})$$

for $h_1 \in H_1$, $h_2 \in H_2$ and $s \in \mathbb{C}$.

Proof. Proving the existence of ϕ amounts to showing that there exists $\phi \in C_c^\infty(G)$ with support in $\overline{U}P$ such that

$$\phi_1(g_1)\phi_2(g_2) = \int_{U\times\overline{U}} \phi(\bar{u}^{-1}um(g_1, g_2))\, d(u\bar{u})$$

for $g_1 \in G_1$ and $g_2 \in G_2$. Let $\phi' \in C_c^\infty(U \times \overline{U})$ have total integral 1 over $U \times \overline{U}$, and define $\phi \in C_c^\infty(G)$ by defining ϕ on $\overline{U}P$ as the composition

$$\overline{U}P \xrightarrow{\sim} U \times \overline{U} \times G_1 \times G_2 \xrightarrow{\phi'\otimes\phi_1\otimes\phi_2} \mathbb{C},$$

and setting ϕ off $\overline{U}P$ to be zero. Then ϕ is contained in $C_c^\infty(G)$, has support in $\overline{U}P$, and the first identity of the lemma holds. It is easy to see that if ϕ_1 and ϕ_2 have support in K_1 and K_2, respectively, then ϕ can be chosen to have support in K.

Now suppose that $\phi \in C_c^\infty(G)$ has support in $\overline{U}P$, and the first identity of the lemma holds. Let $s \in \mathbb{C}$. To prove the second identity of the lemma, we note first that by Lemma 5.1, the first identity can be rewritten as

$$\Phi_{\phi_1}(i_1(g_1,1),s)\Phi_{\phi_2}(i_2(g_2,1),s) = \int_{U\times\overline{U}} \Phi_\phi(i(u,1)i'_1 i_1(g_1,1)i'_2 i_2(g_2,1)i(1,\bar{u}),s)\,d(u\bar{u})$$

for $g_1 \in G_1$, $g_2 \in G_2$. Thus, the second identity holds for $h_1 \in i_1(G_1 \times 1)$ and $h_2 \in i_2(G_2 \times 1)$. Next, we show that the second identity holds for $h_1 = p_1 i_1(g_1,1) \in P'_1 i_1(G_1 \times 1)$ and $h_2 = p_2 i_2(g_2,1) \in P'_2 i_2(G_2 \times 1)$. By Lemma 5.2,

$$\begin{aligned}
&\int_{U\times\overline{U}} |\Phi_\phi(i(u,1)i'_1(h_1)i'_2(h_2)i(1,\bar{u}),s)|\,d(u\bar{u})\\
&= \int_{\overline{U}}(\int_U |\Phi_\phi(i(u,1)i'_1(p_1)i'_2(p_2)i'_1 i_1(g_1,1)i'_2 i_2(g_2,1)i(1,\bar{u}),s)|\,du)\,d\bar{u}\\
&= \int_{\overline{U}} |\alpha_{\mu_1,\mu_2}(p_1,s)\alpha_\chi(p_2,s)| \int_U |\Phi_\phi(i(u,1)i'_1 i_1(g_1,1)i'_2 i_2(g_2,1)i(1,\bar{u}),s)|\,du)\,d\bar{u}\\
&= |\alpha_{\mu_1,\mu_2}(p_1,s)\alpha_\chi(p_2,s)| \int_{U\times\overline{U}} |\Phi_\phi(i(u,1)i'_1 i_1(g_1,1)i'_2 i_2(g_2,1)i(1,\bar{u}),s)|\,d(u\bar{u}).
\end{aligned}$$

Since the last integral is finite, so is the first. A similar computation without absolute values now shows that

$$\int_{U\times\overline{U}} \Phi_\phi(i(u,1)i'_1(h_1)i'_2(h_2)i(1,\bar{u}),s)\,d(u\bar{u})$$

is

$$\alpha_{\mu_1,\mu_2}(p_1,s)\alpha_\chi(p_2,s)\Phi_{\phi_1}(i_1(g_1,1),s)\Phi_{\phi_2}(i_1(g_2,1),s) = \Phi_{\phi_1}(h_1,s)\Phi_{\phi_2}(h_2,s).$$

Finally, suppose that $h_1 \in H_1$ and $h_2 \in H_2$, but $h_1 \notin P'_1 i_1(G_1 \times 1)$ or $h_2 \notin P'_2 i_2(G_2 \times 1)$. Since $\Phi_{\phi_1}(h_1,s)\Phi_{\phi_2}(h_2,s) = 0$, it will suffice to show that the integrand of our integral is identically zero. Since $P'_1 i_1(G_1 \times 1)$ and $P'_2 i_2(G_2 \times 1)$ are dense in H_1 and H_2, respectively, there exist sequences $\{h_1(n)\} = \{p_1(n)i_1(g_1(n),1)\}$ and $\{h_2(n)\} = \{p_2(n)i_2(g_2(n),1)\}$ such that $p_1(n) \in P'_1$, $p_2(n) \in P'_2$, $g_1(n) \in G_1$, $g_2(n) \in G_2$ and $h_1(n) \to h_1$ and $h_2(n) \to h_2$. Let C, $\overline{C}$, C_1 and C_2 be compact subsets of U, $\overline{U}$, G_1 and G_2, respectively, such that ϕ has support in $\overline{C}Cm(C_1 \times C_2)$. Assume that $g_1(n) \in C_1$ for only finitely many n. For any n such that $g_1(n) \notin C_1$, a computation from above shows that

$$\int_{U\times\overline{U}} |\Phi_\phi(i(u,1)i'_1(h_1(n))i'_2(h_2(n))i(1,\bar{u}),s)|\,d(u\bar{u})$$

is

$$|\alpha_{\mu_1,\mu_2}(p_1(n),s)\alpha_\chi(p_2(n),s)| \int_{U\times\overline{U}} |\phi(\bar{u}^{-1}um(g_1(n),g_2(n)))|\,d(u\bar{u}) = 0.$$

Since Φ_ϕ is continuous, this implies that $\Phi_\phi(i(u,1)i_1'(h_1(n))i_2'(h_2(n))i(1,\bar{u}),s) = 0$ for $u \in U$ and $\bar{u} \in \overline{U}$ for all n such that $g_1(n) \notin C_1$. Taking limits, we get

$$\Phi_\phi(i(u,1)i_1'(h_1)i_2'(h_2)i(1,\bar{u}),s) = 0$$

for all $u \in U$ and $\bar{u} \in \overline{U}$, proving our claim. Suppose that $g_1(n) \in C_1$ for infinitely many n. Passing to a subsequence, we may assume that $g_1(n) \in C_1$ for all n, and by a similar argument, that $g_2(n) \in C_2$ for all n. Since C_1 and C_2 are compact, we may assume that $g_1(n) \to k_1 \in C_1$ and $g_2(n) \to k_2 \in C_2$. This implies that $\{p_1(n)\}$ and $\{p_2(n)\}$ converge to elements p_1 and p_2 of P_1' and P_2', respectively. Hence, $h_1 = p_1 i_1(k_1,1) \in P_1' i_1(G_1 \times 1)$ and $h_2 = p_2 i_2(k_2,1) \in P_2' i_2(G_2 \times 1)$, contradicting our hypothesis. □

Remark 7.4. An argument similar to the proof of Lemma 7.3 shows that if $\phi_1 \in C_c^\infty(G_1)$ and $\phi_2 \in C_c^\infty(G_2)$ are such that $\phi_1 \geq 0$ and $\phi_2 \geq 0$, then there exists $\phi \in C_c^\infty(G)$ with support in $\overline{U}P$ and $\phi \geq 0$ such that the second identity of Lemma 7.3 holds and additionally,

$$|\Phi_{\phi_1}(h_1,s)\Phi_{\phi_2}(h_2,s)| = \int_{U\times\overline{U}} |\Phi_\phi(i(u,1)i_1'(h_1)i_2'(h_2)i(1,\bar{u}),s)|\, d(u\bar{u})$$

for $h_1 \in H_1$, $h_2 \in H_2$ and $s \in \mathbb{C}$; if ϕ_1 and ϕ_2 have support in K_1 and K_2, respectively, then ϕ can be chosen to have support in K. We will need this later on for an application of Fubini's theorem.

As we indicated at the beginning of this section, we will translate the last equality of in Lemma 7.3 to obtain a similar equality for sections which produce zeta integrals of the needed form. The following lemma describes the elements by which we may translate.

Lemma 7.5. *The subgroup of elements of $i_1'(H_1)$ which centralize $i(1\times\overline{U})$ is generated by $i_1' i_1(G_1 \times 1)$ and $i_1'(w)$, where*

$$w = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in H_1.$$

Proof. This follows by a direct but nontrivial computation. □

The next lemma will be needed when $F = \mathbb{R}$.

Lemma 7.6. *let $r \in C_c^\infty(-\infty,\infty)$. Then*

$$\Gamma_r(s) = \int_0^\infty r(x)x^{s-1}\, dx$$

converges absolutely to a holomorphic function in $\mathrm{Re}(s) > 0$. Moreover, Γ_r has an meromorphic continuation to $\mathbb{C}$ with at most simple poles contained in $\{0,-1,-2,\dots\}$. Hence, there exists an entire function g on $\mathbb{C}$ such that

$$\Gamma_r(s) = g(s)\Gamma(s)$$

for $s \in \mathbb{C}$. Here, $\Gamma(s)$ the usual gamma function. If $r(0) \neq 0$, then Γ_r has a simple pole at $s = 0$.

We come now to the main result of this section.

Proposition 7.7. *Let notation be as in the main technical lemma, and as at the beginning of Section 6. There exist a μ_1, μ_2-section Φ_1 for G_1, a χ-section Φ_2 for G_2, a χ-section Φ for G, and matrix coefficients f_1 and f_2 for δ and π', respectively, such that*

$$\Phi_1(i_1(g_1, 1), s)\Phi_2(i_2(g_2, 1), s) = \int_{U\times\overline{U}} \Phi(i(\bar{u}^{-1}um(g_1, g_2), 1), s)\, d(u\bar{u})$$

for $g_1 \in G_1$, $g_2 \in G_2$ and $s \in \mathbb{C}$, Φ_1, Φ_2 and Φ are standard if F is nonarchimedean, and:

(1) *If F is nonarchimedean, then there exist constants A and B, with $A \neq 0$, such that*

$$Z(s - 1/2, f_1 \otimes \delta_P^{-1/2}, \Phi_1) = Aq^{Bs}L(s, \chi\delta^{-1}), \qquad Z(s, f_2, \Phi_2) = 1;$$

In particular, $Z(s - 1/2, f_1 \otimes \delta_P^{-1/2}, \Phi_1)$ has a simple pole at s_0.

(2) *If $F = \mathbb{R}$, then there exists an entire function g such that $g(s_0) \neq 0$ and*

$$Z(s - 1/2, f_1 \otimes \delta_P^{-1/2}, \Phi_1) = g(s)\Gamma(s - s_0), \qquad Z(s, f_2, \Phi_2) = 1.$$

In particular, $Z(s - 1/2, f_1 \otimes \delta_P^{-1/2}, \Phi_1)$ has a simple pole at s_0.

Proof. First we make some definitions. Let $f_1 = \delta$. If F is nonarchimedean, fix a positive integer m such that $m \geq c(\chi)$, and let $\phi_1 \in C_c^\infty(F^\times)$ be the characteristic function of $1 + \mathfrak{p}^m$. If F is archimedean, fix a positive number ϵ such that $0 < \epsilon < 1$ and let $\phi_1 \in C_c^\infty(F^\times)$ have support in $(1 - \epsilon, 1 + \epsilon)$ and $\phi_1(1) \neq 0$. Let f_2 be a nonzero coefficient of π', and let $\phi_2 \in C_c^\infty(G_2)$ be such that

$$\int_{G_2} \phi_2(g)f_2(g)\, dg = 1;$$

if F is nonarchimedean, let ϕ_2 have support in K_2. Let $\Phi_2 = \Phi_{\phi_2}$; then evidently $Z(s, f_2, \Phi_2) = 1$. Finally, let $\phi \in C_c^\infty(G)$ be as in Lemma 7.3; if F is nonarchimedean, we may assume that ϕ has support in K.

Now assume F is nonarchimedean. Let $\Phi_1 = w\Phi_{\phi_1}$ and $\Phi = i_1'(w)\Phi_\phi$, where w is as in Lemma 7.5. Then Φ_1, Φ_2 and Φ are standard, and the first identity of the proposition holds by Lemma 7.3 and Lemma 7.5. It remains to compute $Z(s, f_1, \Phi_1)$. Let $x \in F^\times$, $x \neq 1$. Then

$$i_1(x, 1)w = \begin{bmatrix} 1 & x \\ 0 & 1 - x \end{bmatrix} i_1(\frac{x}{x - 1}, 1).$$

Recalling that $\delta_P^{-1/2}(x) = |x|^{-n}$, it follows that for $\mathrm{Re}(s)$ sufficiently large,

$$Z(s - 1/2, f_1 \otimes \delta_P^{-1/2}, \Phi_1) = \int_{F^\times} \Phi_{\phi_1}(i_1(x, 1)w, s - 1/2)\delta(x)|x|^{-n}\, d^\times x$$

$$=\chi(-1)\int_{F^\times}|x-1|^{-s}\phi_1'(\frac{x}{x-1})(\chi^{-1}\delta)(x)\,d^\times x.$$

where $\phi_1' = \chi|\cdot|^{-n}\phi_1$. We have $x/(x-1)\in 1+\mathfrak{p}^m$ if and only if $|x|\geq|\pi_F|^{-m}$. Also, if $|x|\geq|\pi_F|^{-m}$, then $|x-1|=|x|$ and $\chi(x/(x-1))=|x/(x-1)|^{-n}=1$. A computation now shows that

$$\begin{aligned}Z(s-1/2,f_1\otimes\delta_P^{-1/2},\Phi_1)=&\chi(-1)\int_{|x|\geq|\pi_F|^{-m}}|x|^{-s}(\chi^{-1}\delta)(x)\,d^\times x\\ =&\chi(-1)(\int_{\mathfrak{O}^\times}(\chi^{-1}\delta)(u)\,du)\frac{(|\pi_F|^s(\chi\delta^{-1})(\pi_F))^m}{1-|\pi_F|^s(\chi\delta^{-1})(\pi_F)}.\end{aligned}$$

This proves (1), since $\chi^{-1}\delta=|\cdot|^{s_0}$ is unramified.

Assume $F=\mathbb{R}$. Again, we let $\Phi_1=w\Phi_{\phi_1}$, $\Phi=i_1'(w)\Phi_\phi$ and the first identity of the proposition holds. As in the nonarchimedean case, for sufficiently large $\mathrm{Re}(s)$,

$$Z(s-1/2,f_1\otimes\delta_P^{-1/2},\Phi_1)=\int_{\mathbb{R}^\times}|x-1|^{-s}\phi_1'(\frac{x}{x-1})\delta(x)|x|^{-1}\,dx,$$

where $\phi_1'=|\cdot|^{-n}\phi_1$. Let $x\in\mathbb{R}^\times$, $x\neq 1$. Then $x/(x-1)\in(1-\epsilon,1+\epsilon)$ if and only if $x\in(-\infty,1-1/\epsilon)\cup(1+1/\epsilon,\infty)$. Hence,

$$\begin{aligned}&Z(s-1/2,f_1\otimes\delta_P^{-1/2},w\Phi_{\phi_1})\\ &=\int_{-\infty}^{1-1/\epsilon}(1-x)^{-s}\phi_1'(\frac{x}{x-1})\delta(x)|x|^{-1}\,dx\\ &\qquad+\int_{1+1/\epsilon}^{\infty}(x-1)^{-s}\phi_1'(\frac{x}{x-1})\delta(x)|x|^{-1}\,dx\\ &=\int_{-\infty}^{1-1/\epsilon}(1-x)^{-s+s_0-1}\phi_1''(\frac{x}{x-1})\,dx+\int_{1+1/\epsilon}^{\infty}(x-1)^{-s+s_0-1}\phi_1''(\frac{x}{x-1})\,dx,\end{aligned}$$

where $\phi_1''=|\cdot|^{s_0-1}\phi_1'$. Now we change variables. In the first integral, let $y=1/(1-x)$; in the second, $y=1/(x-1)$. Then

$$Z(s-1/2,f_1\otimes\delta_P^{-1/2},w\Phi_{\phi_1})=\int_0^\epsilon r(y)y^{(s-s_0)-1}\,dy,$$

where $r:(-\infty,\infty)\to\mathbb{C}$ is defined by $r(y)=\phi_1''(1-y)+\phi_1''(1+y)$. The function r is smooth, has support in $(-\epsilon,\epsilon)$ and and $r(0)\neq 0$. Our claim now follows from Lemma 7.6. $\square$

Remark 7.8. In the nonarchimedean case, the argument in the proof of Proposition 7.7 actually shows that for any quasi-character $\delta = \delta_1$, not just those of the form $\chi|\cdot|^{s_0}$, there exist f_1, f_2, Φ_1, Φ_2 and Φ, and constants A and B, $A \neq 0$, such that the first identity of Proposition 7.7 holds, $Z(s, f_2, \Phi_2) = 1$, and $Z(s, f_1 \otimes \delta_P^{-1/2}, \Phi_1) = Aq^{Bs}L(s, \chi\delta^{-1})$. This is clear from the argument in the proof of Proposition 7.7 if $\chi^{-1}\delta$ is unramified. If $\chi^{-1}\delta$ is ramified then $L(s, \chi\delta^{-1}) = 1$, and we can easily pick f_1, f_2, Φ_1, Φ_2 and Φ such that the first identity from Proposition 7.7 holds, and $Z(s, f_1 \otimes \delta_P^{-1/2}, \Phi_1) = Z(s, f_2, \Phi_2) = 1$.

Remark 7.9. One can verify that $\Phi_1, \Phi_2, \Phi, f_1$ and f_2 can be chosen that Proposition 7.7 holds, and additionally the identity from Remark 7.4 holds. See Remark 7.4. Again, we will need this for an application of Fubini's theorem.

Finally, to close this section we give the example mentioned at the beginning of this section which shows that in solving (4.1) we are not dealing with an intertwining operator. Let $n = k = 1$, so that $G = \mathrm{Sp}(1, F) = \mathrm{Sl}(2, F)$, $G_1 = \mathrm{Gl}(1, F) = F^\times$ and G_2 is absent. Write

$$u(b) = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \qquad \bar{u}(c) = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$$

for $b, c \in F$. A computation shows that for $a \in F^\times$ and $b, c \in F$,

$$i(u(b), 1)i(1, \bar{u}(c))i_1' \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = i_1' \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} i(u(b/a^2), 1)i(1, \bar{u}(a^2c)).$$

Hence, for $\Phi \in \mathrm{I}_H(\chi)$ and $s \in \mathbb{C}$,

$$\Phi(i(u(b), 1)i(1, \bar{u}(c))i_1' \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, s) = \Phi(i(u(b/a^2), 1)i(1, \bar{u}(a^2c)), s).$$

Now if the question asked at the beginning of this section had a positive answer, then integrating the last equation over $U \times \overline{U} = F \times F$ shows that for $\mathrm{Re}(s) > x_0$, $a \in F^\times$ and $\Phi \in \mathrm{I}_H(\chi)$,

$$\begin{aligned} \Phi'(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, s) =& \Phi'(1, s) \\ \alpha_{\mu_1,\mu_2}(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, s)\Phi'(1, s) =& \Phi'(1, s) \\ |a|^2\Phi'(1, s) =& \Phi'(1, s); \end{aligned}$$

since there exist $\Phi \in \mathrm{I}_H(\chi)$ such that $\Phi'(1, s) \neq 0$ for all s, this is a contradiction.

8. Proof of the main technical lemma

We can now give the proof of the main technical lemma. As we mentioned in the outline of the proof in Section 4, in the nonarchimedean case the proof just combines step one (Theorem 6.10) with step two (Proposition 7.7). If $F = \mathbb{R}$, things are a bit more complicated, since step two does not generally produce $K \times K$-finite sections which are required by step one. To get around this difficulty we will use a density argument. In the following proof, if h is a function holomorphic at a point w, then $\mathrm{ord}_w h$ will denote the order of vanishing of h at w.

Proof of Lemma 1.4, the main technical lemma. Suppose F is nonarchimedean. Let Φ_1, Φ_2, Φ, f_1 and f_2 be as in Proposition 7.7. By Proposition 7.7, Remark 7.9, and Fubini's theorem,

$$\int_{G_1 \times G_2 \times U \times \overline{U}} \Phi(i(\bar{u}^{-1} u m(g_1, g_2), 1), s) f_1(g_1) f_2(g_2) \delta_P(m)^{-1/2} d(g_1 g_2 u \bar{u})$$

converges absolutely for $s \in \mathbb{C}$. By Theorem 6.10, there exists a coefficient f of π such that

$$Aq^{Bs} L(s, \chi\delta^{-1}) = Z(s - 1/2, f_1 \otimes \delta_P^{-1/2}, \Phi_1) Z(s - 1/2, f_2, \Phi_2) = Z(s - 1/2, f, \Phi).$$

This proves the main technical lemma in the nonarchimedean case.

Suppose $F = \mathbb{R}$. Again, let Φ_1, Φ_2, Φ, f_1 and f_2 be as in Proposition 7.7. By Proposition 7.7, Remark 7.9, and Fubini's theorem,

$$g(s)\Gamma(s - s_0) = Z(s - 1/2, f_1 \otimes \delta_P^{-1/2}, \Phi_1) Z(s - 1/2, f_2, \Phi_2) = Z(s - 1/2, f_1, f_2, \Phi)$$

for $\mathrm{Re}(s)$ sufficiently large. It follows that $Z(s, f_1, f_2, \Phi)$ has a meromorphic continuation to $\mathbb{C}$, with a simple pole at $s_0' = s_0 - 1/2$. Let (Φ_n) be a sequence of K'-finite elements of $\mathrm{I}_H(\chi)$ converging to Φ. By Theorem 6.7, for each n there exists a coefficient f_n for π such that

$$Z(s, f_1, f_2, \Phi_n) = Z(s, f_n, \Phi_n).$$

In particular, each $Z(s, f_1, f_2, \Phi_n)$ has a meromorphic continuation to $\mathbb{C}$, since each ordinary zeta integral $Z(s, f_n, \Phi_n)$ does. Since every K'-finite χ-section is a finite linear combination of K'-finite standard χ-sections with coefficients which are entire functions, to complete the proof it will suffice to show that for some n, $Z(s, f_1, f_2, \Phi_n)$ has a pole at s_0'. Suppose that each $Z(s, f_1, f_2, \Phi_n)$ is holomorphic at s_0'; we will obtain a contradiction.

Let M be a sufficiently large positive integer so that $\mathrm{Re}(s_0') > \sigma_0 + \sigma_0' - cM$; here the notation is as in Lemma 6.6. Let $p(s)$ and $X(s)$ be as in Corollary 6.9. Consider the sequence of holomorphic functions

$$Z_n(s) = p(s) Z(s, f_1, f_2, \Phi_n) = Z(s, f_1, f_2, X(s)\Phi_n)$$

in the half plane $\mathrm{Re}(s) > \max(\sigma_0 + \sigma_0' - cM, -\rho_{2n})$. We note that s_0' is contained in this half plane since $\mathrm{Re}(s_0) > 0$. Since each $Z(s, f_1, f_2, \Phi_n)$ is holomorphic at s_0',

$$\mathrm{ord}_{s_0'} Z_n(s) \geq \mathrm{ord}_{s_0'} p(s) \text{ for all } n.$$

Also, since (Φ_n) is a Cauchy sequence in $\mathrm{I}_H(\chi)$, so is $(X(s)\Phi_n)$; it follows by the inequality from Lemma 6.6 that the sequence $(Z_n(s))$ is uniformly Cauchy on closed disks in $\mathrm{Re}(s) > \max(\sigma_0 + \sigma_0' - cM, -\rho_{2n})$. Hence, $(Z_n(s))$ converges uniformly on closed disks in $\mathrm{Re}(s) > \max(\sigma_0 + \sigma_0' - cM, -\rho_{2n})$ to a holomorphic function $Z(s)$. By, for example, Roché's theorem,

$$\mathrm{ord}_{s_0'} Z(s) \geq \mathrm{ord}_{s_0'} Z_n(s) \text{ for sufficiently large } n.$$

Thus,

$$\mathrm{ord}_{s_0'} Z(s) \geq \mathrm{ord}_{s_0'} p(s).$$

On the other hand, using $\Phi_n \to \Phi$ and applying the inequality from Lemma 6.6 with $M = 0$ we have

$$\lim_{n\to\infty} Z_n(s) = \lim_{n\to\infty} p(s)Z(s, f_1, f_2, \Phi_n) = p(s)Z(s, f_1, f_2, \Phi)$$

for $\mathrm{Re}(s) > \sigma_0 + \sigma_0'$. This implies that

$$Z(s) = p(s)Z(s, f_1, f_2, \Phi)$$

in $\mathrm{Re}(s) > \sigma_0 + \sigma_0'$, and hence in $\mathrm{Re}(s) > \max(\sigma_0 + \sigma_0' - cM, -\rho_{2n})$ by the identity principle. Now $Z(s, f_1, f_2, \Phi)$ has a pole at s_0', so that

$$\mathrm{ord}_{s_0'} p(s) > \mathrm{ord}_{s_0'} Z(s).$$

We now have $\mathrm{ord}_{s_0'} p(s) > \mathrm{ord}_{s_0'} p(s)$, a contradiction. $\square$.

Remark 8.1. In the nonarchimedean case, the same argument using Remark 7.8 shows that the claim in Remark 1.5 holds.

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Department of Mathematics, University of Idaho, Moscow ID 83844, USA
E-mail address: brooksr@uidaho.edu