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Local Newforms for GSp(4)

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A Summary

The local theory of new- and oldforms for representations of GL(2) is a tool for studying automorphic forms on GL(2) and their applications. This theory singles out, in infinite-dimensional representations, certain vectors which encode information. Thus, this theory lies at the intersection of representation theory, modular forms theory, and applications to number theory. See the work of Casselman [Ca2]; for more information and references, see [Sch1]. The paper [JPSS] generalized some aspects of the theory for GL(2) to GL(n) for generic representations.

This work presents a local theory of new- and oldforms for representations of GSp(4) with trivial central character. This theory resembles the GL(2) theory, but also has some new features. Our theory considers vectors fixed by the paramodular subgroups $\mathrm{K}(\mathfrak{p}^n)$ as defined below. Paramodular groups, their modular forms, and their application to abelian surfaces with polarizations of type $(1, N)$ have been considered for about fifty years. At the same time, the literature perhaps shows a greater emphasis on Siegel modular forms defined with respect to the Siegel congruence subgroup $\Gamma_0(N)$. Nevertheless, in hindsight, it seems clear that the paramodular subgroups are good analogues of the congruence subgroups underlying the new- and oldforms theory for GL(2) and GL(n). In combination with the structure of the discrete spectrum of $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$, the results of this work lead to a satisfactory theory of new- and oldforms for paramodular Siegel modular forms of genus 2. This is discussed in our paper [RS]. We intend to consider this topic again in a later work. This introduction is divided into three parts. The first part briefly reviews the GL(2) theory, the second part summarizes our main results, and the final part delineates the three methods used to prove the main theorems.

Before beginning, we mention some comments that apply to the entire body of this work. First, as far as we know, our theory of new- and oldforms is novel and is unanticipated by the existing framework of general conjectures. Second, as concerns methods and assumptions, this work contains complete proofs of all results, does not depend on any conjectures, and does not use global methods. And finally, this work makes no assumptions about the residual characteristic of the underlying non-archimedean local field.

The GL(2) Theory

The purpose of this work is to demonstrate the existence of a new- and oldforms theory for GSp(4). We begin by recalling the relevant new- and oldforms theory for GL(2), since this is the archetype for our collection of theorems.

First we require some definitions. Let F be a nonarchimedean local field of characteristic zero with ring of integers $\mathfrak{o}$, let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$, and let q be the number of elements of $\mathfrak{o}/\mathfrak{p}$. Fix a generator ϖ for $\mathfrak{p}$. Let ψ be a non-trivial character of F with conductor $\mathfrak{o}$. For each non-negative integer n let $\Gamma_0(\mathfrak{p}^n)$ be the subgroup of k in $\mathrm{GL}(2,F)$ such that $\det(k)$ is in $\mathfrak{o}^\times$ and

$$k \in \begin{bmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o} \end{bmatrix}.$$

The group $\Gamma_0(\mathfrak{p}^n)$ is normalized by the *Atkin–Lehner element* of level $\mathfrak{p}^n$

$$u_n = \begin{bmatrix} & 1 \\ -\varpi^n & \end{bmatrix}.$$

Note that u_n^2 lies in the center of $\mathrm{GL}(2,F)$.

Next, we consider representations. Recall that an irreducible, admissible representation of $\mathrm{GL}(2,F)$ with trivial central character is either generic, in which case it is infinite-dimensional, or non-generic, in which case it is one-dimensional. The theory of new- and oldforms is mainly about generic representations, and we consider them first. However, since our goal is to provide motivation for the case of GSp(4), and since non-generic representations play a much larger role in the local and global representation theory of GSp(4), we will also treat the case of non-generic, i.e., one-dimensional, representations at the end of this section.

Let π be a generic, irreducible, admissible representation of $\mathrm{GL}(2,F)$ with trivial central character. Let $\mathcal{W}(\pi,\psi)$ be the Whittaker model of π with respect to ψ, and let

$$Z(s,W) = \int\limits_{F^\times} W(\begin{bmatrix} a & \\ & 1 \end{bmatrix})|a|^{s-1/2}\,d^\times a$$

be the zeta integral of $W \in \mathcal{W}(\pi,\psi)$. Zeta integrals satisfy a functional equation involving the element u_0, and the theory of zeta integrals assigns to π an L-factor $L(s,\pi)$ and an ε-factor $\varepsilon(s,\pi)$; see Chapter 6 of [G] for a summary. Let W_F' be the Weil–Deligne group of F. If $\varphi : W_F' \to \mathrm{GL}(2,\mathbb{C})$ is the L-parameter of π, then $L(s,\pi) = L(s,\varphi)$ and $\varepsilon(s,\pi) = \varepsilon(s,\varphi)$. For the definitions of $L(s,\varphi)$ and $\varepsilon(s,\varphi)$ see the end of Sect. 2.4. The following is the main theorem about newforms for GL(2) that is relevant for our purposes.

Theorem (GL(2) Generic Newforms Theorem). *Let (π, V) be a generic, irreducible, admissible representation of* $\mathrm{GL}(2,F)$ *with trivial central character. For each non-negative integer n, let $V(n)$ be the subspace of V of vectors W such that $\pi(k)W = W$ for all k in $\Gamma_0(\mathfrak{p}^n)$. Then the following statements hold:*

i) For some n the space $V(n)$ is non-zero.
ii) If N_π is the minimal n such that $V(n)$ is non-zero, then $\dim V(N_\pi) = 1$.
iii) Assume $V = \mathcal{W}(\pi, \psi)$. There exists W_π in $V(N_\pi)$ such that

$$Z(s, W_\pi) = L(s, \pi).$$

If (π, V) is a generic, irreducible, admissible representation of $\mathrm{GL}(2,F)$ with trivial central character, then we call N_π the *level* of π; in some references, N_π is called the conductor of π. Any non-zero element of the one-dimensional space $V(N_\pi)$ is called a *newform*, and the elements of the spaces $V(n)$ for $n > N_\pi$ are called *oldforms*.

A corollary of the GL(2) Generic Newforms Theorem is the computation of the ε-factor of a generic representation. Let π be a generic, irreducible, admissible representation of $\mathrm{GL}(2,F)$ with trivial central character. Since $u_{N_\pi}^2$ lies in the center of $F^\times$, and since the space $V(N_\pi)$ is one-dimensional, any non-zero element of $V(N_\pi)$ is an eigenvector of $\pi(u_{N_\pi})$ with eigenvalue $\varepsilon_\pi = \pm 1$. As a consequence of the functional equation for zeta integrals and the GL(2) Newforms Theorem we obtain the following corollary.

Corollary (ε-factors of Generic GL(2) Representations). *Let (π, V) be a generic, irreducible, admissible representation of* $\mathrm{GL}(2,F)$ *with trivial central character. Then $\varepsilon(s,\pi) = \varepsilon_\pi q^{-N_\pi(s-1/2)}$.*

This result computes the ε-factor of a generic representation in terms of invariants of a newform that make no reference to a specific kind of model: can $L(s,\pi)$ also be computed in this way? This is possible using a Hecke operator. Let n be a non-negative integer, and let $\mathcal{H}(\Gamma_0(\mathfrak{p}^n))$ be the Hecke algebra of $\Gamma_0(\mathfrak{p}^n)$, i.e., the vector space of left and right $\Gamma_0(\mathfrak{p}^n)$-invariant, compactly supported functions on $\mathrm{GL}(2,F)$ with product given by convolution. Let (π, V) be a smooth representation of $\mathrm{GL}(2,F)$ with trivial central character; we do not assume V is the Whittaker model of π. Then $\mathcal{H}(\Gamma_0(\mathfrak{p}^n))$ acts on $V(n)$ by

$$\pi(f)v = \int\limits_{\mathrm{GL}(2,F)} f(g)\pi(g)v\,dg,$$

where the Haar measure on $\mathrm{GL}(2,F)$ assigns $\Gamma_0(\mathfrak{p}^n)$ volume 1. We will use the operator $\pi(f)$ on $V(n)$ corresponding to the characteristic function f of

$$\Gamma_0(\mathfrak{p}^n)\begin{bmatrix}\varpi & \\ & 1\end{bmatrix}\Gamma_0(\mathfrak{p}^n).$$

We will write $T_1 = \pi(f)$.

Theorem (GL(2) Hecke Eigenvalues and L-functions). *Let (π, V) be a generic, irreducible, admissible representation of* $\mathrm{GL}(2,F)$ *with trivial central character, and let $W \in V(N_\pi)$ be a newform, i.e., a non-zero element of the one-dimensional space $V(N_\pi)$. Then W is an eigenform for T_1; let*

$$T_1 W = \lambda_\pi W.$$

i) Assume $N_\pi = 0$, so that π is unramified. Then

$$L(s,\pi) = \frac{1}{1 - \lambda_\pi q^{-1/2} q^{-s} + q^{-2s}}.$$

ii) Assume $N_\pi = 1$. Then

$$L(s,\pi) = \frac{1}{1 - \lambda_\pi q^{-1/2} q^{-s}}.$$

iii) Assume $N_\pi \geq 2$. Then $\lambda_\pi = 0$, and $L(s,\pi) = 1$.

The last result of the theory for generic representations asserts that vectors in the spaces $V(n)$ for $n > N_\pi$ are obtained by repeatedly applying two level raising operators to a newform and taking linear combinations. For n a non-negative integer, define $\beta' : V(n) \to V(n+1)$ by $\beta'(v) = v$. Also, define $\beta : V(n) \to V(n+1)$ to be the Atkin–Lehner conjugate of β', i.e., define $\beta = \pi(u_{n+1}) \circ \beta' \circ \pi(u_n)$, so that

$$\beta = \pi(\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix}).$$

Theorem (GL(2) Oldforms Theorem). *Let (π, V) be a generic, irreducible, admissible representation of* $\mathrm{GL}(2,F)$ *with trivial central character. Then, for any integer $n \geq N_\pi$,*

$$\dim V(n) = n - N_\pi + 1.$$

If $W \in V(N_\pi)$ is non-zero, then the space $V(n)$ for $n \geq N_\pi$ is spanned by the linearly independent vectors

$$\beta'^i \beta^j W, \quad i,j \geq 0,\ i + j = n - N_\pi.$$

In particular, all oldforms can be obtained by applying level raising operators to the newform and taking linear combinations.

Finally, similar results hold for non-generic representations which admit non-zero vectors fixed by some $\Gamma_0(\mathfrak{p}^n)$. Again, any non-generic, irreducible, admissible representations of $\mathrm{GL}(2,F)$ with trivial central character is one-dimensional, and is thus of the form $\alpha \circ \det$ for some character α of $F^\times$. Let $\pi = \alpha \circ \det$, where α is a character of $F^\times$. Then π admits a non-zero vector fixed by $\Gamma_0(\mathfrak{p}^n)$ for some non-negative integer n if and only if α is unramified. Assume α is unramified. Obviously, $V(n)$ is non-zero and one-dimensional for all non-negative integers n. The quantities N_π, ε_π and λ_π from above are all defined since they are model-independent. We have

$$N_\pi = 0, \quad \varepsilon_\pi = 1, \quad \lambda_\pi = (q+1)\alpha(\varpi).$$

Though the theory of zeta integrals for generic representations does not apply, the Langlands correspondence assigns ε-factors and L-factors to all irreducible, admissible representations of $\mathrm{GL}(2,F)$. These assignments coincide with the assignments made by the theory of zeta integrals for generic representations. If $\varphi_\pi : W_F' \to \mathrm{GL}(2,\mathbb{C})$ is the L-parameter assigned to π, then a computation shows that $\varepsilon(s,\varphi_\pi)$ and $L(s,\varphi_\pi)$ can be expressed by exactly the same formulas as in the generic setting:

$$\varepsilon(s,\varphi_\pi) = \varepsilon_\pi q^{-N_\pi(s-1/2)}, \quad L(s,\varphi_\pi) = \frac{1}{1-\lambda_\pi q^{-1/2}q^{-s}+q^{-2s}}.$$

It is trivial that the elements of $V(n)$ for $n \geq N_\pi = 0$ are obtained from a newform by applying the level raising operator β'. Though it is obvious, we note also that, in contrast to the case of generic representations, β and β' do not produce linearly independent vectors.

Main Results

In analogy to the GL(2) theory, this work considers vectors in irreducible, admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character that are fixed by the paramodular groups $\mathrm{K}(\mathfrak{p}^n)$, as defined below. Such vectors are called paramodular, as are representations which admit non-zero paramodular vectors. Briefly summarized, our work has three main results. First, a theory of new- and oldforms exists for generic representations of GSp(4) with trivial central character, and this theory strongly resembles the GL(2) theory described above. In particular, all generic representations with trivial central character are paramodular. Second, the two essential aspects of the generic theory also hold for arbitrary paramodular representations π of GSp(4): there is uniqueness at the minimal paramodular level, and all oldforms are obtained from a newform by applying certain level raising operators and taking linear combinations. Third, newforms in paramodular representations encode important canonical information. If the language of the conjectural Langlands correspondence is used, then our results, which do not depend on or use any conjectures, indicate that a newform in a paramodular representation π determines the ε_π-factor $\varepsilon(s,\varphi_\pi)$ and the L-factor $L(s,\varphi_\pi)$ of the L-parameter φ_π of π. In this section we will discuss the main results in the order mentioned, beginning with the theorems about generic representations. Readers desiring to see additional data should consult the tables in the appendix. These tables explicitly describe important objects and quantities for each irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. The basis for these tables is the Sally-Tadić classification [ST] of non-supercuspidal, irreducible, admissible representations of $\mathrm{GSp}(4,F)$ in the form of Table A.1. Methods and proofs will be discussed in the next section.

First we need some general definitions. Throughout this work, $\mathrm{GSp}(4,F)$ is the group of g in $\mathrm{GL}(4,F)$ such that ${}^tgJg=\lambda J$ for some λ in $F^\times$, where

$$J=\begin{bmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{bmatrix}.$$

The element λ is unique, and will be denoted by $\lambda(g)$. If $n\geq 0$ is a non-negative integer, then the *paramodular group* $\mathrm{K}(\mathfrak{p}^n)$ of level $\mathfrak{p}^n$ is the subgroup of $k\in\mathrm{GSp}(4,F)$ such that $\lambda(k)$ is in $\mathfrak{o}^\times$ and

$$k\in\begin{bmatrix} \mathfrak{o}&\mathfrak{o}&\mathfrak{o}&\mathfrak{p}^{-n}\\ \mathfrak{p}^n&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\ \mathfrak{p}^n&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\ \mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{o} \end{bmatrix}.$$

The first paramodular group $\mathrm{K}(\mathfrak{p}^0)$ is just $\mathrm{GSp}(4,\mathfrak{o})$, a maximal compact, open subgroup of $\mathrm{GSp}(4,F)$. The second paramodular group $\mathrm{K}(\mathfrak{p}^1)$ is the other maximal compact, open subgroup of $\mathrm{GSp}(4,F)$, up to conjugacy. Note that, in contrast to the case of the Hecke subgroups in $\mathrm{GL}(2,F)$, $\mathrm{K}(\mathfrak{p}^n)$ is not contained in $\mathrm{K}(\mathfrak{p}^m)$ for any pair of distinct non-negative integers n and m. The paramodular group $\mathrm{K}(\mathfrak{p}^n)$ is normalized by the *Atkin–Lehner element*

$$u_n=\begin{bmatrix} &&1&\\ &&&-1\\ \varpi^n&&&\\ &-\varpi^n&& \end{bmatrix}.$$

Suppose that (π,V) is an irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. If n is a non-negative integer, then we define $V(n)$ to be the subspace of vectors v in V such that $\pi(k)v=v$ for $k\in\mathrm{K}(\mathfrak{p}^n)$. The elements of $V(n)$ are called *paramodular vectors*. We say that π is *paramodular* if $V(n)\neq 0$ for some n. If π is paramodular, then we define N_π to be the minimal n such that $V(n)$ is non-zero, and we call N_π the *paramodular level* of π.

Generic Representations. Now we will discuss our results for generic representations. Let (π,V) be a generic, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. Again, let ψ be a non-trivial character of F with conductor $\mathfrak{o}$, fix $c_1,c_2\in\mathfrak{o}^\times$, and define the Whittaker model $\mathcal{W}(\pi,\psi_{c_1,c_2})$ of π with respect to a certain character ψ_{c_1,c_2} of the upper-triangular subgroup of $\mathrm{GSp}(4,F)$ as in Section 2.1. A theory of zeta integrals, introduced by Novodvorsky [N], exists for generic representations of $\mathrm{GSp}(4,F)$. If $W\in\mathcal{W}(\pi,\psi_{c_1,c_2})$, then the zeta integral of W is

$$Z(s,W)=\int\limits_{F^\times}\int\limits_{F}W(\begin{bmatrix}a&&&\\&a&&\\&x&1&\\&&&1\end{bmatrix})|a|^{s-3/2}\,dx\,d^\times a.$$

See Section 2.6 for more theory and references. As regards basic facts, this theory of zeta integrals is similar to the theory of zeta integrals for generic representations of $\mathrm{GL}(2,F)$. In particular, zeta integrals satisfy a functional equation involving u_0, and the theory associates to π an L-factor $L(s,\pi)$ and an ε-factor $\varepsilon(s,\pi)$. The work [Tak] computed the factors $L(s,\pi)$ for all generic, irreducible, admissible representations π of $\mathrm{GSp}(4,F)$. The factor $L(s,\pi)$ is sometimes called the spin L-function of π, and is of the form $1/Q(q^{-s})$, where $Q(X)\in\mathbb{C}[X]$ is a polynomial of at most degree four such that $Q(0)=1$. If the conjectural Langlands correspondence for $\mathrm{GSp}(4,F)$ exists, and if $\varphi_\pi : W_F'\to\mathrm{GSp}(4,\mathbb{C})$ is the L-parameter of π according to this correspondence, then it is conjectured that $L(s,\pi)=L(s,\varphi_\pi)$ and $\varepsilon(s,\pi)=\varepsilon(s,\varphi_\pi)$. Here, φ_π is regarded as a four-dimensional representation of the Weil–Deligne group W_F'; we have $\varepsilon(s,\varphi_\pi)=\varepsilon_{\varphi_\pi}q^{-a(\varphi_\pi)(s-1/2)}$, where $\varepsilon_{\varphi_\pi}=\pm1$, and $a(\varphi_\pi)$ is a non-negative integer. We call $a(\varphi_\pi)$ the *conductor* of φ_π. The following theorem is an analogue of the corresponding GL(2) result described above.

Theorem 7.5.4 (Generic Main Theorem). *Let (π,V) be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character. Then the following statements hold:*

i) There exists an n such that $V(n)\neq0$, i.e., π is paramodular.
ii) If N_π is the minimal n such that $V(n)\neq0$, then $\dim V(N_\pi)=1$.
iii) Assume $V=\mathcal{W}(\pi,\psi_{c_1,c_2})$. There exists $W_\pi\in V(N_\pi)$ such that

$$Z(s,W_\pi)=L(s,\pi).$$

One immediate consequence of this theorem is that paramodular representations exist and include generic representations. If π is a generic, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character, then we call the non-zero elements of $V(N_\pi)$ *newforms*; the above theorem asserts that a newform for π is essentially unique. The elements of $V(n)$ for $n>N_\pi$ are called *oldforms*.

Just as for GL(2), if π is a generic, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character, then the ε-factor and L-factor of π can be computed in terms of universal invariants of a newform, i.e., invariants of a newform that do not depend on a specific model for π. These formulas for $\varepsilon(s,\pi)$ and $L(s,\pi)$ involve the level N_π, the Atkin–Lehner eigenvalue of a newform, and the Hecke eigenvalues of a newform. The formula for $\varepsilon(s,\pi)$, and its derivation from Theorem 7.5.4, are identical to those of the GL(2) theory.

Corollary 7.5.5 (ε-factors of Generic Representations). *Let (π, V) be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4, F)$ *with trivial central character. Let N_π be the paramodular level of π as in Theorem 7.5.4, and let ε_π be the eigenvalue of the Atkin–Lehner involution $\pi(u_{N_\pi})$ on the one-dimensional space $V(N_\pi)$. Then*

$$\varepsilon(s, \pi) = \varepsilon_\pi q^{-N_\pi(s-1/2)}.$$

The formula for $L(s, \pi)$ in terms of model-independent invariants of a newform requires two Hecke operators. Let n be a non-negative integer, and let $\mathcal{H}(\mathrm{K}(\mathfrak{p}^n))$ be the Hecke algebra of $\mathrm{K}(\mathfrak{p}^n)$, i.e., the vector space of left and right $\mathrm{K}(\mathfrak{p}^n)$-invariant, compactly supported functions on $\mathrm{GSp}(4, F)$ with product given by convolution. Suppose that (π, V) is a smooth representation of $\mathrm{GSp}(4, F)$ with trivial central character; no assumption is made about V. Then $\mathcal{H}(\mathrm{K}(\mathfrak{p}^n))$ acts on $V(n)$ via the formula

$$\pi(f)v = \int\limits_{\mathrm{GSp}(4,F)} f(g)\pi(g)v\,dg.$$

Here the Haar measure on $\mathrm{GSp}(4, F)$ gives $\mathrm{K}(\mathfrak{p}^n)$ volume 1. We will use the operators on $V(n)$ induced by the characteristic functions of

$$\mathrm{K}(\mathfrak{p}^n)\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix}\mathrm{K}(\mathfrak{p}^n) \quad \text{and} \quad \mathrm{K}(\mathfrak{p}^n)\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&\varpi&\\&&&1\end{bmatrix}\mathrm{K}(\mathfrak{p}^n).$$

These operators will be called $T_{0,1}$ and $T_{1,0}$, respectively. Motivation for the consideration of these Hecke operators is provided below in the section on methods and proofs.

Theorem 7.5.3 (Hecke Eigenvalues and L-functions). *Let (π, V) be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4, F)$ *with trivial central character. Let W be a newform of π, i.e., a non-zero element of the one-dimensional space $V(N_\pi)$. Let*

$$T_{0,1}W = \lambda_\pi W, \quad T_{1,0}W = \mu_\pi W,$$

where λ_π and μ_π are complex numbers.

i) Assume $N_\pi = 0$, so that π is unramified. Then

$$L(s, \pi) = \frac{1}{1 - q^{-3/2}\lambda_\pi q^{-s} + (q^{-2}\mu_\pi + 1 + q^{-2})q^{-2s} - q^{-3/2}\lambda_\pi q^{-3s} + q^{-4s}}.$$

ii) Assume $N_\pi = 1$, and let $\pi(u_1)W = \varepsilon_\pi W$, where $\varepsilon_\pi = \pm 1$ is the Atkin–Lehner eigenvalue of W. Then

$$L(s, \pi) = \frac{1}{1 - q^{-3/2}(\lambda_\pi + \varepsilon_\pi)q^{-s} + (q^{-2}\mu_\pi + 1)q^{-2s} + \varepsilon_\pi q^{-1/2}q^{-3s}}.$$

iii) Assume $N_\pi \geq 2$. Then

$$L(s,\pi) = \frac{1}{1 - q^{-3/2}\lambda_\pi q^{-s} + (q^{-2}\mu_\pi + 1)q^{-2s}}.$$

This theorem exhibits two new phenomena not present in the GL(2) theory. First, when $N_\pi = 1$, the formula for $L(s,\pi)$ involves not just a Hecke eigenvalue, but also the Atkin–Lehner eigenvalue ε_π. Second, in contrast to the GL(2) theory, it is not true that $L(s,\pi) = 1$ if N_π is sufficiently large. There are examples of π such that N_π is arbitrarily large and $\mu_\pi = 0$; for such π we have $L(s,\pi) \neq 1$ by iii) of Theorem 7.5.3.

Oldforms in generic representations of $\mathrm{GSp}(4,F)$ also exhibit a new phenomenon. Just as in the GL(2) case, oldforms for GSp(4) are obtained from a newform via level raising operators; however, the GSp(4) case requires an extra operator, and the spaces of oldforms have an additional summand. Let (π, V) be a smooth representation of $\mathrm{GSp}(4,F)$ with trivial central character. The first two level raising operators, called θ' and θ, are analogues of the GL(2) operators β' and β. The operator $\theta' : V(n) \to V(n+1)$ is the natural trace operator, and is the analogue of the GL(2) level raising operator β'. The operator $\theta : V(n) \to V(n+1)$ is the Atkin–Lehner conjugate of θ', and is thus defined by $\theta = \pi(u_{n+1}) \circ \theta' \circ \pi(u_n)$. This operator is the analogue of β. The third operator, $\eta : V(n) \to V(n+2)$, skips one level and does not have a GL(2) analogue. It is defined by

$$\eta = \pi(\begin{bmatrix} \varpi^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}).$$

Theorem 7.5.6 (Generic Oldforms Theorem). *Let (π, V) be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character. Let N_π be the paramodular level of π and let W_π be the newform as in Theorem 7.5.4. Then, for any integer $n \geq N_\pi$,*

$$\dim V(n) = \left[\frac{(n - N_\pi + 2)^2}{4}\right].$$

For $n \geq N_\pi$, the space $V(n)$ is spanned by the linearly independent vectors

$$\theta'^i \theta^j \eta^k W_\pi, \quad i,j,k \geq 0,\ i + j + 2k = n - N_\pi.$$

In particular, all oldforms are obtained by applying level raising operators to the newform and taking linear combinations.

An alternative formulation of this theorem exposes the similarities and differences between oldforms for GL(2) and oldforms in generic representations of $\mathrm{GSp}(4,F)$. Theorem 7.5.6 is equivalent to the statement that for $n \geq N_\pi$

the space $V(n)$ is the direct sum of the subspace spanned by the linearly independent vectors

$$\theta'^i \theta^j W_\pi, \quad i, j \geq 0, i + j = N_\pi - n,$$

and the subspace $\eta V(n-2)$, so that

$$\dim V(n) = n - N_\pi + 1 + \dim V(n-2).$$

Stated this way, we see that oldforms in generic representations of $\mathrm{GSp}(4, F)$ have a structure similar to the structure of oldforms in generic representations of $\mathrm{GL}(2)$, with one difference: in the case of $\mathrm{GSp}(4)$, the space $V(n-2)$ also contributes to $V(n)$ via η. The subspace $\eta V(n-2)$ can be characterized as the subspace of W in $V(n)$ such that $Z(s, W) = 0$. We call this characterization the *η Principle*, and discuss it in the next section. Vectors W in $V(n)$ such that $Z(s, W) = 0$ are *degenerate*. Degenerate vectors do not exist in the $\mathrm{GL}(2)$ theory, and are a new phenomenon for $\mathrm{GSp}(4)$.

Arbitrary Representations. This work also treats arbitrary paramodular, irreducible, admissible representations of $\mathrm{GSp}(4, F)$ with trivial central character. We prove that the two basic principles of the generic theory hold for arbitrary paramodular representations. These principles are essential for global applications. First of all, there is uniqueness at the minimal paramodular level:

Theorem 7.5.1 (Uniqueness at Minimal Level). *Let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character. Assume that π is paramodular, and let N_π be the minimal paramodular level. Then $\dim V(N_\pi) = 1$.*

If π is a paramodular, irreducible, admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character, then we call the non-zero elements of $V(N_\pi)$ *newforms*; the theorem asserts that newforms in paramodular representations are essentially unique. The elements of $V(n)$ for $n > N_\pi$ are called *oldforms*. Global applications will require the following theorem. This second basic principle asserts that oldforms are obtained from a newform by applying level raising operators:

Theorem 7.5.7 (Oldforms Principle). *Let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character. Assume that π is paramodular. If v is a non-zero element of the one-dimensional space $V(N_\pi)$ and $n \geq N_\pi$, then the space $V(n)$ is spanned by the (not necessarily linearly independent) vectors*

$$\theta'^i \theta^j \eta^k v, \quad i, j, k \geq 0, \ i + j + 2k = n - N_\pi.$$

In other words, all oldforms can be obtained from the newform v by applying level raising operators and taking linear combinations.

In fact, we have determined a basis for $V(n)$ among the spanning set of vectors $\theta'^i\theta^j\eta^k W$ for all paramodular representations and all n. By Theorem 7.5.6, if π is generic, then the spanning vectors $\theta'^i\theta^j\eta^k W$, where $i,j,k \geq 0$ and $i+j+2k = n-N_\pi$, form a basis, and the dimension of $V(n)$ is $[(n-N_\pi+2)^2/4]$. This characterizes generic representations: a representation is generic if and only if the representation is paramodular and $\dim V(n) = [(n-N_\pi+2)^2/4]$ for $n \geq N_\pi$. The bases for $V(n)$ for non-generic, paramodular representations also follow general schemes. There are four patterns for non-generic, paramodular representations. First, it can happen that the vectors $\theta'^i\eta^k W$ where $i,k \geq 0$ and $i+2k = n-N_\pi$ form a basis for $V(n)$, so that $\dim V(n) = [(n-N_\pi+2)/2]$ for $n \geq N_\pi$. This occurs if and only if π is paramodular and of type IIb, IVb, Vb, Vc, VIc, VId or XIb. The second possibility is that the vectors $\theta'^i\theta^j W$ where $i,j \geq 0$ and $i+j = n-N_\pi$ form a basis for $V(n)$, and hence $\dim V(n) = n - N_\pi + 1$ for $n \geq N_\pi$. This happens if and only if π is paramodular and of type IIIb or IVc. Third, the vectors $\eta^k W$ where $k \geq 0$ and $2k = n - N_\pi$ form a basis for $V(n)$, so that $\dim V(n) = (1+(-1)^n)/2$ for $n \geq N_\pi$. This occurs if and only if π is paramodular and of type Vd. Finally, it can happen that the vectors $\theta'^i W$ where $i = n - N_\pi$ form a basis for $V(n)$, and thus $\dim V(n) = 1$ for $n \geq N_\pi$. This last possibility happens exactly for quadratic unramified twists of the trivial representation, i.e., π is paramodular and of type IVd. See Table A.12 for the dimensions of the spaces $V(n)$ for all irreducible, admissible representations of $\mathrm{GSp}(4,F)$ with trivial character.

Information Carried by a Newform. Finally, our results show that a newform in a paramodular representation carries important canonical information. Let π be a paramodular, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. We saw above that if π is generic, then $\varepsilon(s,\pi)$ and $L(s,\pi)$ can be expressed in terms of the model-independent invariants N_π, ε_π, λ_π and μ_π. Thus, if π is generic, then a newform for π contains all the information present in $\varepsilon(s,\pi)$ and $L(s,\pi)$. Next, assume that π is non-generic. Then the theory of zeta integrals is not available, but based on the generic case it is natural to conjecture the following: if φ_π is the conjectural L-parameter of π, then $\varepsilon(s,\varphi_\pi)$ and $L(s,\varphi_\pi)$ can be expressed in terms of N_π , ε_π, λ_π and μ_π via the same formulas in Corollary 7.5.5 and Theorem 7.5.3. Of course, verifying this conjecture requires knowing φ_π; this appears to be a problem since the Langlands correspondence for $\mathrm{GSp}(4,F)$ is conjectural, so that the L-parameters of general representations are not known. However, it turns out that the desiderata of the conjectural Langlands correspondence, in combination with the classification of induced representations from [ST], do determine the L-parameters of some representations of $\mathrm{GSp}(4,F)$, namely those that are non-supercuspidal. The following theorem implies that any non-generic, paramodular representation is of this type, and is even non-tempered.

Theorem 7.5.8 (Tempered Representations). *Let π be an irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character. Assume π is tempered. Then π is paramodular if and only if π is generic.*

The next theorem shows that the conjecture of the last paragraph is true when π is non-generic; in fact, the conjecture is true for all paramodular π that are non-supercuspidal.

Theorem 7.5.9 (Non-supercuspidal Newforms and L- and ε-factors). *Let (π,V) be a paramodular, non-supercuspidal, irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character. Let $\varphi_\pi : W_F' \to$* $\mathrm{GSp}(4,\mathbb{C})$ *be the L-parameter assigned to π as in Sect. 2.4. Let N_π be the minimal paramodular level of π, and let $v \in V(N_\pi)$ be a non-zero vector. Let ε_π be the Atkin–Lehner eigenvalue of v, and let λ_π and μ_π be the Hecke eigenvalues of v, defined by $T_{0,1}v = \lambda_\pi v$ and $T_{1,0}v = \mu_\pi v$. Then*

$$\varepsilon(s,\varphi_\pi) = \varepsilon_\pi q^{-N_\pi(s-1/2)}.$$

i) Assume $N_\pi = 0$, so that π is unramified. Then

$$L(s,\varphi_\pi)=\frac{1}{1-q^{-3/2}\lambda_\pi q^{-s}+(q^{-2}\mu_\pi+1+q^{-2})q^{-2s}-q^{-3/2}\lambda_\pi q^{-3s}+q^{-4s}}.$$

ii) Assume $N_\pi = 1$. Then

$$L(s,\varphi_\pi) = \frac{1}{1-q^{-3/2}(\lambda_\pi+\varepsilon_\pi)q^{-s}+(q^{-2}\mu_\pi+1)q^{-2s}+\varepsilon_\pi q^{-1/2}q^{-3s}}.$$

iii) Assume $N_\pi \geq 2$. Then

$$L(s,\varphi_\pi) = \frac{1}{1-q^{-3/2}\lambda_\pi q^{-s}+(q^{-2}\mu_\pi+1)q^{-2s}}.$$

To close this section we remark that if the conjectural Langlands correspondence for $\mathrm{GSp}(4,F)$ has the expected properties, then paramodular representations are well behaved with respect to L-packets. An examination of the L-parameters for $\mathrm{GSp}(4,F)$ along with the desiderata of the Langlands correspondence conjecture show that the L-packets with more than one element should be tempered. It is conjectured that a tempered L-packet contains a unique generic element. Therefore, by Theorem 7.5.8, the following is true if the Langlands correspondence for $\mathrm{GSp}(4,F)$ has the expected properties: any L-packet for $\mathrm{GSp}(4,F)$ contains at most one paramodular representation.

Methods and Proofs

Having described the main results of this work, we will now give an overview of the proofs. A chief feature of the arguments in this work is the use of three methods: P_3-theory, double coset decompositions, and Hecke operators. Below, we will describe each method and its consequences; these methods are implemented in the same sequence in the body of this work. In addition, two other tools are used throughout this work to study paramodular vectors. The first is the use of the level raising operators θ', θ and η, and more generally, level changing operators. The second tool is zeta integrals. Thus, in this work level raising operators and zeta integrals play roles beyond their original purposes of accounting for oldforms and defining L-functions and ε-factors, respectively. Finally, there is another organizational element that should be kept in mind while reading this exposition. This is the partition of the irreducible, admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character into certain subclasses. It is possible to partition the representations of $\mathrm{GSp}(4,F)$ in two important ways: first, a representation is generic or non-generic; second, a representation is non-supercuspidal or supercuspidal. Thus, one obtains four subclasses:

Generic non-supercuspidal representations	Generic supercuspidal representations
Non-generic non-supercuspidal representations	Non-generic supercuspidal representations

Fig. 1.1. Partition of the representations of $\mathrm{GSp}(4,F)$.

All non-supercuspidal representations of $\mathrm{GSp}(4,F)$ have been classified. See [ST] and Table A.1. We will often use this classification of non-supercuspidal representations; however, we do not require any explicit constructions of supercuspidal representations of $\mathrm{GSp}(4,F)$.

P_3-theory. In this work we use a Kirillov-type theory for $\mathrm{GSp}(4)$ to prove key results about paramodular vectors. The basic idea is to map an irreducible, admissible representation (π, V) of $\mathrm{GSp}(4,F)$ with trivial central character to a certain associated smooth representation of the subgroup

$$P_3 = \begin{bmatrix} * & * & * \\ * & * & * \\ & & 1 \end{bmatrix}$$

of $\mathrm{GL}(3,F)$. Results about paramodular vectors in V are then obtained from knowledge about the associated P_3 representation. We call this method P_3-theory, and it is analogous to the familiar $\mathrm{GL}(n)$ technique. In the setting of $\mathrm{GL}(n)$, every irreducible, admissible representation τ of $\mathrm{GL}(n,F)$ defines an associated smooth representation $\tau|_{P_n}$ of the analogous subgroup P_n of $\mathrm{GL}(n,F)$ by restriction. The representation $\tau|_{P_n}$ is of finite length, and this association plays an important role in the representation theory of $\mathrm{GL}(n,F)$.

For example, it can be used to prove the existence of Kirillov models for generic, irreducible, admissible representations of $\mathrm{GL}(n,F)$. See [BZ]. As we shall see in a moment, for GSp(4) the definition of the associated representation, and its relationship to the original representation, are more complicated than restriction. Nevertheless, P_3-theory is a powerful technique, and we make two important applications to the study of paramodular vectors that form a basis for the rest of this work. First, we use P_3-theory to prove that generic, irreducible, admissible representations of $\mathrm{GSp}(4,F)$ are paramodular; in the other direction, we use P_3-theory to prove that certain families of non-generic representations are non-paramodular. Second, we use P_3-theory in an essential way to prove the η Principle. The η Principle characterizes degenerate paramodular vectors in generic representations, that is, paramodular vectors W such that $Z(s,W)$ vanishes. As such, the η Principle removes an obstacle to the productive use of zeta integrals in the study of paramodular vectors. An immediate corollary of these results is a lower bound on the dimension of the space of paramodular vectors of level $\mathfrak{p}^n$ in a generic representation; later on we prove that this lower bound is actually the dimension of the space of paramodular vectors of level $\mathfrak{p}^n$. We describe these applications below; the double coset and Hecke operators methods that we subsequently delineate will build on these results. We were inspired to use P_3-theory for GSp(4) by the global Lemma 6.2 of [PS]. This lemma provides the key homomorphism described below; see also [GPSR], Part B, Chapter II, for an example of the use of P_n-theory in the study of representations of $\mathrm{SO}(2n+1)$.

In the $\mathrm{GSp}(4,F)$ context, the starting point for the definition of the associated P_3 representation is the observation that there is a natural isomorphism

$$P_3 \xrightarrow{\sim} Q/ZZ^J,$$

where Z is the center of $\mathrm{GSp}(4,F)$, Q is the Klingen parabolic subgroup of $\mathrm{GSp}(4,F)$, and Z^J is the normal subgroup of Q defined as

$$Q = \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & * \end{bmatrix}, \quad Z^J = \begin{bmatrix} 1 & & & * \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. Then, via this isomorphism, the space V_{Z^J} is a smooth representation of P_3; here, V_{Z^J} is the quotient of V by the subspace $V(Z^J)$ consisting of all $\mathbb{C}$ linear combinations of vectors of the form $v - \pi(z)v$ for v in V and z in Z^J. The P_3 representation V_{Z^J} has finite length. The irreducible, smooth representations of P_3 are classified in [BZ], and any non-zero, irreducible, smooth representation of P_3 is obtained from a non-zero irreducible, admissible representations of $\mathrm{GL}(0,F) = 1$, $\mathrm{GL}(1,F)$ or $\mathrm{GL}(2,F)$ via an induction process. We denote them by

$$\tau(\mathbf{1}) = \tau_{\mathrm{GL}(0)}^{P_3}(\mathbf{1}), \quad \tau(\chi) = \tau_{\mathrm{GL}(1)}^{P_3}(\chi), \quad \tau(\rho) = \tau_{\mathrm{GL}(2)}^{P_3}(\rho).$$

Here, $\mathbf{1}$ is the non-trivial irreducible, admissible representation of $\mathrm{GL}(0,F) = 1$, i.e., the trivial representation, χ is a character of $F^\times$, and ρ is a non-zero, irreducible, admissible representation of $\mathrm{GL}(2,F)$. The representation V_{Z^J} thus has a filtration of finite length whose irreducible subquotients are of the above form. In fact, there exists a sequence of P_3 subspaces of V_{Z^J}

$$0 \subset V_2 \subset V_1 \subset V_0 = V_{Z^J}$$

such that V_2 is either 0 or $\tau(\mathbf{1})$, the non-zero irreducible subquotients of V_1/V_2, if they exist, are of the form $\tau(\chi)$, and the non-zero irreducible subquotients of V_0/V_1, if they exist, are of the form $\tau(\rho)$. The representation π is generic if and only if V_2 is non-zero, and π is supercuspidal if and only if $V_{Z^J} = V_2$. Thus, π is generic and supercuspidal if and only if V_{Z^J} is non-zero and $V_{Z^J} \cong \tau(\mathbf{1})$, and π is non-generic and supercuspidal if and only if $V_{Z^J} = 0$. In addition, the structure of $L(s,\pi)$ is reflected in the filtration of V_1/V_2 by irreducible subquotients when π is generic. More precisely, there is a bijection between the poles of $L(s,\pi)$ and those irreducible subquotients of V_1/V_2 of the form $\tau(\chi)$, where χ is an unramified character of $F^\times$. Each pole of $L(s,\pi)$, counted for multiplicity, induces a certain linear functional on V_{Z^J}, and each irreducible subquotient of V_1/V_2 of the form $\tau(\chi)$ with χ unramified also induces a particular linear functional on V_{Z^J}: the linear functionals associated to corresponding poles and irreducible subquotients coincide. For more, see Section 4.2. Finally, the filtrations of all irreducible, admissible representations π of $\mathrm{GSp}(4,F)$ with trivial central character can be computed, and appear in Tables A.5 and A.6.

With these facts in place, our first application of P_3-theory to paramodular vectors concerns non-existence and existence. As the next proposition shows, if (π, V) is an irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character, then the associated representation V_{Z^J} sees all of the paramodular vectors in V. This is the initial indication that P_3-theory might be useful in the study of paramodular vectors. The proof of this proposition uses the fact that non-zero paramodular vectors of distinct levels are linearly independent; in the statement V_{para} is the vector space spanned by all paramodular vectors at all levels.

Proposition 3.4.2. *Let (π, V) be a smooth representation of $\mathrm{GSp}(4,F)$ such that the center of $\mathrm{GSp}(4,F)$ acts trivially. Assume that the subspace of vectors of V fixed by $\mathrm{Sp}(4,F)$ is trivial. Let $p : V \to V_{Z^J}$ be the projection map. If $v \in V(n)$ for some non-negative integer $n \geq 0$ and $p(v) = 0$, then $v = 0$. More generally, if $v \in V_{\mathrm{para}}$ and $p(v) = 0$, then v is a linear combination of vectors of the form*

$$qw - \theta' w + \eta w,$$

where $W \in V(m)$ for some non-negative integer $m \geq 0$.

We use this proposition to prove that certain families of non-generic representations are not paramodular. Let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. If V contains a non-zero paramodular vector v, then it follows from Proposition 3.4.2 that some non-zero irreducible subquotient of the P_3 representation V_{Z^J} contains a non-zero vector invariant under the group $P_3(\mathfrak{o})$. As we mentioned above, there are three types of non-zero irreducible, smooth representations of P_3, and such a representation admits a non-zero vector invariant under $P_3(\mathfrak{o})$ if and only if the representation is of the form $\tau(\gamma)$, where γ is an unramified, irreducible, admissible representation of $\mathrm{GL}(0,F)=1$, $\mathrm{GL}(1,F)$ or $\mathrm{GL}(2,F)$. Thus, if none of the non-zero irreducible subquotients of V_{Z_J} are of this form, then π is not paramodular. By this, we deduce that if π is non-generic and supercuspidal, then π is non-paramodular. Using Tables A.5 and A.6, which list the P_3-filtrations of all irreducible admissible representations, we also conclude that a number of other families of non-generic representations are also not paramodular: see Theorem 3.4.3. By later arguments, the π that can be proven to be non-paramodular using this argument turn out to be exactly the non-paramodular representations.

Turning to existence, we also use P_3-theory to prove that generic representations are paramodular. Let (π,V) be a generic, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. By Proposition 3.4.2, if V does contain a non-zero paramodular vector, then its image in V_{Z^J} is non-zero and invariant under $P_3(\mathfrak{o})$. Thus, to prove the existence of a non-zero paramodular vector in V, one might start with an appropriate non-zero $P_3(\mathfrak{o})$-invariant vector in V_{Z^J}. Indeed, since π is generic, the P_3 representation $\tau(\mathbf{1})$ is contained in V_{Z^J}, and we construct non-zero paramodular vectors in V from appropriate $P_3(\mathfrak{o})$-invariant vectors in $\tau(\mathbf{1})$. Readers familiar with GL(2) theory will recognize an analogy to the existence argument for non-zero $\Gamma_0(\mathfrak{p}^n)$-invariant vectors in generic representations of $\mathrm{GL}(2,F)$ with trivial central character using Kirillov models. However, due to the nature of the paramodular group, the GSp(4) case uses more than smoothness and involves level raising operators and zeta integrals.

Our second application of P_3-theory is a proof of the η Principle. Let π be a generic, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character, let $V=\mathcal{W}(\pi,\psi_{c_1,c_2})$ be the Whittaker model of π, and let W be a non-zero paramodular vector in V of level n. Algebraic manipulations prove that if W is in the image of the level raising operator η, i.e., $n\geq 2$ and $W=\eta W_1$ for some paramodular vector of level $n-2$, then $Z(s,W)=0$. The η Principle asserts that the converse is true, so that this is the only way that degenerate vectors can arise.

Theorem 4.3.7 (η Principle). *Let π be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character, and let $V=\mathcal{W}(\pi,\psi_{c_1,c_2})$. Let $n\geq 0$ be an integer. If W is non-zero and degenerate, then $n\geq 2$, and there exists $W_1\in V(n-2)$ such that $W=\eta W_1$.*

To understand how P_3-theory helps to prove the η Principle, again let W be a non-zero degenerate paramodular vector in V of level n; for simplicity, assume that $n \geq 2$. Using that η is given by the action of a single element of $\mathrm{GSp}(4,F)$, a computation shows that $W = \eta W_1$ for a paramodular vector W_1 of level $n-2$ if and only if W is invariant under the group of elements

$$\begin{bmatrix} 1 & y & & \\ & 1 & & \\ & & 1 & -y \\ & & & 1 \end{bmatrix}, \quad y \in \mathfrak{p}^{-1},$$

or equivalently and more conveniently, the single vector

$$W' = qW - \sum_{x \in \mathfrak{p}^{-1}/\mathfrak{o}} \pi(\begin{bmatrix} 1 & y & & \\ & 1 & & \\ & & 1 & -y \\ & & & 1 \end{bmatrix})W$$

is zero. Since we are working in the Whittaker model of π, we thus need to prove that $W'(g) = 0$ for all $g \in \mathrm{GSp}(4,F)$. Using little more than the definition of degeneracy, one can verify that in fact $W'(q) = 0$ for all $q \in Q$. If our group were $\mathrm{GL}(2)$ or $\mathrm{GL}(n)$, then we could immediately conclude that $W' = 0$ using one of the fundamental theorems of $\mathrm{GL}(n)$ Kirillov theory. To proceed in our $\mathrm{GSp}(4)$ situation we use the appropriate substitute, which is P_3-theory. The first step is to prove that the image $p(W')$ of W' in V_{Z^J} is zero. The argument for this proceeds by pushing down the vector $p(W')$ in the sequence $0 \subset V_2 \subset V_1 \subset V_{Z^J}$ described above. First of all, due to some basic properties of $p(W')$ and the fact that the irreducible subquotients of V_{Z^J}/V_1 are all of $\mathrm{GL}(2)$-type, we must have $p(W') \in V_1$. The next stage proves that $p(W') \in V_2$. The argument for this uses the degeneracy of W and the bijection between the poles of the irreducible subquotients of V_1/V_2 of the form $\tau(\chi)$ where χ is an unramified character of $F^\times$. Now that $p(W') \in V_2$, we prove that $p(W') = 0$, i.e., $W' \in V(Z^J)$. This step considers the sequence of Q representations

$$0 \to V(Z^J) \to p^{-1}(V_2) \to \{U|_Q : U \in p^{-1}(V_2)\} \to 0.$$

The point is that this sequence is exact because V_2 is irreducible: as π is generic, V_2 is isomorphic to the unique non-zero irreducible P_3 representation $\tau(\mathbf{1})$ of $\mathrm{GL}(0)$-type. Since W' vanishes on Q, by exactness we obtain $W' \in V(Z^J)$, or equivalently, $p(W') = 0$; this completes the role of P_3-theory in the analysis of W'. The final step is to prove that $W' = 0$ using $p(W') = 0$. This algebraic argument again uses level raising operators.

The results obtained from P_3-theory imply a lower bound for the dimension of the space of paramodular vectors of fixed level in a generic representation. Let (π, V) be a generic, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. By the first application of P_3-theory,

π is paramodular. Let N_π be the smallest non-negative integer n such that $V(n)$ is non-zero, i.e., let N_π be the paramodular level of π, and let W be a non-zero element of $V(N_\pi)$. Applying the commuting level raising operators θ', θ and η creates the vectors

$$\theta'^i \theta^j \eta^k W.$$

These vectors lie in $V(n)$ with $n = i + j + 2k$. We use zeta integrals to prove that these vectors are linearly independent for distinct triples i, j and k and fixed n. The key point is that the η Principle implies that $Z(s, W)$ is non-zero: this invariant sees W. Another important ingredient is the compatibility of zeta integrals with level raising operators. If U is a paramodular vector in V, then

$$Z(s, \theta' U) = qZ(s, U), \quad Z(s, \theta U) = q^{-s+3/2} Z(s, U), \quad Z(s, \eta U) = 0.$$

With linear independence, we obtain a lower bound for the dimension of $V(n)$ by counting the number of solutions to $n = i + j + 2k$. The result is

$$\dim V(n) \geq \left[\frac{(n - N_\pi + 2)^2}{4} \right] = (n - N_\pi) + 1 + \left[\frac{(n - N_\pi)^2}{4} \right] \quad \text{for} \quad n \geq N_\pi.$$

As is asserted by Theorem 7.5.6, this lower bound turns out to be the dimension of $V(n)$. This, and the full statement of Theorem 7.5.6, follow from the application of the next two methods.

The contributions of P_3-theory can be conveniently summarized using the above mentioned partition of of the irreducible, admissible representations of $\mathrm{GSp}(4, F)$ into four classes:

Generic non-supercuspidal representations	:	η Principle, dim. lower bound	Generic supercuspidal representations	:	η Principle, dim. lower bound
Non-generic non-supercuspidal representations	:	Some families are not paramodular	Non-generic supercuspidal representations	:	Not paramodular

Fig. 1.2. Major contributions of P_3-theory.

Since they are not paramodular, non-generic supercuspidal representations require no further consideration. Non-generic induced representations that are not paramodular also require no more examination. However, the fact that they admit no paramodular vectors will play a role in our next topic.

Double coset decompositions. The second method deployed in this work is the use of double coset decompositions in the analysis of non-supercuspidal representations. In combination with some of the P_3-theory results, the main results proven using double coset decompositions are dimension formulas for all of the spaces of paramodular vectors of fixed level in all non-supercuspidal, irreducible, admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character, and the Oldforms Principle for all paramodular, non-supercuspidal representations. A consequence is the uniqueness at the minimal paramodular level, and, using double coset methods, we also compute the Atkin–Lehner eigenvalues of the newforms in all non-supercuspidal representations. A corollary of this is the formula expressing the ε-factor of the L-parameter of a paramodular non-supercuspidal representation in terms of the minimal paramodular level and the Atkin–Lehner eigenvalue as in Theorem 7.5.9.

The basic idea is straightforward. By fundamental theory, every non-supercuspidal, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character is an irreducible subquotient of a representation parabolically induced from an irreducible supercuspidal representation on the Levi component of a proper parabolic subgroup of $\mathrm{GSp}(4,F)$. Parabolically induced representations, in turn, have models as subspaces of functions on $\mathrm{GSp}(4,F)$ that transform on the left according to representations of the Borel, Klingen, or Siegel parabolic subgroups, which we denote by B, Q and P, respectively. Paramodular vectors of level n in such representations are thus determined by their values on the elements of one of the double coset spaces

$$B\backslash\mathrm{GSp}(4,F)/\mathrm{K}(\mathfrak{p}^n),\quad Q\backslash\mathrm{GSp}(4,F)/\mathrm{K}(\mathfrak{p}^n),\quad P\backslash\mathrm{GSp}(4,F)/\mathrm{K}(\mathfrak{p}^n).$$

Thus, given explicit double coset representatives, paramodular vectors in non-supercuspidal representations can be studied in a relatively concrete setting.

The implementation of this idea begins with an initial observation and computation. Thanks to the work [ST], the composition series of representations parabolically induced from an irreducible, supercuspidal representation on the Levi component of a proper parabolic subgroup of $\mathrm{GSp}(4,F)$ are known, and there is a resulting classification of the non-supercuspidal, irreducible, admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character. See Sect. 2.2. A representation parabolically induced from an irreducible, supercuspidal representation on the Levi component of B, Q or P can have either one, two, or four irreducible subquotients. Conveniently, however, every non-supercuspidal, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ can be explicitly realized as the second or fourth element of a short exact sequence

$$0\to\pi\to\Pi\to\pi'\to 0$$

where π and π' are irreducible, admissible representations of $\mathrm{GSp}(4,F)$, and Π is parabolically induced from an irreducible, admissible representation on the Levi component of Q or P. See Sect. 2.2 and especially (2.9), (2.10) and (2.11). Thus, the first step is to compute the paramodular vectors of all levels in all such representations Π with trivial central character. When Π is induced from the Siegel parabolic P this is carried out in Theorem 5.2.2, and when Π is induced from the Klingen parabolic Q this appears in Theorem 5.4.2. In particular, if Π is paramodular, then the space of paramodular vectors in Π at the minimal level is one-dimensional and spanned by a vector supported on an explicit coset, so that we can speak of an essentially unique newform in Π.

The first application is to the determination of the dimensions of spaces of paramodular vectors of fixed level in all non-supercuspidal, irreducible, admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character. The analysis begins by noting that, by P_3-theory, representations of type VIb, VIIb and IXb are never paramodular. Next, if the representation is induced from the Siegel or Klingen parabolic subgroup, then the dimensions are immediately computed by the theorems mentioned in the last paragraph; these are the representations of type I, IIa, IIb, IIIa, IIIb, VII and X. To handle the remaining types we first of all note that several are the second or fourth elements of a short exact sequence as above such that the remaining elements of the sequence have known paramodular dimensions; again, the middle element Π is covered by the theorems mentioned in the last paragraph. For such types, the paramodular dimensions follow immediately by subtraction. The types covered by this technique are IVa, IVb, IVc, VIa, VIc, VId and IXa. The remaining types are now Va, Vb, Vc, Vd, XIa and XIb. To deal with these cases we continue to use short exact sequences; now, however, the second and fourth elements both have unknown paramodular dimensions. The appropriate short exact sequences involve what we call Saito–Kurokawa representations. To define these representations, let π be an infinite-dimensional, irreducible, admissible representation of $\mathrm{GL}(2,F)$ with trivial central character such that $\pi \not\cong \nu^{3/2} \times \nu^{-3/2}$, and let σ be a character of $F^\times$ such that $\sigma^2 = 1$. Then the representation $\nu^{1/2}\pi \rtimes \nu^{-1/2}\sigma$ has a unique, non-zero, irreducible quotient $Q(\nu^{1/2}\pi, \nu^{-1/2})$ and a unique, non-zero, irreducible subrepresentation $G(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$, so that there is short exact sequence of representations of $\mathrm{GSp}(4,F)$ with trivial central character:

$$0 \to G(\nu^{1/2}\pi, \nu^{-1/2}\sigma) \to \nu^{1/2}\pi \rtimes \nu^{-1/2}\sigma \to Q(\nu^{1/2}\pi, \nu^{-1/2}\sigma) \to 0.$$

We call $Q(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ a *Saito–Kurokawa representation*. This representation is non-generic, and many non-generic paramodular representations are Saito–Kurokawa. The representation $G(\nu^{1/2}\pi, \mu^{-1/2}\sigma)$ is generic, and we refer to it as the generic companion of $Q(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$. The Saito–Kurokawa representations are those of type IIb, Vb, Vc, VIc and XIb; their generic companions are the representations of type IIa, Va, VIa and XIa. To analyze the paramodular vectors in $Q(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ and $G(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ we use our explicit knowledge of the paramodular vectors in representations of the form $\nu^s\pi \rtimes \nu^{-s}\sigma$ for $s \in \mathbb{C}$ and the level raising operators θ, θ' and η. With an understanding of the paramodular dimensions in Saito–Kurokawa representations and their companions, only representations of type Vd remain. Such representations now fit into the first type of short exact sequences as above, completing the computation of paramodular dimensions for all non-supercuspidal representations.

The second application of double cosets is the proof of the Oldforms Principle for paramodular, non-supercuspidal, irreducible, admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character. To begin, generic non-supercuspidal representations are dealt with using a combination of P_3-theory and the now available paramodular dimensions. If (π, V) is generic and W is a non-zero element of $V(N_\pi)$, then by P_3-theory the vectors $\theta'^i\theta^i\eta^kW$ in $V(n)$ for $i+j+2k=n$ are linearly independent and have cardinality $[(n-N_\pi+2)^2/4]$; by the above, this is, in fact, the dimension of $V(n)$, proving the Oldforms Principle for π. The majority of the remaining paramodular, non-supercuspidal representations are Saito–Kurokawa. For these representations the Oldforms Principle is proven in conjunction with the computation of paramodular dimensions. The remaining types of representations are organized into pairs: IIIb and IVc; IVb and VId; and IVd and Vd. Paramodular representations of type IIIb and IVc are irreducible subquotients of $\nu^s \rtimes \nu^{-s/2}\mathbf{1}_{\mathrm{GSp}(2)}$ for some $s \in \mathbb{C}$. We prove the Oldforms Principle for these representations by computing the matrices of θ and θ' in the standard bases for $V(n)$ and $V(n+1)$ corresponding to the double cosets that support paramodular vectors. Representations of type IVb and VId are treated similarly, with $\nu^s\mathbf{1}_{\mathrm{GL}(2)} \rtimes \nu^{-s}$ replacing $\nu^s \rtimes \nu^{-s/2}\mathbf{1}_{\mathrm{GSp}(2)}$. Turning to the remaining pair, the Oldforms Principle is clear for paramodular representations of type Vd since the sequence of paramodular dimensions in these representations is $1,0,1,0,\ldots$ and η is injective. Similarly, the paramodular representations of type IVd are of the form $\sigma 1_{\mathrm{GSp}(4)}$ for σ an unramified character of $F^\times$ such that $\sigma^2=1$. For these representations the paramodular dimensions are $1,1,1,\ldots$ and the Oldforms Principle follows from the injectivity of θ'.

Finally, we use the double coset method to compute the Atkin–Lehner eigenvalues of the newforms in paramodular, non-supercuspidal, irreducible, admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character. As in the previous applications, the initial computation considers representations Π parabolically induced from an irreducible, admissible representation on the Levi component of Q or P. When the inducing data is infinite-dimensional, the Atkin–Lehner eigenvalue of the newform in such a Π is computed in Lemma 5.7.1. Case-by-case arguments as above then determine the Atkin–Lehner eigenvalue in all paramodular, non-supercuspidal representations. The table below summarizes the results obtained using double cosets.

Generic non-supercuspidal : representations	dim $V(n)$ for all n, Oldforms Principle, Atkin-Lehner eigenvalues	Generic supercuspidal representations	: *
Non-generic non-supercuspidal : representations	dim $V(n)$ for all n, Oldforms Principle, Atkin–Lehner eigenvalues	Non-generic supercuspidal representations	: *

Fig. 1.3. Major contributions of the double coset method.

The double coset method also serves as a tool in the computation of Hecke eigenvalues, and is thus involved in our next method.

Hecke operators. We use Hecke operators to prove the remaining assertions of this monograph. Let π be an irreducible, admissible representation $\mathrm{GSp}(4,F)$ with trivial central character. After the application of the first two methods, the key issues are now: when π is generic, the computation of the zeta integrals of Hecke eigenforms in $V(N_\pi)$ in terms of eigenvalues; when π is generic and supercuspidal, the proof of uniqueness at the minimal paramodular level and the Oldforms Principle; and when π is paramodular, the computation of the applicable L-factor of π in terms of the Hecke eigenvalues of the newform.

The first step is to prove that the zeta integral of a Hecke eigenform at the minimal paramodular level in a generic representation is recursively computed by its eigenvalues. Assume that π is generic, and let W be a non-zero element of the non-zero space $V(N_\pi)$; recall that, except if π is non-supercuspidal, we do not know at this point that $V(N_\pi)$ is one-dimensional. By the η Principle, W is determined by the numbers

$$c_{i,j} = W(\Delta_{i,j}), \quad \Delta_{i,j} = \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix}$$

for $i, j \geq 0$. In fact, the η Principle asserts that W is determined by just the numbers $c_{0,j}$ for $j \geq 0$, and computing $Z(s,W)$ amounts to determining these numbers. The key observation now is that Theorem 7.5.4 predicts that $Z(s,W)$ should be of the form $1/P(q^{-s})$, and therefore the numbers $c_{0,j}$ for $j \geq 0$ should be recursively determined. One approach to proving this, and thus computing $Z(s,W)$, is to apply endomorphisms T of $V(N_\pi)$ to W. Since $V(N_\pi)$ should be one-dimensional, one might further assume that W is an eigenform for T. If T had a convenient formula, then an equality $TW = aW$ would imply a relation between the numbers $c_{i,j}$ for $i, j \geq 0$. What endomorphisms T should be considered? Let

$$g = \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \quad \text{and} \quad g' = \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix}.$$

Then $(\pi(g)W)(\Delta_{i,j}) = c_{i,j+1}$ and $(\pi(g)W)(\Delta_{i,j}) = c_{i+1,j}$ for all i and j. Thus, $\pi(g)$ and $\pi(g')$ translate j and i by a unit, respectively. If $\pi(g)$ and $\pi(g')$ preserved $V(N_\pi)$, then the theory would be simple. Instead, however, $\pi(g)$ and $\pi(g')$ must be composed with the projection of V onto $V(N_\pi)$. The results are the operators $T_{0,1}$ and $T_{1,0}$, respectively; note that the actions of $\pi(g)$ and $\pi(g')$ on i and j explain the names of $T_{0,1}$ and $T_{1,0}$. By computing left coset representatives we obtain workable formulas for $T_{0,1}$ and $T_{1,0}$, and further computations prove that if $T_{0,1}W = \lambda W$ and $T_{1,0}W = \mu W$ for some complex numbers λ and μ, then the sequence $c_{0,j}$ for $j \geq 0$ is recursively defined. The result is that if W in $V(N_\pi)$ is an eigenform, then

$$Z(s,W) = (1 - q^{-1})W(1)L(s,\lambda,\mu),$$

where $L(s,\lambda,\mu)$ is defined by the formulas in i), ii) and iii) of Theorem 7.5.3 or Theorem 7.5.9. An immediate consequence of this formula and the η Principle is that if $L(s,\pi) = 1$, then necessarily $L(s,\lambda,\mu) = 1$, so that $Z(s,W) = (1 - q^{-1})W(1)$, $\lambda = 0$ and $\mu = -q^2$.

We use this formula, along with some additional facts about $T_{0,1}$ and $T_{1,0}$, to prove uniqueness at the minimal paramodular level when π is generic and supercuspidal. The first extra ingredient asserts that $T_{0,1}$ and $T_{1,0}$ commute as endomorphisms of $V(N_\pi)$. In fact, we prove the more general assertion that if π is any irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character and $n \geq 2$, then

$$T_{0,1}T_{1,0} - T_{1,0}T_{0,1} = q^2(\theta\delta_1 - \theta'\delta_2),$$

where both sides are regarded as endomorphisms of $V(n)$. Here, δ_1 and δ_2 are certain level lowering operators from $V(n)$ to $V(n-1)$. When π is unitary, e.g., when π is supercuspidal, then we also prove that the endomorphisms $T_{0,1}$ and $T_{1,0}$ of $V(n)$ are self-adjoint and hence diagonalizable. With these two additional facts, the proof of uniqueness at the minimal level for generic, supercuspidal representations is succinct: Since the endomorphisms $T_{0,1}$ and $T_{1,0}$ of $V(N_\pi)$ are simultaneously diagonalizable, the space $V(N_\pi)$ is the direct sum of the common eigenspaces. By the last paragraph, since $L(s,\pi) = 1$, there is only one eigenspace, and for this eigenspace $\lambda = 0$ and $\mu = -q^2$; therefore, for any $W \in V(N_\pi)$, we have $Z(s,W) = (1 - q^{-1})W(1)$. Uniqueness follows now from the η Principle, which asserts that at the minimal paramodular level there are no vectors with trivial zeta integral except the zero vector.

With all of this theory, the proof of the Oldforms Principle for supercuspidal, generic representations is also compact. As explained in the discussion on P_3-theory, to prove the Oldforms Principle for π it suffices to prove that the dimension of $V(n)$ is at most $[(n-N_\pi+2)^2/4]$ for $n \geq N_\pi$, so that, in fact, this is the dimension of $V(n)$. As a consequence of $Z(s,W_\pi) = 1 = L(s,\pi)$ for an appropriate normalization W_π of the essentially unique newform in $V(N_\pi)$, we have $\varepsilon(s,\pi) = \varepsilon_\pi q^{-N_\pi(s-1/2)}$. Using this in combination with the functional equation one can prove that if $W \in V(n)$, then $Z(s,W)$ is a polynomial in q^{-s} of degree at most $n - N_\pi$. Therefore, the dimension of $V(n)$ is at most $n - N_\pi + 1$ plus the dimension of the space of degenerate vectors in $V(n)$. By the η Principle, this last dimension is the dimension of $V(n-2)$; by induction this is in turn $[(n-2-N_\pi+2)^2/4]$. Adding, the dimension of $V(n)$ is at most $[(n - N_\pi + 2)^2/4]$, as desired.

Finally, by computing the eigenvalues of the newforms in all paramodular, irreducible, admissible representations π of $\mathrm{GSp}(4,F)$ with trivial central character, we verify that $L(s,\pi) = L(s,\lambda_\pi,\mu_\pi)$ if π is generic and

$L(s,\varphi_\pi) = L(s,\lambda_\pi,\mu_\pi)$ if π is non-supercuspidal and φ_π is the L-parameter of π. Hecke eigenvalues for generic π with $L(s,\pi) = 1$ have already been computed. To compute the Hecke eigenvalues for the remaining paramodular representations we apply two subsidiary computations. First, we compute the Hecke eigenvalues of the newform in representations parabolically induced from an irreducible, admissible representation on the Levi component of Q or P. As it happens, in the case of the Klingen parabolic subgroup Q, we need only consider the representation $\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$ for χ and σ unramified characters of $F^\times$ with $\chi\sigma^2 = 1$; this representation has minimal paramodular level $N_\pi = 2$. Second, to compute the Hecke eigenvalues for the generic companions of some Saito–Kurokawa representations we use commutation relations between $T_{0,1}$ and $T_{1,0}$ and the level raising operators θ and θ'. For level $n \geq 2$, the relations are:

$$\begin{aligned} T_{0,1} \circ \theta' &= \theta' \circ T_{0,1} + q^2\theta - \eta \circ \delta_2, \\ T_{0,1} \circ \theta &= \theta \circ T_{0,1} + q^2\theta' - \eta \circ \delta_1, \\ T_{1,0} \circ \theta' &= \theta' \circ T_{1,0} + q^3\theta' - q\,\eta \circ \delta_3, \\ T_{1,0} \circ \theta &= q\,T_{0,1} \circ \theta' - q^2(q+1)\,\theta. \end{aligned}$$

Again, δ_1, δ_2 and δ_3 are certain natural level lowering operators from $V(n)$ to $V(n-1)$.

The next chart lists the results proven using Hecke operators. On the next page, another chart gives an overview of the implications between the chief results of this work.

Generic non-supercuspidal representations	:	Formula for $Z(s,W)$, $L(s,\pi) = L(s,\lambda_\pi,\mu_\pi)$, $L(s,\varphi_\pi) = L(s,\lambda_\pi,\mu_\pi)$	Generic supercuspidal representations	:	Formula for $Z(s,W)$, Uniqueness, $L(s,\pi) = L(s,\lambda_\pi,\mu_\pi)$, Oldforms Principle
Non-generic non-supercuspidal representations	:	$L(s,\varphi_\pi) = L(s,\lambda_\pi,\mu_\pi)$	Non-generic supercuspidal representations	:	$*$

Fig. 1.4. Major contributions of Hecke operators.

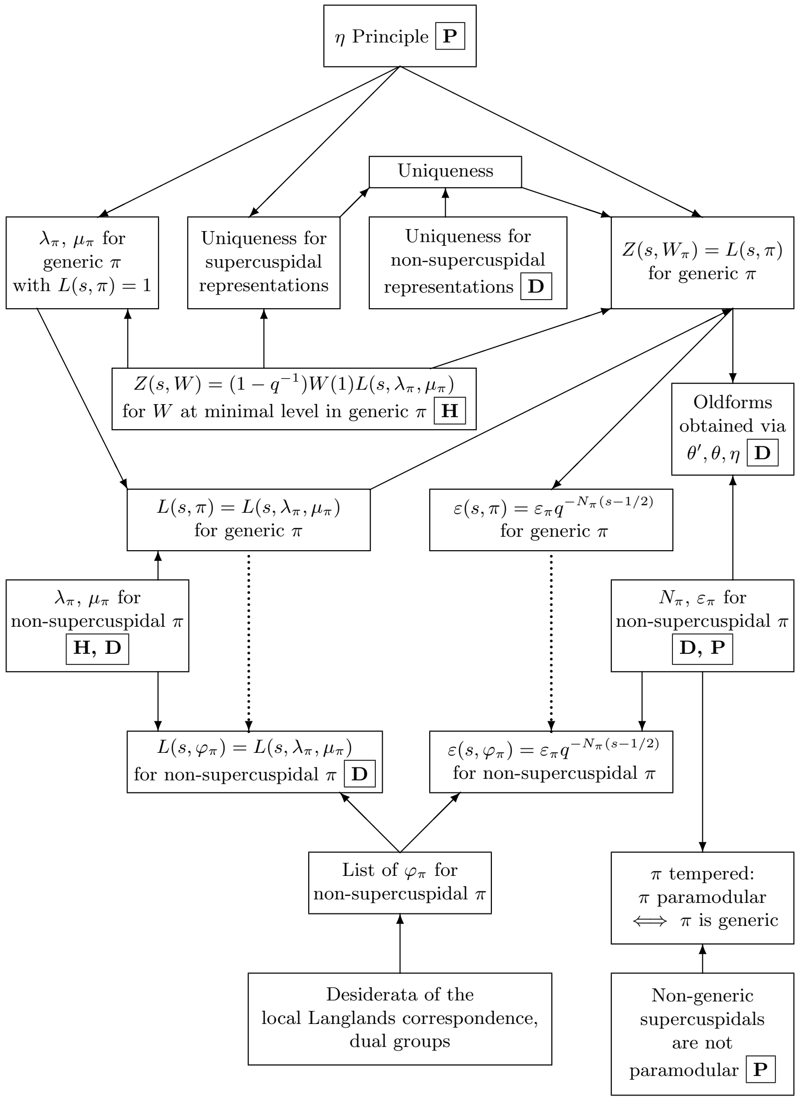


Fig. 1.5. Relations between major results. The symbols $\boxed{\mathbf{P}}$, $\boxed{\mathbf{D}}$ and $\boxed{\mathbf{H}}$ indicate the use of P_3-theory, double coset decompositions, and Hecke operators, respectively. The factor $L(s, \lambda_\pi, \mu_\pi)$ is defined by the formulas in i), ii) and iii) of Theorem 7.5.3 or Theorem 7.5.9. The dashed arrows indicate motivation.

2

Representation Theory

This chapter has two main themes. The first theme concerns the non-supercuspidal representations of GSp(4) over a non-archimedean local field. These representations were classified by Sally and Tadić and we summarize their work. Using this classification, we find it convenient to list and name all non-supercuspidal representations in the form of the fundamental Table A.1. With this list at hand we consider the conjectural local Langlands correspondence for GSp(4) as regards non-supercuspidal representations. As it happens, the desiderata of the conjectural local Langlands correspondence determine the L-parameters attached to all non-supercuspidal representations, and we describe these L-parameters explicitly.

The second main theme of this chapter concerns a type of Kirillov theory, called P_3-theory, for representations of GSp(4) with trivial central character, and the application of this theory to the theory of zeta integrals for generic representations. We give complete definitions and proofs of the basic theory of zeta integrals.

2.1 Definitions

The definitions made in this section will be used throughout this monograph. In addition, we also make some basic observations.

The Base Field F

Until the end of this monograph F is a nonarchimedean local field of characteristic zero. We denote by $\mathfrak{o}$ the ring of integers of F, and we let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$. Let q be the number of elements of $\mathfrak{o}/\mathfrak{p}$. Once and for all, we fix a generator ϖ for the ideal $\mathfrak{p}$. If x is in $F^\times$, then we define $v(x)$ to be the unique integer such that $x = u\varpi^{v(x)}$ for some unit u in $\mathfrak{o}^\times$. We write $\nu(x)$ or $|x|$ for the normalized absolute value of x; thus $\nu(\varpi) = q^{-1}$. Throughout this work we use the Haar measure on F that assigns $\mathfrak{o}$ volume

1, and we use the Haar measure on $F^\times$ defined by $d^\times x = dx/|\cdot|$, where dx is our Haar measure on F. We use the convention that $1+\varpi^0\mathfrak{o}$ means $\mathfrak{o}^\times$. We fix a continuous homomorphism $\psi : F \to \mathbb{C}^1$ such that $\psi(\mathfrak{o}) = 1$ but $\psi(\mathfrak{p}^{-1}) \neq 1$. If $\chi : F^\times \to \mathbb{C}^\times$ is a continuous homomorphism, then we let $a(\chi)$ be the smallest non-negative integer n such that $\chi(1+\varpi^n\mathfrak{o}) = 1$. We say that $a(\chi)$ is the *conductor* of χ. If S is a set and $A_1,\ldots,A_n$ are subsets of S, then $S = A_1 \sqcup \cdots \sqcup A_n$ means that S is the union of $A_1,\ldots,A_n$ and the A_i are mutually disjoint.

GSp(4) and its Parabolic Subgroups

We define $\mathrm{GSp}(4,F)$ to be the subgroup of $\mathrm{GL}(4,F)$ consisting of all g such that ${}^tgJg = \lambda J$ for some $\lambda \in F^\times$. If such a λ exists for g, then it is unique, we denote it by $\lambda(g)$, and call it the *multiplier* of g. Here, J is the element

$$J = \begin{bmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{bmatrix}$$

of $\mathrm{GL}(4,F)$. As will be the case whenever we write matrices, blank entries are taken to be the zero of F. Several subgroups of $\mathrm{GSp}(4,F)$ will be important. The subgroup of $\mathrm{GSp}(4,F)$ of all elements g such that $\lambda(g) = 1$ is called $\mathrm{Sp}(4,F)$. The subgroup of $\mathrm{GSp}(4,F)$ comprised of all elements

$$\begin{bmatrix} z&&&\\ &z&&\\ &&z&\\ &&&z \end{bmatrix}, \qquad z \in F^\times$$

will be denoted by Z. This subgroup lies in the center of $\mathrm{GSp}(4,F)$. The *Borel subgroup* B of $\mathrm{GSp}(4,F)$ consists of all upper triangular matrices in $\mathrm{GSp}(4,F)$. It is convenient to write this definition as

$$B = \begin{bmatrix} *&*&*&*\\ &*&*&*\\ &&*&*\\ &&&* \end{bmatrix}.$$

We will define other subgroups of $\mathrm{GSp}(4,F)$ using this notation. Every element of B can be written in the form

$$g = \begin{bmatrix} a&&&\\ &b&&\\ &&cb^{-1}&\\ &&&ca^{-1} \end{bmatrix} \begin{bmatrix} 1&&&\\ &1&x&\\ &&1&\\ &&&1 \end{bmatrix} \begin{bmatrix} 1&\lambda&\mu&\kappa\\ &1&&\mu\\ &&1&-\lambda\\ &&&1 \end{bmatrix}$$

where $a,b,c \in F^\times$ and $x,\lambda,\mu,\kappa \in F$. Evidently, $\lambda(g) = c$. The subgroup of B of elements which have 1 in every entry on the main diagonal will be denoted by U. The diagonal subgroup of $\mathrm{GSp}(4,F)$ will be called T. The *Siegel parabolic subgroup* of $\mathrm{GSp}(4,F)$ is

$$P = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{bmatrix}.$$

Every element of P can be written in the form

$$p = \begin{bmatrix} a & b & & \\ c & d & & \\ & & \lambda a/\Delta & -\lambda b/\Delta \\ & & -\lambda c/\Delta & \lambda d/\Delta \end{bmatrix} \begin{bmatrix} 1 & & \mu & \kappa \\ & 1 & x & \mu \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

where $\Delta = ad - bc \in F^\times$, $\lambda \in F^\times$, and $\mu,\kappa,x \in F$. We have $\lambda(p) = \lambda$. We will sometimes write

$$A' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} {}^tA^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{for } A \in \mathrm{GL}(2,F). \tag{2.1}$$

Explicitly, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A' = \frac{1}{ad-bc} \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}.$$

Using this notation, a general element in the Levi subgroup of P can be written as

$$p = \begin{bmatrix} A & \\ & \lambda A' \end{bmatrix}, \qquad A \in \mathrm{GL}(2,F),\ \lambda \in F^\times.$$

The *Klingen parabolic subgroup* of $\mathrm{GSp}(4,F)$ is

$$Q = \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & * \end{bmatrix}.$$

Every element of Q can be written in the form

$$q = \begin{bmatrix} t & & & \\ & a & b & \\ & c & d & \\ & & & \Delta t^{-1} \end{bmatrix} \begin{bmatrix} 1 & \lambda & \mu & \kappa \\ & 1 & & \mu \\ & & 1 & -\lambda \\ & & & 1 \end{bmatrix}$$

where $\Delta = ad - bc \in F^\times$, $t \in F^\times$ and $\lambda,\mu,\kappa \in F$. A computation shows that $\lambda(q) = \Delta$. The *Jacobi subgroup* of Q is

$$G^J = \begin{bmatrix} 1 & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & 1 \end{bmatrix}.$$

The Jacobi subgroup is contained in $\mathrm{Sp}(4,F)$. The subgroup

$$Z^J = \begin{bmatrix} 1 & & & * \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

is the center of the Jacobi subgroup G^J. More generally, Z^J is a normal subgroup of the Klingen parabolic subgroup Q.

Weyl Group Elements

Certain elements of $\mathrm{GSp}(4,F)$ will play an important role in this work. Let $N(T)$ be the normalizer in $\mathrm{GSp}(4,F)$ of $T = T(F)$. Then $W = N(T)/T$ is a group of order eight. The images of the elements

$$s_1 = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}, \qquad s_2 = \begin{bmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{bmatrix}$$

in W generate W. Representatives for the eight elements of W are

$$s_1, \quad s_2, \quad s_2s_1s_2, \quad s_1s_2s_1$$

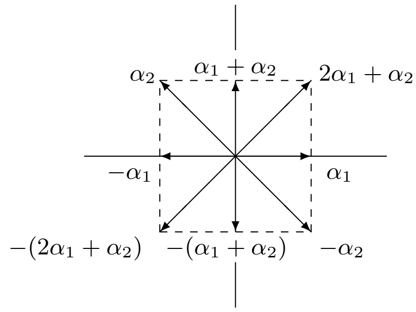
and

$$1, \quad s_1s_2, \quad s_2s_1, \quad s_1s_2s_1s_2.$$

It is worthwhile to explicitly state $s_2s_1s_2$ and $s_1s_2s_1$,

$$s_2s_1s_2 = \begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix}, \qquad s_1s_2s_1 = \begin{bmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ -1 & & & \end{bmatrix}.$$

We may identify W with the subgroup of the orthogonal group of the real plane that maps the unit square into itself, i.e., the dihedral group of order eight. This is illustrated in the following diagram. The element corresponding to s_1 is the reflection sending α_1 to $-\alpha_1$ and the element corresponding to s_2 is the reflection sending α_2 to $-\alpha_2$.



The Paramodular Group and Other Congruence Subgroups

This monograph considers the vectors in representations of $\mathrm{GSp}(4, F)$ fixed by a certain family of compact open subgroups of $\mathrm{GSp}(4, F)$. For a nonnegative integer n we define $\mathrm{K}(\mathfrak{p}^n)$ to be the subgroup of all k in $\mathrm{GSp}(4, F)$ such that $\det(k) \in \mathfrak{o}^\times$ and

$$k \in \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-n} \\ \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o} \end{bmatrix}$$

and call $\mathrm{K}(\mathfrak{p}^n)$ the *paramodular group* of level $\mathfrak{p}^n$. The first group $\mathrm{K}(\mathfrak{p}^0)$ in the family is $\mathrm{GSp}(4, \mathfrak{o})$, a maximal compact subgroup of $\mathrm{GSp}(4, F)$. The second group $\mathrm{K}(\mathfrak{p}^1)$ is the other maximal compact subgroup of $\mathrm{GSp}(4, F)$, up to conjugacy, and it is this group that is sometimes known in the literature as the paramodular group. The paramodular group $\mathrm{K}(\mathfrak{p}^n)$ is normalized by the *Atkin–Lehner element*

$$u_n = \begin{bmatrix} & & 1 & \\ & & & -1 \\ \varpi^n & & & \\ & -\varpi^n & & \end{bmatrix} \tag{2.2}$$

of level $\mathfrak{p}^n$. Note that u_n^2 lies in the center of $\mathrm{GSp}(4, F)$. For any nonnegative integer n the paramodular group $\mathrm{K}(\mathfrak{p}^n)$ contains the Weyl group element s_2. Additionally, $\mathrm{K}(\mathfrak{p}^n)$ contains the important element

$$t_n = \begin{bmatrix} & & & -\varpi^{-n} \\ & 1 & & \\ & & 1 & \\ \varpi^n & & & \end{bmatrix} \in \mathrm{K}(\mathfrak{p}^n). \tag{2.3}$$

Two other families of congruence subgroups of $\mathrm{GSp}(4, F)$ will be important. We define the *Siegel congruence subgroup* of level $\mathfrak{p}^n$ to be the subgroup $\mathrm{Si}(\mathfrak{p}^n)$ of all k in $\mathrm{GSp}(4, F)$ such that $\det(k) \in \mathfrak{o}^\times$ and

$$k \in \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} \end{bmatrix}. \tag{2.4}$$

We define the *Klingen congruence subgroup* of level $\mathfrak{p}^n$ to be the subgroup $\mathrm{Kl}(\mathfrak{p}^n)$ of $\mathrm{GSp}(4, F)$ consisting of all k such that $\det(k) \in \mathfrak{o}^\times$ and

$$k \in \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o} \end{bmatrix}. \tag{2.5}$$

The Siegel congruence subgroup $\mathrm{Si}(\mathfrak{p}^n)$ is normalized by u_n, but the Klingen congruence subgroup $\mathrm{Kl}(\mathfrak{p}^n)$ is not. In fact, the group generated by $\mathrm{Kl}(\mathfrak{p}^n)$ and $u_n \mathrm{Kl}(\mathfrak{p}^n) u_n^{-1}$ is the paramodular group $\mathrm{K}(\mathfrak{p}^n)$ (see Lemma 3.3.1 for a more precise statement). It is easy to prove that the *Iwahori factorization* holds for the Siegel and Klingen congruence subgroups, i.e., for any $n \geq 1$,

$$\begin{aligned} \mathrm{Si}(\mathfrak{p}^n) &= \begin{bmatrix} 1 &&& \\ & 1 && \\ \mathfrak{p}^n & \mathfrak{p}^n & 1 & \\ \mathfrak{p}^n & \mathfrak{p}^n && 1 \end{bmatrix} \begin{bmatrix} \mathfrak{o} & \mathfrak{o} && \\ \mathfrak{o} & \mathfrak{o} && \\ && \mathfrak{o} & \mathfrak{o} \\ && \mathfrak{o} & \mathfrak{o} \end{bmatrix} \begin{bmatrix} 1 && \mathfrak{o} & \mathfrak{o} \\ & 1 & \mathfrak{o} & \mathfrak{o} \\ && 1 & \\ &&& 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 && \mathfrak{o} & \mathfrak{o} \\ & 1 & \mathfrak{o} & \mathfrak{o} \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} \mathfrak{o} & \mathfrak{o} && \\ \mathfrak{o} & \mathfrak{o} && \\ && \mathfrak{o} & \mathfrak{o} \\ && \mathfrak{o} & \mathfrak{o} \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ \mathfrak{p}^n & \mathfrak{p}^n & 1 & \\ \mathfrak{p}^n & \mathfrak{p}^n && 1 \end{bmatrix} \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \mathrm{Kl}(\mathfrak{p}^n) &= \begin{bmatrix} 1 &&& \\ \mathfrak{p}^n & 1 && \\ \mathfrak{p}^n && 1 & \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n & 1 \end{bmatrix} \begin{bmatrix} \mathfrak{o}^\times &&& \\ & \mathfrak{o} & \mathfrak{o} & \\ & \mathfrak{o} & \mathfrak{o} & \\ &&& \mathfrak{o}^\times \end{bmatrix} \begin{bmatrix} 1 & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ & 1 && \mathfrak{o} \\ && 1 & \mathfrak{o} \\ &&& 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ & 1 && \mathfrak{o} \\ && 1 & \mathfrak{o} \\ &&& 1 \end{bmatrix} \begin{bmatrix} \mathfrak{o}^\times &&& \\ & \mathfrak{o} & \mathfrak{o} & \\ & \mathfrak{o} & \mathfrak{o} & \\ &&& \mathfrak{o}^\times \end{bmatrix} \begin{bmatrix} 1 &&& \\ \mathfrak{p}^n & 1 && \\ \mathfrak{p}^n && 1 & \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n & 1 \end{bmatrix}. \end{aligned} \tag{2.7}$$

In this work, the Klingen congruence subgroup will play a much bigger role than the Siegel congruence subgroup.

General Representation Theory

In this monograph we will use the following definitions concerning representation theory. Let G be a group of td-type, as in [Car], with a countable basis.

Let $\mathcal{S}(G)$ be the complex vector space of all locally constant, compactly supported complex valued functions on G. A representation of G is a complex vector space V along with a homomorphism π of $\mathrm{GSp}(4,F)$ into the group $\mathrm{Aut}(V)$ of invertible $\mathbb{C}$-linear endomorphisms of V. Such a representation will be written as a pair (π,V) or just π. A representation (π,V) is *smooth* if every vector in V is fixed by an open-compact subgroup K of G. We say that (π,V) is *admissible* if π is smooth and the multiplicities of each K-type are finite for each such compact-open subgroup K of G. A smooth representation (π,V) is *irreducible* if the only G subspaces of V are 0 and V. If (π,V) is a smooth representation of G, then an *irreducible constituent* or *irreducible subquotient* of π is an irreducible representation of G that is isomorphic to W/W', where $W'\subset W\subset V$ are G subspaces of V. The smooth *contragredient* representation of π is $\pi^\vee$, and if π admits a central character, then we denote it by ω_π. A *character* of G is a smooth one-dimensional representation of G, i.e., a continuous homomorphism from G to $\mathbb{C}^\times$. We let $\mathbf{1}_G$ denote the *trivial* representation of G, i.e., the trivial character of G. If π is a representation of G and G is contained as a normal subgroup in another group G' of td-type, then we let $g'\pi$ be the representation of G with the same space as π and action defined by $(g'\pi)(g)=\pi(g'^{-1}gg')$. A representation of G is *unitary* if there exists a nondegenerate G-invariant Hermitian form on the space of π. Let H be a closed subgroup of G, and let (π,V) be a smooth representation of H. Then $\text{c-Ind}_H^G\pi$ is the representation of G whose space is the vector space of all functions $f:G\to V$ such that $f(hg)=\pi(h)f(g)$ for $h\in H$ and $g\in G$, there exists a compact, open subgroup K of G such that $f(gk)=f(g)$ for $k\in K$ and $g\in G$, and there exists a compact subgroup $X\subset G$ such that f vanishes off of HX. Suppose G is unimodular, and M and U are closed subgroups of G such that M normalizes U, $M\cap U=1$, $P=MU$ is closed in G, U is unimodular, and $P\backslash G$ is compact. Fix a Haar measure du on U. For $p\in P$ let $\delta_P(p)$ be the positive number such that for all $f\in\mathcal{S}(U)$,

$$\int\limits_U f(p^{-1}up)\,du=\delta_P(p)\int\limits_U f(u)\,du.$$

We call $\delta_P:P\to\mathbb{C}^\times$ the *modular character* of P. Suppose that σ is a smooth representation of M. Then $\mathrm{Ind}_P^G\sigma$ is the representation of G by right translation on the complex vector space of smooth functions f on G with values in σ such that $f(mug)=\delta_P(m)^{1/2}\sigma(m)f(g)$ for $m\in M$, $u\in U$ and $g\in G$. If π is a smooth representation of G, then the *normalized Jacquet module* $\mathrm{R}_U(\pi)$ is the smooth representation of M defined by $\mathrm{R}_U(\pi)=\pi_U\otimes\delta_P^{-1/2}$, where π_U is the quotient of π by the $\mathbb{C}$-subspace generated by the vectors $v-\pi(u)v$ for $v\in\pi$ and $u\in U$. We define $\overline{\mathrm{R}}_U(\pi)=R_U(\pi^\vee)^\vee$. If π is a smooth representation of G and σ is a smooth representation of M, then we have Frobenius reciprocity: $\mathrm{Hom}_G(\pi,\mathrm{Ind}_P^G\sigma)\cong\mathrm{Hom}_M(\mathrm{R}_U(\pi),\sigma)$ and $\mathrm{Hom}_G(\mathrm{Ind}_P^G\sigma,\pi^\vee)\cong\mathrm{Hom}_M(\sigma,\mathrm{R}_U(\pi)^\vee)$. If π is admissible we also have $\mathrm{Hom}_G(\mathrm{Ind}_P^G\sigma,\pi)\cong\mathrm{Hom}_M(\sigma,\overline{\mathrm{R}}_U(\pi))$.

Generic Representations of $\mathrm{GSp}(4, F)$

Let ψ be our fixed non-trivial additive character of F. Every other such character is then of the form $x \mapsto \psi(cx)$ for a uniquely determined element $c \in F^\times$. Fix $c_1, c_2 \in F^\times$, and consider the character ψ_{c_1,c_2} of $U(F)$, the unipotent radical of the Borel subgroup $B(F)$, given by

$$\psi_{c_1,c_2}(\begin{bmatrix} 1 & x & * & * \\ & 1 & y & * \\ & & 1 & -x \\ & & & 1 \end{bmatrix}) = \psi(c_1 x + c_2 y).$$

An irreducible, admissible representation π of $\mathrm{GSp}(4, F)$ is called *generic* if $\mathrm{Hom}_{U(F)}(\pi, \psi_{c_1,c_2}) \neq 0$. This definition does not depend on the choice of c_1 or c_2. If π is generic, then there exists a *Whittaker model* for π with respect to ψ_{c_1,c_2}, i.e., π can be realized as a space of functions $W : \mathrm{GSp}(4, F) \to \mathbb{C}$ that satisfy the transformation property

$$W(\begin{bmatrix} 1 & x & * & * \\ & 1 & y & * \\ & & 1 & -x \\ & & & 1 \end{bmatrix} g) = \psi(c_1 x + c_2 y) W(g), \qquad \text{all } g \in \mathrm{GSp}(4, F),$$

and $\mathrm{GSp}(4, F)$ acts on this space by right translations. By [Rod], such a Whittaker model is unique. We denote it by $\mathcal{W}(\pi, \psi_{c_1,c_2})$. *In this work, when working with $\mathcal{W}(\pi, \psi_{c_1,c_2})$, we will always assume that c_1 and c_2 are in $\mathfrak{o}^\times$.*

Contragredients of Representations of $\mathrm{GSp}(4, F)$

Suppose that π is an irreducible, admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character. Then the contragredient of π is π: $\pi^\vee \cong \pi$. This follows from the Theorem on p. 91 of [MVW]. For this, we note that by Appendix A.7 we can identify π with an irreducible representation of $\mathrm{SO}(5, F)$, and that $\mathrm{O}(5, F)$ is the direct product of $\mathrm{SO}(5, F)$ with the subgroup $\{1, -1\}$.

A Useful Identity

We will often use the following identity for $x \in F^\times$:

$$\begin{bmatrix} 1 & \\ x & 1 \end{bmatrix} = \begin{bmatrix} 1 & x^{-1} \\ & 1 \end{bmatrix} \begin{bmatrix} -x^{-1} & \\ & -x \end{bmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} 1 & x^{-1} \\ & 1 \end{bmatrix}. \tag{2.8}$$

2.2 Parabolically Induced Representations

The irreducible, admissible representations of $\mathrm{GSp}(4, F)$ come in two classes. The first class consists of all those representations that can be obtained as subquotients of parabolically induced representations from one of the parabolic subgroups B, P or Q. Thanks to the work of Sally and Tadić in [ST], these representations have been classified and described, and we reproduce their list. The second class consists of all the other representations; these representations are called *supercuspidal*. In this work we will not need explicit descriptions of the supercuspidal representations.

Parabolic Induction

First we explain parabolic induction from B. Let χ_1, χ_2 and σ be characters of $F^\times$, and consider the character of $B(F)$ given by

$$\begin{bmatrix} a & * & * & * \\ & b & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{bmatrix} \longmapsto \chi_1(a)\chi_2(b)\sigma(c).$$

The representation of $\mathrm{GSp}(4,F)$ obtained by normalized parabolic induction of this character of $B(F)$ is denoted by $\chi_1 \times \chi_2 \rtimes \sigma$. The standard model of this representation consists of all locally constant functions $f:\ \mathrm{GSp}(4,F) \to \mathbb{C}$ that satisfy the transformation property

$$f(hg) = |a^2 b||c|^{-3/2}\chi_1(a)\chi_2(b)\sigma(c)f(g) \quad \text{for all } h = \begin{bmatrix} a & * & * & * \\ & b & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{bmatrix}.$$

Note here that the modular character of B is given by $\delta_B(h) = |a|^4|b|^2|c|^{-3}$. The group $\mathrm{GSp}(4,F)$ acts on this space by right translations. The central character of $\chi_1 \times \chi_2 \rtimes \sigma$ is $\chi_1\chi_2\sigma^2$.

We turn to parabolic induction from P. Let (π, V) be an admissible representation of $\mathrm{GL}(2,F)$, and let σ be a character of $F^\times$. Then we denote by $\pi \rtimes \sigma$ the representation of $\mathrm{GSp}(4,F)$ obtained by normalized parabolic induction from the representation of $P(F)$ on V given by

$$\begin{bmatrix} A & * \\ & cA' \end{bmatrix} \longmapsto \sigma(c)\pi(A)$$

(see (2.1) for the A' notation). Since the modular character of P is given by $\delta_P(\begin{bmatrix} A & * \\ & cA' \end{bmatrix}) = |\det(A)|^3|c|^{-3}$, the standard space of $\pi \rtimes \sigma$ consists of all locally constant functions $f:\ \mathrm{GSp}(4,F) \to V$ that satisfy the transformation property

$$f(hg) = |\det(A)c^{-1}|^{3/2}\sigma(c)\pi(A)f(g) \quad \text{for all } h = \begin{bmatrix} A & * \\ & cA' \end{bmatrix} \in P(F).$$

If π has central character ω_π, then the central character of $\pi \rtimes \sigma$ is $\omega_\pi\sigma^2$.

Finally we consider parabolic induction from Q. Let χ be a character of $F^\times$, and let (π, V) be an admissible representation of $\mathrm{GL}(2,F)$ (for systematic reasons, this $\mathrm{GL}(2,F)$ should really be considered as the symplectic similitude group $\mathrm{GSp}(2,F)$). Then we denote by $\chi \rtimes \pi$ the representation of $\mathrm{GSp}(4,F)$ obtained by normalized parabolic induction from the representation of $Q(F)$ on V given by

$$\begin{bmatrix} t & * & * & * \\ & a & b & * \\ & c & d & * \\ & & & \Delta t^{-1} \end{bmatrix} \longmapsto \chi(t)\pi(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) \qquad (\Delta = ad - bc).$$

The standard space of $\chi \rtimes \pi$ consists of all locally constant functions $f : \mathrm{GSp}(4,F) \to V$ that satisfy the transformation property

$$f(hg) = |t^2(ad-bc)^{-1}|\chi(t)\pi(\begin{bmatrix} a & b \\ c & d \end{bmatrix})f(g) \quad \text{for all } h = \begin{bmatrix} t & * & * & * \\ & a & b & * \\ & c & d & * \\ & & & \Delta t^{-1} \end{bmatrix},$$

because the modular character of Q is given by $\delta_Q(h) = |t|^4|ad-bc|^{-2}$. If π has central character ω_π, then the central character of $\chi \rtimes \pi$ is $\chi\omega_\pi$.

Twisting

If τ is a character of $F^\times$ and (π, V) is a representation of $\mathrm{GSp}(4,F)$, then we define a new representation $\tau\pi$ on the same space V by

$$(\tau\pi)(g) = \tau(\lambda(g))\pi(g),$$

where λ is the multiplier homomorphism $\mathrm{GSp}(4,F) \to F^\times$. We call $\tau\pi$ the *twist* of the representation π by the character τ. The central character of $\tau\pi$ is the central character of π multiplied by τ^2. One checks easily that twisting has the following effect on representations parabolically induced from B, P and Q, respectively:

$$\begin{aligned} \tau(\chi_1 \times \chi_2 \rtimes \sigma) &\cong \chi_1 \times \chi_2 \rtimes \tau\sigma, \\ \tau(\pi \rtimes \sigma) &\cong \pi \rtimes \tau\sigma, \\ \tau(\chi \rtimes \pi) &\cong \chi \rtimes \tau\pi. \end{aligned}$$

In these formulas, π is a representation of $\mathrm{GL}(2,F)$ as before.

The List of Representations

In this section, using the results of [ST], we describe a useful listing of the non-supercuspidal, irreducible, admissible representations of $\mathrm{GSp}(4,F)$. The basis for this list is the fact that every non-supercuspidal, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ is an irreducible constituent (irreducible subquotient) of a parabolically induced representation with proper supercuspidal inducing data. Given this, one might try to classify the non-supercuspidal, irreducible, admissible representations of $\mathrm{GSp}(4,F)$ by doing the following: First, write down all the supercuspidal inducing data for the Borel, Klingen and Siegel parabolic subgroups of $\mathrm{GSp}(4,F)$. Second, determine the Langlands classification data of all the constituents of all the resulting parabolically induced representations. Third, find all the possible ways in which a fixed non-supercuspidal, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ arises as a constituent in the second step; the results of this step may show that some of the supercuspidal inducing data from the first step is redundant from the point of view of listing all representations. Finally, write down a list of the non-supercuspidal, irreducible, admissible representations of $\mathrm{GSp}(4,F)$ that includes all such representations and is as convenient as possible. In, fact, the paper [ST] has carried out the difficult aspects of this procedure. The resulting list appears in tabulated form in Table A.1, and to understand the table the reader should consult the following.

First of all, it turns out that eleven groups of supercuspidal inducing data are required. Below, we list these eleven groups and the corresponding constituents of the parabolically induced representations. Groups I to VI contain representations supported in B, i.e., these representations are constituents of induced representations of the form $\chi_1 \times \chi_2 \rtimes \sigma$. Groups XII, XIII and IX contain representations supported in Q, i.e., they are constituents of induced representations of the form $\chi \rtimes \pi$, where π is a supercuspidal representation of $\mathrm{GL}(2,F)$. Finally, groups X and XI contain representations supported in P, i.e., they are constituents of induced representations of the form $\pi \rtimes \sigma$, where π is a supercuspidal representation of $\mathrm{GL}(2,F)$. In this work, if π is a constituent of the parabolically induced representation with supercuspidal inducing data of Group A, where "A" is the name of the group, then we will say that π *belongs to Group A* or that π is a *Group A representation.* Sometimes, in this context, we will use the word "Type" instead of "Group".

Group I. These are the *irreducible* representations of the form $\chi_1 \times \chi_2 \rtimes \sigma$, where χ_1, χ_2 and σ are characters of $F^\times$. The induced representation $\chi_1 \times \chi_2 \rtimes \sigma$ is irreducible if and only if $\chi_1 \neq \nu^{\pm 1}$, $\chi_2 \neq \nu^{\pm 1}$ and $\chi_1 \neq \nu^{\pm 1}\chi_2^{\pm 1}$.

Group II. Let χ be a character of $F^\times$ such that $\chi \neq \nu^{\pm 3/2}$ and $\chi^2 \neq \nu^{\pm 1}$. Then $\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$ decomposes into two irreducible constituents

$$\mathrm{IIa} : \chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma \qquad \text{and} \qquad \mathrm{IIb} : \chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$$

by [ST] Lemmas 3.3 and 3.7. Here $\mathrm{St}_{\mathrm{GL}(2)}$ denotes the Steinberg representation and $\mathbf{1}_{\mathrm{GL}(2)}$ denotes the trivial representation of $\mathrm{GL}(2,F)$. Note that these are the two constituents of the standard induced representation $\nu^{1/2} \times \nu^{-1/2}$ of $\mathrm{GL}(2,F)$. The representation IIa is a subrepresentation and IIb is a quotient of $\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$. These representations can also be written as Langlands quotients,

$$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma = \begin{cases} L(\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma) & \text{if } e(\chi) = 0, \\ L(\chi\mathrm{St}_{\mathrm{GL}(2)}, \sigma) & \text{if } 0 < e(\chi), \end{cases}$$

$$\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma = \begin{cases} L(\nu^{1/2}\chi, \nu^{1/2}\chi^{-1}, \nu^{-1/2}\chi\sigma) & \text{if } 0 \le e(\chi) < 1/2, \\ L(\nu^{1/2}\chi, \nu^{-1/2}\chi \rtimes \sigma) & \text{if } e(\chi) = 1/2, \\ L(\nu^{1/2}\chi, \nu^{-1/2}\chi, \sigma) & \text{if } 1/2 < e(\chi). \end{cases}$$

Here, the real number $e(\chi)$ is the *exponent* of the character χ, defined by $|\chi(x)| = |x|^e$ for all $x \in F^\times$.

Group III. Let χ be a character of $F^\times$ such that $\chi \neq 1_{F^\times}$ and $\chi \neq \nu^{\pm 2}$. Then $\chi \times \nu \rtimes \nu^{-1/2}\sigma$ decomposes into two irreducible constituents

$$\text{IIIa} : \chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)} \qquad \text{and} \qquad \text{IIIb} : \chi \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}$$

by [ST] Lemmas 3.4 and 3.9. The representation IIIa is a subrepresentation and IIIb is a quotient of $\chi \times \nu \rtimes \nu^{-1/2}\sigma$. Written as Langlands quotients we have

$$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)} = \begin{cases} L(\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}) & \text{if } e(\chi) = 0, \\ L(\chi, \sigma\mathrm{St}_{\mathrm{GSp}(2)}) & \text{if } 0 < e(\chi), \end{cases}$$

$$\chi \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)} = \begin{cases} L(\nu, \chi \rtimes \nu^{-1/2}\sigma) & \text{if } e(\chi) = 0, \\ L(\chi, \nu, \nu^{-1/2}\sigma) & \text{if } 0 < e(\chi). \end{cases}$$

Group IV. These are the four irreducible constituents of $\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma$, where σ is an arbitrary character of $F^\times$. We shall need more precise information about the way this induced representation decomposes. By [ST] Lemma 3.5,

$$\begin{aligned} \nu^2 \times \nu \rtimes \nu^{-3/2}\sigma &= \underbrace{\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{-3/2}\sigma}_{\text{sub}} + \underbrace{\nu^{3/2}\mathbf{1}_{\mathrm{GL}(2)} \rtimes \nu^{-3/2}\sigma}_{\text{quot}} \\ &= \underbrace{\nu^2 \rtimes \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)}}_{\text{sub}} + \underbrace{\nu^2 \rtimes \nu^{-1}\sigma\mathbf{1}_{\mathrm{GSp}(2)}}_{\text{quot}}, \end{aligned}$$

and each of the four representations on the right is reducible and has two irreducible constituents as shown in the following table. The quotients appear at the bottom and on the right.

	$\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{-3/2}\sigma$	$\nu^{3/2}\mathbf{1}_{\mathrm{GL}(2)} \rtimes \nu^{-3/2}\sigma$
$\nu^2 \rtimes \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$L(\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$
$\nu^2 \rtimes \nu^{-1}\sigma\mathbf{1}_{\mathrm{GSp}(2)}$	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$

(2.9)

Here $\mathrm{St}_{\mathrm{GSp}(4)}$ is the Steinberg representation of $\mathrm{GSp}(4, F)$, and $\mathbf{1}_{\mathrm{GSp}(4)}$ is the trivial representation of $\mathrm{GSp}(4, F)$.

Group V. These are the four irreducible constituents of $\nu\xi \times \xi \rtimes \nu^{-1/2}\sigma$, where ξ is a non-trivial quadratic character of $F^\times$, and where σ is an arbitrary character of $F^\times$. In this case, by [ST] Lemma 3.6,

$$\nu\xi \times \xi \rtimes \nu^{-1/2}\sigma = \underbrace{\nu^{1/2}\xi\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma}_{\text{sub}} + \underbrace{\nu^{1/2}\xi\,\mathbf{1}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma}_{\text{quot}}$$
$$= \underbrace{\nu^{1/2}\xi\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \xi\nu^{-1/2}\sigma}_{\text{sub}} + \underbrace{\nu^{1/2}\xi\,\mathbf{1}_{\mathrm{GL}(2)} \rtimes \xi\nu^{-1/2}\sigma}_{\text{quot}}.$$

Each of the representations on the right side has two constituents as indicated in the following table. The quotients appear at the bottom and on the right.

	$\nu^{1/2}\xi\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\xi\sigma$	$\nu^{1/2}\xi\,\mathbf{1}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\xi\sigma$
$\nu^{1/2}\xi\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma$	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	$L(\nu^{1/2}\xi\,\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$
$\nu^{1/2}\xi\,\mathbf{1}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma$	$L(\nu^{1/2}\xi\,\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\xi\sigma)$	$L(\nu\xi,\xi \rtimes \nu^{-1/2}\sigma)$

(2.10)

Here $\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$ is an essentially square integrable representation. Note that the representations Vb and Vc are twists of each other, but they are not equivalent.

Group VI. These are the four irreducible constituents of $\nu \times 1_{F^\times} \rtimes \nu^{-1/2}\sigma$, where σ is an arbitrary character of $F^\times$. By [ST] Lemma 3.8,

$$\nu \times 1_{F^\times} \rtimes \nu^{-1/2}\sigma = \underbrace{\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma}_{\text{sub}} + \underbrace{\nu^{1/2}\,\mathbf{1}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma}_{\text{quot}}$$
$$= \underbrace{1_{F^\times} \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}}_{\text{sub}} + \underbrace{1_{F^\times} \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}}_{\text{quot}},$$

and each representation on the right side is again reducible. Their constituents are summarized in the following table, with the quotients appearing at the bottom and on the right.

	$\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma$	$\nu^{1/2}\,\mathbf{1}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma$
$1_{F^\times} \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$\tau(S,\nu^{-1/2}\sigma)$	$\tau(T,\nu^{-1/2}\sigma)$
$1_{F^\times} \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}$	$L(\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$L(\nu,1_{F^\times} \rtimes \nu^{-1/2}\sigma)$

(2.11)

The representations $\tau(S,\nu^{-1/2}\sigma)$ and $\tau(T,\nu^{-1/2}\sigma)$ are essentially tempered but not square integrable.

Group VII. These are the irreducible representations of the form $\chi \rtimes \pi$, where π is a supercuspidal representation of $\mathrm{GL}(2,F)$. The condition for irreducibility is that $\chi \neq 1_{F^\times}$ and $\chi \neq \xi\nu^{\pm 1}$, where ξ is a character of order 2 such that $\xi\pi \cong \pi$.

Group VIII. If π is a supercuspidal representation of $\mathrm{GL}(2,F)$, then $1_{F^\times} \rtimes \pi$ is a direct sum of two essentially tempered representations $\tau(S,\pi)$ (type VIIIa) and $\tau(T,\pi)$ (type VIIIb).

Group IX. If ξ is a non-trivial quadratic character of $F^\times$ and π is a supercuspidal representation of $\mathrm{GL}(2,F)$ such that $\xi\pi \cong \pi$, then $\nu\xi \rtimes \nu^{-1/2}\pi$ has two irreducible constituents. There is an essentially square-integrable subrepresentation $\delta(\nu\xi, \nu^{-1/2}\pi)$ (type IXa), and there is a non-tempered quotient $L(\nu\xi, \nu^{-1/2}\pi)$ (type IXb).

Group X. These are the irreducible representations of the form $\pi \rtimes \sigma$, where π is a supercuspidal representation of $\mathrm{GL}(2,F)$ and σ is a character of $F^\times$. The condition for irreducibility is that π is not of the form $\nu^{\pm 1/2}\rho$ with ρ a supercuspidal representation of $\mathrm{GL}(2,F)$ with trivial central character.

Group XI. If π is a supercuspidal representation of $\mathrm{GL}(2,F)$ with trivial central character, and σ is a character of $F^\times$, then $\nu^{1/2}\pi \rtimes \nu^{-1/2}\sigma$ has two irreducible constituents. It contains an irreducible and essentially square-integrable subrepresentation $\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ (type XIa), and it has a non-tempered quotient $L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ (type XIb).

The main result about these groups of representations is the following:

Theorem 2.2.1 (Sally-Tadić). *Let π be a non-supercuspidal, irreducible, admissible representation of* $\mathrm{GSp}(4,F)$. *Then π belongs to Group I, II, III, IV, V, VI, VII, VIII, IX, X or XI. Moreover, if π belongs to Group A, and π also belongs to Group B, then $A = B$.*

Proof. This follows from the results of [ST]. □

It is important to realize that the representations of a particular group will not always be distinct for different choices of the supercuspidal inducing data. For example, the Group I representation $\chi_1 \times \chi_2 \rtimes \sigma$ corresponding to the data (χ_1, χ_2, σ) is isomorphic to the Group I representation $\chi_2 \times \chi_1 \rtimes \sigma$ corresponding to the data (χ_2, χ_1, σ). However, all such equivalences are described in [ST].

As we mentioned above, Table A.1 lists the representations of Groups I, II, III, IV, V, VI, VII, VIII, IX, X and XI. Besides this, the table has some additional columns. The "tempered" column in Table A.1 states the conditions on the inducing data under which a representation is tempered. The "ess. L^2" column indicates which of the tempered representations are essentially square-integrable. The "generic" column indicates the generic representations; see Sect. 2.1 for the definition of generic representations. In each of the full induced representations given in the third column of Table A.1 there is exactly one generic constituent, and it is always a subrepresentation. These results follow from [ST] and other basic references.

2.3 Dual Groups

Having described the classification of the non-supercuspidal, irreducible, admissible representations of $\mathrm{GSp}(4,F)$, we will now use the desiderata of the conjectural local Langlands correspondence for $\mathrm{GSp}(4)$ to assign L-parameters to all such representations. One of the desiderata is an algorithm that assigns

L-packets to non-discrete series L-parameters. Implementation of this algorithm requires detailed knowledge of dual groups, which will be recalled in this section.

Let G be a connected, reductive, linear, algebraic group over an algebraically closed field $\bar{F}$. We choose a maximal torus T and a Borel subgroup B containing T. Let $X^*(T)$ be the group of algebraic homomorphisms $T \to \mathbb{G}_m$, and let $X_*(T)$ be the group of algebraic homomorphisms $\mathbb{G}_m \to T$. There is a canonical pairing

$$\langle\ ,\ \rangle :\ X^*(T) \times X_*(T) \longrightarrow \mathbb{Z}$$

given by composition. Let $\Phi \subset X^*(T)$ be the root system with respect to T. For $\alpha \in \Phi$ let $\alpha^\vee$ be the corresponding coroot; we have $\langle \alpha, \alpha^\vee \rangle = 2$. Let $\Phi^\vee \subset X_*(T)$ be the set of coroots. Let $\Delta \subset \Phi$ be the basis of Φ determined by the choice of B, and let $\Delta^\vee = \{\alpha^\vee :\ \alpha \in \Delta\}$. Then

$$\Psi = (X^*(T),\, \Delta,\, X_*(T),\, \Delta^\vee)$$

is the based root datum of the group G with respect to the choices of T and B. By definition, a *dual group* of G is a pair $(\hat{G}, \iota)$ consisting of a connected, reductive, linear, complex, algebraic group $\hat{G}$, equipped with a choice of maximal torus $\hat{T}$ and Borel subgroup $\hat{B} \supset \hat{T}$, and an isomorphism ι from the root datum $(X^*(\hat{T}),\, \hat{\Delta},\, X_*(\hat{T}),\, \hat{\Delta}^\vee)$ to the dual $(X_*(T),\, \Delta^\vee,\, X^*(T),\, \Delta)$ of the root datum of G. By definition, ι is a pair $(i, i^\vee)$, where

$$i :\ X^*(\hat{T}) \longrightarrow X_*(T), \qquad i^\vee :\ X^*(T) \longrightarrow X_*(\hat{T})$$

are isomorphisms of abelian groups such that

$$\langle y, i(x) \rangle = \langle x, i^\vee(y) \rangle\hat{} \tag{2.12}$$

for $x \in X^*(\hat{T})$, $y \in X^*(T)$, such that $i(\hat{\Delta}) = \Delta^\vee$, $i^\vee(\Delta) = \hat{\Delta}^\vee$, and such that the diagram

$$\begin{array}{ccc} \hat{\Phi} & \xrightarrow{\ \vee\ } & \hat{\Phi}^\vee \\ {\scriptstyle i}\big\downarrow & & \big\uparrow{\scriptstyle i^\vee} \\ \Phi^\vee & \xrightarrow[\ \vee\ ]{} & \Phi \end{array}$$

commutes.

If $\hat{M}$ is a standard Levi subgroup of $\hat{G}$, then ι determines a standard Levi subgroup M of G such that $\hat{M}$ is a dual group of M. More precisely, let $\hat{P}$ be a standard parabolic subgroup of $\hat{G}$ with Levi subgroup $\hat{M}$. Then $\hat{T}$ is a maximal torus of $\hat{M}$, and $\hat{M} \cap \hat{B}$ is a Borel subgroup of $\hat{M}$. The corresponding based root datum of $\hat{M}$ is

$$\hat{\Psi}_{\hat{P}} = (X^*(\hat{T}),\, \hat{\Delta}_{\hat{P}},\, X_*(\hat{T}),\, \hat{\Delta}_{\hat{P}}{}^\vee),$$

where $\hat{\Delta}_{\hat{P}}$ is the subset of $\hat{\Delta}$ corresponding to $\hat{P}$. Let P be the parabolic subgroup of G corresponding to the image of $\hat{\Delta}_{\hat{P}}$ under the composition of the bijections $\hat{\Phi} \xrightarrow{i} \Phi^\vee \xrightarrow{\vee} \Phi$. Let Δ_P be this image. In this situation we say that P and $\hat{P}$ are *dual*; this provides a bijection between the sets of standard parabolic subgroups of G and $\hat{G}$. Let M be the Levi subgroup of P containing T. Then T is a maximal torus of M, and $M \cap B$ is a Borel subgroup of M. The corresponding based root datum of M is

$$\Psi_P = (X^*(T),\, \Delta_P,\, X_*(T),\, \Delta_P^\vee).$$

The maps i and $i^\vee$ define an isomorphism, which we also call ι, between the root datum of $\hat{M}$ and the dual of the root datum of M. Hence $(\hat{M}, \iota)$ is a dual group of M.

We will now consider these definitions in detail for GSp(4). We choose T and B as before, so that T is the subgroup of diagonal matrices inside $\mathrm{GSp}(4, \bar{F})$, and B is the subgroup of upper triangular matrices. We write a typical element of T as

$$t(a,b,c) = \begin{bmatrix} a & & & \\ & b & & \\ & & cb^{-1} & \\ & & & ca^{-1} \end{bmatrix}.$$

We define $e_1, e_2, e_3 \in X^*(T)$ by

$$e_1(t(a,b,c)) = a, \qquad e_2(t(a,b,c)) = b, \qquad e_3(t(a,b,c)) = c.$$

Then $X^*(T) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3$. Define further $f_1, f_2, f_3 \in X_*(T)$ by

$$f_1(x) = t(x,1,1), \qquad f_2(x) = t(1,x,1), \qquad f_3(x) = t(1,1,x).$$

The duality between roots and coroots is $\langle e_i, f_j \rangle = \delta_{i,j}$. The positive roots and their corresponding coroots are as follows:

$$\begin{aligned} \alpha_1 &= e_1 - e_2, & \alpha_1^\vee &= f_1 - f_2, \\ \alpha_2 &= 2e_2 - e_3, & \alpha_2^\vee &= f_2, \\ \alpha_1 + \alpha_2 &= e_1 + e_2 - e_3, & (\alpha_1 + \alpha_2)^\vee &= f_1 + f_2, \\ 2\alpha_1 + \alpha_2 &= 2e_1 - e_3, & (2\alpha_1 + \alpha_2)^\vee &= f_1. \end{aligned}$$

With our choice of B we get the system of simple roots $\Delta = \{\alpha_1, \alpha_2\}$ and the corresponding coroots $\Delta^\vee = \{\alpha_1^\vee, \alpha_2^\vee\}$.

We make similar choices and definitions for the group $\mathrm{GSp}(4, \mathbb{C})$. The diagonal torus will be denoted by $\hat{T}$, and the upper triangular subgroup by $\hat{B}$. The basis for $X^*(\hat{T})$ is $\hat{e}_1, \hat{e}_2, \hat{e}_3$ and the basis for $X_*(\hat{T})$ is $\hat{f}_1, \hat{f}_2, \hat{f}_3$.

Lemma 2.3.1. *Using the above notation concerning the groups* $\mathrm{GSp}(4, \bar{F})$ *and* $\mathrm{GSp}(4, \mathbb{C})$*, there exist exactly two isomorphisms of based root data*

$$(X^*(\hat{T}),\, \hat{\Delta},\, X_*(\hat{T}),\, \hat{\Delta}^\vee) \longrightarrow (X_*(T),\, \Delta^\vee,\, X^*(T),\, \Delta).$$

Proof. If we identify the free abelian groups involved with $\mathbb{Z}^3$ via the above choices of coordinates, the isomorphism $i:\ X^*(\hat{T}) \to X_*(T)$ is given by an element $S \in \mathrm{GL}(3,\mathbb{Z})$, and the isomorphism $i^\vee : X^*(T) \longrightarrow X_*(\hat{T})$ is given by an element $S^\vee \in \mathrm{GL}(3,\mathbb{Z})$. The condition (2.12) is equivalent to ${}^tS = S^\vee$. The conditions $i(\hat{\Delta}) = \Delta^\vee$, $i^\vee(\Delta) = \hat{\Delta}^\vee$, together with the fact that $S \in \mathrm{GL}(3,\mathbb{Z})$, imply, after some calculations, that either

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & & 1 \\ 1 & 1 & 2 \end{bmatrix} \tag{2.13}$$

or

$$S = \begin{bmatrix} & & -1 \\ & -1 & -1 \\ -1 & -1 & -2 \end{bmatrix}. \tag{2.14}$$

One checks that with either choice the remaining conditions for an isomorphism of root data are also satisfied. □

In the following, we work with the isomorphism ι determined by the choice (2.13) for the matrix S, so that $(\mathrm{GSp}(4,\mathbb{C}),\iota)$ is our fixed choice of dual group for $\mathrm{GSp}(4,\bar{F})$. The fixed choice of a dual group determines dual groups for the Levi subgroups of our standard parabolics in $\mathrm{GSp}(4,\bar{F})$, as follows. Using a previous notation, we have $\Delta_P = \{\alpha_1\} \subset \Delta$ and $\Delta_Q = \{\alpha_2\} \subset \Delta$. Letting $\hat{P}$ and $\hat{Q}$ be standard parabolics in $\mathrm{GSp}(4,\mathbb{C})$ dual to P and Q, respectively, we have $\Delta_{\hat{P}} = \{\hat{\alpha}_2\} \subset \hat{\Delta}$ and $\Delta_{\hat{Q}} = \{\hat{\alpha}_1\} \subset \hat{\Delta}$. Therefore

$$\hat{P} = \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & * \end{bmatrix}, \quad M_{\hat{P}} = \begin{bmatrix} * & & & \\ & * & * & \\ & * & * & \\ & & & * \end{bmatrix}, \quad \hat{Q} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{bmatrix}, \quad M_{\hat{Q}} = \begin{bmatrix} * & * & & \\ * & * & & \\ & & * & * \\ & & * & * \end{bmatrix}$$

in $\mathrm{GSp}(4,\mathbb{C})$. Note that $\hat{P}$ is now the Klingen parabolic subgroup of $\mathrm{GSp}(4,\mathbb{C})$, and $\hat{Q}$ is the Siegel parabolic subgroup. As explained above, $M_{\hat{P}}$ is the dual group of M_P, and $M_{\hat{Q}}$ is the dual group of M_Q; the involved maps of root data are inherited from ι. Also, $\Delta_B = \emptyset$, $\Delta_{\hat{B}} = \emptyset$, B and $\hat{B}$ are dual, and $\hat{T}$ is the dual group of T; again, the map of root data comes from ι.

We fix isomorphisms between the Levi subgroups and products of general linear groups, as follows.

$$\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C}) \longrightarrow M_{\hat{P}}, \tag{2.15}$$

$$(g,t) \longmapsto \begin{bmatrix} t & & \\ & g & \\ & & t^{-1}\det(g) \end{bmatrix},$$

$$\mathrm{GL}(2,\bar{F}) \times \mathrm{GL}(1,\bar{F}) \longrightarrow M_P, \tag{2.16}$$

$$(g,t) \longmapsto \begin{bmatrix} g & \\ & tg' \end{bmatrix}.$$

See (2.1) for the g' notation. We consider $\mathrm{GL}(2,\bar{F})\times\mathrm{GL}(1,\bar{F})$ as a linear, reductive group over $\bar{F}$. We fix the Borel subgroup consisting of upper triangular matrices, and the maximal torus $T_2\times T_1$ consisting of diagonal matrices in each group. Let the root datum of $\mathrm{GL}(2,\bar{F})\times\mathrm{GL}(1,\bar{F})$ be

$$\Xi=(X^*(T_2\times T_1),\,\Sigma,\,X_*(T_2\times T_1),\,\Sigma^\vee).$$

If we define

$$\tilde{e}_1(\begin{bmatrix}a&\\&b\end{bmatrix},t)=a,\qquad \tilde{e}_2(\begin{bmatrix}a&\\&b\end{bmatrix},t)=b,\qquad \tilde{e}_3(\begin{bmatrix}a&\\&b\end{bmatrix},t)=t,$$
$$\tilde{f}_1(x)=(\begin{bmatrix}x&\\&1\end{bmatrix},1),\qquad \tilde{f}_2(x)=(\begin{bmatrix}1&\\&x\end{bmatrix},1),\qquad \tilde{f}_3(x)=(\begin{bmatrix}1&\\&1\end{bmatrix},x),$$

then $X^*(T_2\times T_1)=\mathbb{Z}\tilde{e}_1\oplus\mathbb{Z}\tilde{e}_2\oplus\mathbb{Z}\tilde{e}_3$, $X_*(T_2\times T_1)=\mathbb{Z}\tilde{f}_1\oplus\mathbb{Z}\tilde{f}_2\oplus\mathbb{Z}\tilde{f}_3$, $\Sigma=\{\tilde{e}_1-\tilde{e}_2\}$ and $\Sigma^\vee=\{\tilde{f}_1-\tilde{f}_2\}$. Similar definitions and comments apply to $\mathrm{GL}(2,\mathbb{C})\times\mathrm{GL}(1,\mathbb{C})$. We will indicate the analogously defined objects with a hat. In particular, the root datum of $\mathrm{GL}(2,\mathbb{C})\times\mathrm{GL}(1,\mathbb{C})$ is

$$\hat{\Xi}=(X^*(\hat{T}_2\times\hat{T}_1),\,\hat{\Sigma},\,X_*(\hat{T}_2\times\hat{T}_1),\,\hat{\Sigma}^\vee).$$

The isomorphism of groups (2.16) induces an isomorphism of root data $\Psi_P\cong\Xi$, and hence $\Psi_P^\vee\cong\Xi^\vee$. The isomorphism of groups (2.15) induces an isomorphism of root data $\hat{\Psi}_{\hat{P}}\cong\hat{\Xi}$. Furthermore, we have the isomorphism $\iota:\hat{\Psi}_{\hat{P}}\to\Psi_P^\vee$. By composing these isomorphisms, we obtain an isomorphism $\kappa:\hat{\Xi}\to\Xi^\vee$, as in the following diagram.

$$\begin{array}{ccc}\Psi_P^\vee & \xleftarrow[\sim]{\ \iota\ } & \hat{\Psi}_{\hat{P}}\\ \sim\downarrow & & \downarrow\sim\\ \Xi^\vee & \xleftarrow[\kappa]{\ \sim\ } & \hat{\Xi}\end{array}$$

In addition, there exists the standard isomorphism $\sigma:\hat{\Xi}\to\Xi^\vee$; it is given by the identity matrix in the bases defined above. We obtain an automorphism of $\hat{\Xi}$ given by $\kappa^{-1}\circ\sigma$. Computations show that this automorphism is induced by the automorphism of $\mathrm{GL}(2,\mathbb{C})\times\mathrm{GL}(1,\mathbb{C})$ defined by

$$(g,x)\longmapsto(xg',\det(g)x^{-1}).\tag{2.17}$$

Other automorphisms of $\mathrm{GL}(2,\mathbb{C})\times\mathrm{GL}(1,\mathbb{C})$ induce $\kappa^{-1}\circ\sigma$, but any such two automorphisms differ by an inner automorphism given by a conjugation with a torus element.

Next we consider the case of the parabolic Q. Similarly to above we fix isomorphisms

$$\mathrm{GL}(2,\mathbb{C})\times\mathrm{GL}(1,\mathbb{C})\longrightarrow M_{\hat{Q}},\tag{2.18}$$

$$(g,t) \longmapsto \begin{bmatrix} g & \\ & tg' \end{bmatrix},$$

$$\mathrm{GL}(2,\bar{F}) \times \mathrm{GL}(1,\bar{F}) \longrightarrow M_Q, \tag{2.19}$$

$$(g,t) \longmapsto \begin{bmatrix} t & & \\ & g & \\ & & t^{-1}\det(g) \end{bmatrix}.$$

The isomorphism of groups (2.19) induces an isomorphism of root data $\Psi_Q \cong \Xi$, and hence $\Psi_Q^\vee \cong \Xi^\vee$. The isomorphism of groups (2.18) induces an isomorphism of root data $\hat{\Psi}_{\hat{Q}} \cong \hat{\Xi}$. Furthermore, we have the isomorphism $\iota : \hat{\Psi}_{\hat{Q}} \to \Psi_Q^\vee$. By composing these isomorphisms, we obtain an isomorphism $\kappa : \hat{\Xi} \to \Xi^\vee$, as in the following diagram.

$$\begin{array}{ccc} \Psi_Q^\vee & \xleftarrow[\sim]{\iota} & \hat{\Psi}_{\hat{Q}} \\ \sim\downarrow & & \downarrow\sim \\ \Xi^\vee & \xleftarrow[\kappa]{\sim} & \hat{\Xi} \end{array}$$

As above, we obtain an automorphism of $\hat{\Xi}$ given by $\kappa^{-1} \circ \sigma$. Computations show that this automorphism is induced by the same automorphism of $\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})$ as defined in (2.17).

Finally we consider the case of the Borel subgroup B. Now we fix the isomorphisms

$$\mathrm{GL}(1,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C}) \longrightarrow \hat{T} \tag{2.20}$$

and

$$\mathrm{GL}(1,\bar{F}) \times \mathrm{GL}(1,\bar{F}) \times \mathrm{GL}(1,\bar{F}) \longrightarrow T \tag{2.21}$$

given by the common formula

$$(a,b,c) \longmapsto \begin{bmatrix} a & & & \\ & b & & \\ & & cb^{-1} & \\ & & & ca^{-1} \end{bmatrix}.$$

Let the root datum of $\mathrm{GL}(1,\bar{F}) \times \mathrm{GL}(1,\bar{F}) \times \mathrm{GL}(1,\bar{F})$ be

$$\Omega = (X^*(T_1 \times T_1 \times T_1),\, \emptyset,\, X_*(T_1 \times T_1 \times T_1),\, \emptyset).$$

If we define

$$\begin{aligned} &\tilde{e}_1(a,b,c) = a, && \tilde{e}_2(a,b,c) = b, && \tilde{e}_3(a,b,c) = c, \\ &\tilde{f}_1(x) = (x,1,1), && \tilde{f}_2(x) = (1,x,1), && \tilde{f}_3(x) = (1,1,x), \end{aligned}$$

then $X^*(T_1{\times}T_1{\times}T_1) = \mathbb{Z}\tilde{e}_1{\oplus}\mathbb{Z}\tilde{e}_2{\oplus}\mathbb{Z}\tilde{e}_3$ and $X_*(T_1{\times}T_1{\times}T_1) = \mathbb{Z}\tilde{f}_1{\oplus}\mathbb{Z}\tilde{f}_2{\oplus}\mathbb{Z}\tilde{f}_3$. Similar definitions and comments apply to $\mathrm{GL}(1,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})$. We will indicate the analogously defined objects with a hat. In particular, the root datum of $\mathrm{GL}(1,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})$ is

$$\hat{\Xi} = (X^*(\hat{T}_1 \times \hat{T}_1 \times \hat{T}_1),\, \emptyset,\, X_*(\hat{T}_1 \times \hat{T}_1 \times \hat{T}_1),\, \emptyset).$$

The isomorphism of groups (2.21) induces an isomorphism of root data $\Psi_B \cong \Omega$, and hence $\Psi_B^\vee \cong \Omega^\vee$. The isomorphism of groups (2.20) induces an isomorphism of root data $\hat{\Psi}_{\hat{B}} \cong \hat{\Omega}$. Furthermore, we have the isomorphism $\iota:\ \hat{\Psi}_{\hat{B}} \to \Psi_B^\vee$. By composing these isomorphisms, we obtain an isomorphism $\kappa:\ \hat{\Omega} \to \Omega^\vee$, as in the following diagram.

$$\begin{array}{ccc} \Psi_B^\vee & \xleftarrow[\sim]{\iota} & \hat{\Psi}_{\hat{B}} \\ {\scriptstyle\sim}\big\downarrow & & \big\downarrow{\scriptstyle\sim} \\ \Omega^\vee & \xleftarrow[\kappa]{\sim} & \hat{\Omega} \end{array}$$

As above, we obtain an automorphism of $\hat{\Omega}$ given by $\kappa^{-1} \circ \sigma$. Computations show that this automorphism is induced by the automorphism of $\mathrm{GL}(1,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})$ defined by

$$(a,b,c) \longmapsto (abc^{-1}, ab^{-1}, ca^{-1}). \tag{2.22}$$

2.4 The Local Langlands Correspondence

The main purpose of this section is to use some of the desiderata of the local Langlands correspondence for GSp(4) to assign L-parameters to the non-supercuspidal representations of $\mathrm{GSp}(4,F)$. After reviewing the two basic principles derived from the desiderata, we present the list of resulting L-parameter assignments. The final part of this section recalls the definitions and essential properties of the L-and ε-factors assigned to representations of the Weil–Deligne group. The degree 4 and degree 5 L- and ε-factors of L-parameters of GSp(4) are tabulated in Tables A.8, A.9, A.10 and A.11. We point out that the statements of this section hold for representations of $\mathrm{GSp}(4,F)$ with arbitrary central character.

Let W_F be the Weil group of F, as in [T]. It comes equipped with an isomorphism

$$r:\ F^\times \longrightarrow W_F^{\mathrm{ab}}.$$

We use the convention of (1.4.1) of [T], so that $r(\varpi)$ acts by the inverse of the map $x \mapsto x^q$ on residue field extensions corresponding to finite unramified extensions of F. Using the isomorphism r we can identify characters of $F^\times$ and characters of W_F. In particular, we have the character ν on W_F, which is the normalized absolute value on $F^\times$. Let $W'_F = \mathbb{C} \rtimes W_F$ be the Weil–Deligne group; see [T]. The multiplication in this semidirect product is $(z,w)(z',w') = (z+\nu(w)z', ww')$. Since

$$W_F^{\mathrm{ab}} \cong W_F'^{\,\mathrm{ab}} \cong F^\times,$$

characters of W_F, W'_F and $F^\times$ can be identified. We shall not distinguish in notation between a character of $F^\times$ and the corresponding characters of W_F and W'_F.

A *representation* of the Weil–Deligne group is a continuous homomorphism $W'_F \to \mathrm{GL}(n,\mathbb{C})$ such that the restriction to $\mathbb{C}$ is complex analytic. There is a bijection between the set of representations φ of W'_F and the set of pairs (ρ, N), where ρ is a continuous homomorphism $W_F \to \mathrm{GL}(n,\mathbb{C})$ and N is a nilpotent $n \times n$ matrix such that

$$\rho(w)N\rho(w)^{-1} = \nu(w)N, \qquad w \in W_F.$$

The bijection is such that $\varphi(zw) = \exp(zN)\rho(w)$.

If (ρ, N) is a representation of W'_F and χ is a character of W_F, then $\chi(\rho,N) := (\chi \otimes \rho, N)$ is a representation of W'_F called the *twist* of (ρ,N) by χ. The representation (ρ,N) is called *admissible* if ρ is a semisimple representation. It is *irreducible* if $N=0$ and ρ is irreducible. These properties are preserved under twisting.

Given an n-dimensional semisimple representation ρ of the Weil group W_F, then the set of admissible representations $\varphi = (\rho', N')$ of the Weil–Deligne group W'_F such that $\rho' = \rho$ has the following structure. Evidently, (ρ,N) is a representation of the Weil–Deligne group if and only if N is an element of the vector space

$$V_\rho^{\mathrm{nil}} = \{N \in \mathfrak{gl}(n,\mathbb{C}) :\ N \text{ nilpotent},\ \rho(w)N\rho(w)^{-1} = \nu(w)N \text{ for } w \in W_F\}.$$

Let $C(\rho) = \{g \in \mathrm{GL}(n,\mathbb{C}) :\ g\rho(w)g^{-1} = \rho(w) \text{ for all } w \in W_F\}$. Then $C(\rho)$ acts on V_ρ^{nil} by conjugation. Two elements N_1 and N_2 of V_ρ^{nil} are in the same $C(\rho)$ orbit if and only if the representations (ρ,N_1) and (ρ,N_2) are equivalent. Therefore, the set of equivalence classes of the $\varphi = (\rho',N')$ with $\rho'=\rho$ is in bijection with the $C(\rho)$ orbits of V_ρ^{nil}.

L-parameters and the Langlands Correspondence

Now let G be a split, connected, reductive, linear, algebraic group over F with root datum Ψ. We consider triples $(\varphi, \hat{G}, \iota)$, where $(\hat{G},\iota)$ is a dual group for G, as defined in Sect. 2.3, and $\varphi :\ W'_F \to \hat{G}$ is a homomorphism, such that:

- φ is continuous;
- $\varphi\big|_{W_F}$ is semisimple, i.e., $\varphi(x)$ is semisimple for $x \in W_F$;
- $\varphi\big|_{\mathbb{C}}$ is algebraic, i.e., given by polynomial entries.

If $\hat{G} \subset \mathrm{GL}(n, \mathbb{C})$ for some n, then φ may be regarded as an admissible representation of the Weil–Deligne group. In this case we may therefore use the definitions and observations made above. Let $(\varphi, \hat{G}, \iota)$ and $(\phi, \tilde{G}, \kappa)$ be two such triples. Let $\hat{\Psi}$ be the root datum of $\hat{G}$, and let $\tilde{\Psi}$ be the root datum of $\tilde{G}$. The isomorphism of root data $\hat{\Psi} \to \tilde{\Psi}$ obtained as a composition

$$\hat{\Psi} \xrightarrow{\iota} \Psi^\vee \xrightarrow{\kappa^{-1}} \tilde{\Psi}$$

is induced from an isomorphism $\gamma : \tilde{G} \to \hat{G}$ (unique up to inner automorphisms determined by torus elements). We call $(\varphi, \hat{G}, \iota)$ and $(\phi, \tilde{G}, \kappa)$ *equivalent* if there exists a $g \in \tilde{G}$ such that the diagram

$$\begin{array}{ccc} W_F' & \xrightarrow{\varphi} & \hat{G} \\ {\scriptstyle\phi}\big\downarrow & & \big\uparrow{\scriptstyle\gamma} \\ \tilde{G} & \xrightarrow[c_g]{} & \tilde{G} \end{array}$$

is commutative. Here, c_g is the inner automorphism determined by g. We define an *L-parameter* for G to be an equivalence class of triples $(\varphi, \hat{G}, \iota)$. Often we will abuse language and say "φ is an L-parameter" for G. These definitions may seem slightly pedantic, but they are important in situations where unfamiliar versions of a dual group arise naturally. See the work [BR], which also recognizes the importance of specifying the isomorphism ι in the definition of the dual group.

Given an L-parameter $\varphi = (\rho, N)$ for GSp(4), we define the *component group* of φ as

$$\mathcal{C}(\varphi) = \mathrm{Cent}(\varphi)/\mathrm{Cent}(\varphi)^0\, \mathbb{C}^\times,$$

where $\mathrm{Cent}(\varphi)$ denotes the centralizer of the image of φ in $\mathrm{GSp}(4, \mathbb{C})$, $\mathrm{Cent}(\varphi)^0$ denotes its identity component, and $\mathbb{C}^\times$ stands for the center of $\mathrm{GSp}(4, \mathbb{C})$. Note that $\mathrm{Cent}(\varphi)$ consists of all $g \in \mathrm{GSp}(4, \mathbb{C})$ that centralize the image of ρ, and for which $\mathrm{Ad}(g)N = N$.

Langlands has conjectured that there exists a partition of the set of irreducible, admissible representations of $G(F)$ into finite sets, and a bijection between the collection of these finite sets and the set of L-parameters of G. This bijection, which is called the *local Langlands correspondence* for G, should satisfy certain desiderata; see [Bo2]. The local Langlands correspondence for $\mathrm{GL}(n)$ is known; see [HT] and [H]. For later use we point out that in the literature a standard representative for each L-parameter for $\mathrm{GL}(n)$ is chosen, namely, $\hat{G} = \mathrm{GL}(n, \mathbb{C})$ and ι is the obvious choice. If the local Langlands correspondence exists for a group G, we let $\Pi(\varphi)$ be the finite set of irreducible, admissible representations of G corresponding to the L-parameter φ. These sets are called *L-packets*.

Two Principles

Below we assign to each non-supercuspidal, irreducible, admissible representation of $\mathrm{GSp}(4, F)$ an L-parameter. To do so, we use two principles, which are desiderata of the local Langlands correspondence.

The first principle concerns non-discrete series representations. One of the desiderata of the local Langlands correspondence is that $\Pi(\varphi)$ consists of essentially square-integrable representations if and only if the image of φ is not contained in any proper Levi subgroup of $\hat{G}$. Such L-parameters are called *discrete series parameters*. The desiderata of the local Langlands conjecture include a procedure for assigning L-packets to non-discrete series L-parameters, *if the local Langlands correspondence is known for the proper Levi subgroups of* G. The idea of the procedure is as follows.

- Fix a representative $(\varphi, \hat{G}, \iota)$ for a given non-discrete series L-parameter of G.
- Determine the unique minimal standard Levi subgroup $\hat{M}$ that contains a conjugate of the image of φ. By replacing our representative by an equivalent one, we may assume that the image of φ is contained in $\hat{M}$.
- As explained earlier, $\hat{M}$ is the dual group of a standard Levi subgroup M of G; the involved isomorphism of root data is inherited from our fixed choice of ι. In this way we may consider φ as a representative for an L-parameter of M.
- Since the local Langlands correspondence for M is assumed to be known, the parameter φ determines an L-packet $\Pi_M(\varphi)$ of representations of $M(F)$. These representations are essentially square-integrable, since $\hat{M}$ was minimal.
- Let M' be the maximal standard Levi subgroup of G containing M with the property that parabolically inducing the representations in $\Pi_M(\varphi)$ from M to M' produces essentially tempered representations. Let $\{\tau_i\}$ be the irreducible constituents of these induced representations.
- For each τ_i there is a Langlands quotient $L(\tau_i)$, which is an irreducible, admissible representation of G. By definition, the L-packet $\Pi(\varphi)$ is the set consisting of all these $L(\tau_i)$.
- This procedure is independent of the choice of representative for the given L-parameter.

This procedure may be applied to $\mathrm{GSp}(4)$, since the local Langlands correspondences for the proper Levi subgroups are known. It turns out that every non–square-integrable, irreducible, admissible, representation of $\mathrm{GSp}(4, F)$ is an element of exactly one packet constructed according to this procedure. Further below we will present the results of this procedure for $\mathrm{GSp}(4)$. Since we are taking a representation-theoretic viewpoint, we will in fact present the results by specifying for each non–square-integrable representation its L-parameter. We will not go through the details of this procedure for all parameters; instead we will consider one illustrative example.

Let μ be an essentially discrete series parameter for GL(2); we use the standard choice of dual group for GL(2). Consider the GSp(4) L-parameter φ given by

$$W_F' \ni w \longmapsto \begin{bmatrix} \det(\mu(w))\mu(w)' & \\ & \mu(w) \end{bmatrix} \in \mathrm{GSp}(4,\mathbb{C}). \tag{2.23}$$

Here, we use the fixed choice of dual group from Sect. 2.3. The image of φ is contained in the Levi $M_{\hat{Q}}$ and cannot be conjugated into the torus $\hat{T}$. Hence φ defines an essentially discrete series L-parameter for M_Q; the dual group of M_Q is inherited from that of GSp(4) as explained in Sect. 2.3. We need to apply the local Langlands correspondence for M_Q; to do so, we identify M_Q with $\mathrm{GL}(2)\times\mathrm{GL}(1)$ as in (2.19), and $M_{\hat{Q}}$ with $\mathrm{GL}(2,\mathbb{C})\times\mathrm{GL}(1,\mathbb{C})$ as in (2.18). Via this identification, we can regard φ as an L-parameter for $\mathrm{GL}(2)\times\mathrm{GL}(1)$. Note, however, that the dual group of $\mathrm{GL}(2)\times\mathrm{GL}(1)$ with the isomorphism inherited from ι is not the standard form of the dual group for $\mathrm{GL}(2)\times\mathrm{GL}(1)$. Working through the definitions, we obtain an L-parameter equivalent to our $\mathrm{GL}(2)\times\mathrm{GL}(1)$ L-parameter φ, but with the standard choice of dual group, by composing φ with the automorphism (2.17) of $\mathrm{GL}(2,\mathbb{C})\times\mathrm{GL}(1,\mathbb{C})$. This composition is given by

$$W_F' \ni w \longmapsto (\mu(w),1) \in \mathrm{GL}(2,\mathbb{C})\times\mathrm{GL}(1,\mathbb{C}). \tag{2.24}$$

Remembering the identification $M_Q \cong \mathrm{GL}(2)\times\mathrm{GL}(1)$, the corresponding singleton L-packet of M_Q consists of the representation

$$\begin{bmatrix} t & & \\ & g & \\ & & t^{-1}\det(g) \end{bmatrix} \longmapsto \pi(g), \tag{2.25}$$

where π is the representation of $\mathrm{GL}(2,F)$ assigned to μ by the local Langlands correspondence for GL(2). If we parabolically induce the representation (2.25) to $\mathrm{GSp}(4,F)$, we obtain a tempered representation $1_{F^\times}\rtimes\pi$; hence, the intermediate group M' mentioned in the outline above is in this case given by the full group GSp(4). To complete the procedure, we need to decompose the induced representation into irreducibles. By [ST], we obtain two irreducible constituents. If π is a twist of $\mathrm{St}_{\mathrm{GL}(2)}$, then we obtain the two representations VIa and VIb. If π is supercuspidal, we obtain VIIIa and VIIIb. In each case we obtain a two-element L-packet attached to the parameter (2.23). It turns out that these are the only non-discrete series L-packets that contain more than one element.

The second principle we will use to assign parameters to representations asserts that the supercuspidal support of a representation determines the semisimple part of its L-parameter. Suppose that a standard parabolic subgroup of $\mathrm{GSp}(4,F)$ and an irreducible, admissible, supercuspidal representation of its Levi subgroup are given; consider the associated parabolically induced representation of $\mathrm{GSp}(4,F)$. Then, according to the principle, the

L-parameters of the irreducible subquotients of this induced representation should all have equivalent semisimple part. To implement this procedure one determines one of the irreducible constituents of the full induced representation whose L-parameter is determined by the first principle; this gives ρ, the common semisimple part of the parameters of the irreducible constituents. Next one uses the remark from the beginning of this section to find the possible representations (ρ', N') of W_F' such that $\rho' = \rho$; in the discussion one has to replace $\mathfrak{gl}(n, \mathbb{C})$ by $\mathfrak{gsp}(4, \mathbb{C})$. Now consider on the one hand the set of irreducible constituents π not covered by the first principle, and on the other hand the set of (ρ, N) that are not discrete series parameters. As it happens, in our GSp(4) case, either both sets are empty or both sets have one element; in the latter situation, the second principle implies that the L-parameter of π is (ρ, N). As an example, we consider the irreducible constituents of $\nu^2 \times \nu \rtimes \nu^{-1/2}\sigma$; one of the constituents is $\sigma\mathrm{St}_{\mathrm{GSp}(4)}$ (type IVa). The common semisimple part is

$$\rho(w) = \begin{bmatrix} (\nu^{3/2}\sigma)(w) & & & \\ & (\nu^{1/2}\sigma)(w) & & \\ & & (\nu^{-1/2}\sigma)(w) & \\ & & & (\nu^{-3/2}\sigma)(w) \end{bmatrix}.$$

One determines that V_ρ^{nil} is spanned by

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & -1 \\ & & & 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 0 & & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{bmatrix}.$$

The centralizer $C(\rho)$ consists of all diagonal matrices in $\mathrm{GSp}(4, \mathbb{C})$. Therefore, we get four orbits represented by

$$0, \qquad \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & -1 \\ & & & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & -1 \\ & & & 0 \end{bmatrix}.$$

Carrying out the first principle shows that the first three representatives correspond to the non-discrete series representations of type IVd, IVb, IVc, respectively. Therefore, the last representative N, which defines a discrete series parameter (ρ, N), corresponds to the remaining irreducible constituent, which is $\sigma\mathrm{St}_{\mathrm{GSp}(4)}$.

The List of Representations and their L-parameters

To present our list of non-supercuspidal representations and their L-parameters, we need a few additional definitions and observations. An irreducible principal series representation $\chi_1 \times \chi_2$ of $\mathrm{GL}(2, F)$ corresponds to (μ, N) with

$$\mu(w) = \begin{bmatrix} \chi_1(w) & \\ & \chi_2(w) \end{bmatrix}, \qquad N = 0.$$

A twist of the trivial representation $\chi\mathbf{1}_{\mathrm{GL}(2)}$ corresponds to (μ, N) with

$$\mu(w) = \begin{bmatrix} \chi(w)\nu(w)^{1/2} & \\ & \chi(w)\nu(w)^{-1/2} \end{bmatrix}, \qquad N = 0.$$

The twisted Steinberg representation $\chi\mathrm{St}_{\mathrm{GL}(2)}$ has the same μ but $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. It is common to denote by sp(2) the representation (μ, N) with

$$\mu(w) = \begin{bmatrix} \nu(w) & \\ & 1 \end{bmatrix}, \qquad N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{2.26}$$

Using this notation, the L-parameter of $\mathrm{St}_{\mathrm{GL}(2)}$ is $\nu^{-1/2}\mathrm{sp}(2)$. We will also require the four-dimensional parameter sp(4) given by

$$\rho(w) = \begin{bmatrix} \nu(w)^3 & & & \\ & \nu(w)^2 & & \\ & & \nu(w) & \\ & & & 1 \end{bmatrix}, \qquad N = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & -1 \\ & & & 0 \end{bmatrix}. \tag{2.27}$$

Note that the image of this parameter is contained in $\mathrm{GSp}(4, \mathbb{C})$. The representations sp(2) and sp(4) are not irreducible, but *indecomposable*, i.e., they cannot be written as a direct sum of proper subrepresentations.

Having made these definitions, we now present the list of L-parameters associated to non-supercuspidal representations of $\mathrm{GSp}(4, F)$. The component groups of all parameters are trivial unless stated otherwise.

Group I

To an irreducible representation of the form $\chi_1 \times \chi_2 \rtimes \sigma$ we attach the L-parameter (ρ, N) with $N = 0$ and ρ given by

$$\rho : W_F \longrightarrow \mathrm{GSp}(4, \mathbb{C}), \tag{2.28}$$
$$w \longmapsto \begin{bmatrix} (\chi_1\chi_2\sigma)(w) & & & \\ & (\chi_1\sigma)(w) & & \\ & & (\chi_2\sigma)(w) & \\ & & & \sigma(w) \end{bmatrix}.$$

As mentioned above, χ_1, χ_2, σ stand both for characters of $F^\times$ and the corresponding characters of W_F.

Group II

Let χ and σ be characters of $F^\times$ such that $\chi^2 \neq \nu^{\pm 1}$ and $\chi \neq \nu^{\pm 3/2}$. Then $\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$ has two irreducible constituents, $\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$ (type IIa) and $\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$ (type IIb). The L-parameters attached to each of these two irreducible representations will have the same semisimple part

$$\begin{aligned} \rho:\ W_F &\longrightarrow \mathrm{GSp}(4,\mathbb{C}), \\ w &\longmapsto \begin{bmatrix} (\chi^2\sigma)(w) &&& \\ & (\nu^{1/2}\chi\sigma)(w) && \\ && (\nu^{-1/2}\chi\sigma)(w) & \\ &&& \sigma(w) \end{bmatrix}. \end{aligned} \tag{2.29}$$

To $\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$ we attach the parameter (ρ, N) with $N = 0$, and to $\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$ we attach the parameter (ρ, N_1) with

$$N_1 = \begin{bmatrix} 0 &&& \\ & 0 & 1 & \\ && 0 & \\ &&& 0 \end{bmatrix}. \tag{2.30}$$

Group III

Let χ and σ be characters of $F^\times$ such that $\chi \neq 1$ and $\chi \neq \nu^{\pm 2}$. Then the induced representation $\chi \times \nu \rtimes \nu^{-1/2}\sigma$ has two irreducible constituents $\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$ (type IIIa) and $\chi \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}$ (type IIIb). The L-parameters attached to each of these two irreducible representations will have the same semisimple part

$$\begin{aligned} \rho:\ W_F &\longrightarrow \mathrm{GSp}(4,\mathbb{C}), \\ w &\longmapsto \begin{bmatrix} (\nu^{1/2}\chi\sigma)(w) &&& \\ & (\nu^{-1/2}\chi\sigma)(w) && \\ && (\nu^{1/2}\sigma)(w) & \\ &&& (\nu^{-1/2}\sigma)(w) \end{bmatrix}. \end{aligned} \tag{2.31}$$

To $\chi \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}$ we attach (ρ, N) with $N = 0$, and to $\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$ we attach the parameter (ρ, N_4), where

$$N_4 = \begin{bmatrix} 0 & 1 && \\ & 0 && \\ && 0 & -1 \\ &&& 0 \end{bmatrix}. \tag{2.32}$$

Group IV

For any character σ of $F^\times$ the induced representation $\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma$ decomposes into the four irreducible constituents of type IV. The L-parameters attached to each of these four representations will have the same semisimple part

$$\rho:\ W_F \longrightarrow \mathrm{GSp}(4,\mathbb{C}), \qquad (2.33)$$
$$w \longmapsto \begin{bmatrix} (\nu^{3/2}\sigma)(w) &&& \\ & (\nu^{1/2}\sigma)(w) && \\ && (\nu^{-1/2}\sigma)(w) & \\ &&& (\nu^{-3/2}\sigma)(w) \end{bmatrix}.$$

The Langlands quotient (type IVd) is a twist of the trivial representation $\sigma\mathbf{1}_{\mathrm{GSp}(4)}$, and to it we attach the parameter (ρ, N) with $N = 0$. To the IVc type representation $L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$ we attach (ρ, N_1) with N_1 as in (2.30). To the IVb type representation $L(\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$ we attach (ρ, N_4) with N_4 as in (2.32). Finally, to $\sigma\mathrm{St}_{\mathrm{GSp}(4)}$ (type IVa) we attach (ρ, N_5) with

$$N_5 = \begin{bmatrix} 0 & 1 && \\ & 0 & 1 & \\ && 0 & -1 \\ &&& 0 \end{bmatrix}. \qquad (2.34)$$

Note that this is $\sigma\nu^{-3/2}\mathrm{sp}(4)$ with $\mathrm{sp}(4)$ as defined in (2.27).

Group V

Let ξ be a non-trivial quadratic character of $F^\times$, and let σ be any character of $F^\times$. Then $\nu\xi \times \xi \rtimes \nu^{-1/2}\sigma$ decomposes into the four irreducible group V representations. The L-parameters attached to each of these four representations have the same semisimple part

$$\rho:\ W_F \longrightarrow \mathrm{GSp}(4,\mathbb{C}), \qquad (2.35)$$
$$w \longmapsto \begin{bmatrix} (\nu^{1/2}\sigma)(w) &&& \\ & (\nu^{1/2}\xi\sigma)(w) && \\ && (\nu^{-1/2}\xi\sigma)(w) & \\ &&& (\nu^{-1/2}\sigma)(w) \end{bmatrix}.$$

To the Langlands quotient $L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$ (type Vd) we attach (ρ, N) with $N = 0$. Note that this representation is invariant under twisting with ξ (see (3.13) further below), and that the same is true for the corresponding parameter. To the Vc type representation $L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$ we attach (ρ, N_2) with

$$N_2 = \begin{bmatrix} 0 & & & 1 \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}. \tag{2.36}$$

To the Vb type representation $L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$ we attach (ρ, N_1) with N_1 as in (2.30). Note that Vb is the ξ-twist of Vc, and the corresponding parameters are also ξ-twists of each other. Finally, to the essentially square-integrable Va type representation $\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$ we attach (ρ, N_3) with

$$N_3 = \begin{bmatrix} 0 & & & 1 \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{bmatrix}. \tag{2.37}$$

If we write

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \times \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \qquad \text{for} \qquad \begin{bmatrix} a_1 & & & b_1 \\ & a_2 & b_2 & \\ & c_2 & d_2 & \\ c_1 & & & d_1 \end{bmatrix}, \tag{2.38}$$

then this parameter can be written as $\sigma\nu^{-1/2}\mathrm{sp}(2) \times \xi\sigma\nu^{-1/2}\mathrm{sp}(2)$. This parameter cannot be conjugated (by an element of $\mathrm{GSp}(4, \mathbb{C})$) into a Levi subgroup of $\mathrm{GSp}(4, \mathbb{C})$. The component group $\mathcal{C}(\rho, N_3)$ has two elements, represented by

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix}.$$

Group VI

Let σ be a character of $F^\times$ and consider the induced representation $\nu \times 1_{F^\times} \rtimes \nu^{-1/2}\sigma$. Its irreducible constituents are the four representations of type VI. The parameters (ρ, N) we attach to these representations will all have the same semisimple part

$$\begin{aligned} \rho:\ W_F &\longrightarrow \mathrm{GSp}(4, \mathbb{C}), \\ w &\longmapsto \begin{bmatrix} (\nu^{1/2}\sigma)(w) & & & \\ & (\nu^{1/2}\sigma)(w) & & \\ & & (\nu^{-1/2}\sigma)(w) & \\ & & & (\nu^{-1/2}\sigma)(w) \end{bmatrix}. \end{aligned} \tag{2.39}$$

To the Langlands quotient $L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$ (type VId) we attach the parameter (ρ, N) with $N = 0$. To the VIc type representation $L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$

we attach the parameter (ρ, N_1). To both of the essentially tempered representations $\tau(S, \nu^{-1/2}\sigma)$ and $\tau(T, \nu^{-1/2}\sigma)$ we attach the parameter (ρ, N_3) with N_3 as in (2.37). Let $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$ and $g = \begin{bmatrix} A & \\ & A' \end{bmatrix} \in \mathrm{Sp}(4, \mathbb{C})$. A computation shows that

$$g \begin{bmatrix} 0 & & & 1 \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{bmatrix} g^{-1} = \begin{bmatrix} 0 & 1 & & \\ & 0 & & 1 \\ & & 0 & \\ & & & 0 \end{bmatrix}.$$

After a further conjugation with the Weyl group element s_2, we see that (ρ, N_3) is equivalent to the parameter (ρ', N_4), where N_4 is defined in (2.32), and

$$\rho'(w) = \begin{bmatrix} (\nu^{1/2}\sigma)(w) & & & \\ & (\nu^{-1/2}\sigma)(w) & & \\ & & (\nu^{1/2}\sigma)(w) & \\ & & & (\nu^{-1/2}\sigma)(w) \end{bmatrix}.$$

In particular, the image of (ρ, N_3) can be conjugated into the Levi component of the Siegel parabolic subgroup. Moreover, the component group $\mathcal{C}(\rho, N)$ has two elements, represented by

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}.$$

Group VII

Let χ be a character of $F^\times$ and π a supercuspidal representation of $\mathrm{GL}(2, F)$. Assume that the induced representation $\chi \rtimes \pi$ is irreducible, so that we get a type VII representation. We attach to $\chi \rtimes \pi$ the L-parameter

$$W'_F \ni w \longmapsto \begin{bmatrix} \chi(w) \det(\mu(w))\mu(w)' & \\ & \mu(w) \end{bmatrix} \in \mathrm{GSp}(4, \mathbb{C}), \tag{2.40}$$

where μ is the parameter of π.

Group VIII

If π is a supercuspidal representation of $\mathrm{GL}(2, F)$, then the induced representation $1_{F^\times} \rtimes \pi$ splits into a direct sum $\tau(S, \pi) \oplus \tau(T, \pi)$. These are the representations of type VIIIa and VIIIb. To both of them we associate the L-parameter (ρ, N) with

$$\rho:\ w \longmapsto \begin{bmatrix} \det(\mu(w))\mu(w)' & \\ & \mu(w) \end{bmatrix} \in \mathrm{GSp}(4, \mathbb{C}), \tag{2.41}$$

where $\mu : W_F \to \mathrm{GL}(2,\mathbb{C})$ is the parameter of π, and $N = 0$. The component group $\mathcal{C}(\rho, N)$ has two elements, represented by

$$\begin{bmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix}&&-1&\\&&&1\\-1&&&\\&1&&\end{bmatrix}.$$

Group IX

Now let π be a supercuspidal representation of $\mathrm{GL}(2,F)$ with L-parameter $\mu : W_F \to \mathrm{GL}(2,\mathbb{C})$, and let ξ be a non-trivial quadratic character of $F^\times$ such that $\xi\pi = \pi$. Then $\nu\xi \rtimes \nu^{-1/2}\pi$ splits into the two components $\delta(\nu\xi, \nu^{-1/2}\pi)$ (type IXa) and $L(\nu\xi, \nu^{-1/2}\pi)$ (type IXb). To the Langlands quotient $L(\nu\xi, \nu^{-1/2}\pi)$ we attach the L-parameter with semisimple part

$$\begin{aligned}\rho :\ W_F &\longrightarrow \mathrm{GSp}(4,\mathbb{C}), \\ w &\longmapsto \begin{bmatrix}\xi(w)\nu^{1/2}(w)\det(\mu(w))\mu'(w) & \\ & \nu^{-1/2}(w)\mu(w)\end{bmatrix},\end{aligned} \tag{2.42}$$

and nilpotent part $N = 0$. To define the L-parameter for $\delta(\nu\xi, \nu^{-1/2}\pi)$, we require the following lemma.

Lemma 2.4.1. *Let π be a supercuspidal representation of* $\mathrm{GL}(2,F)$ *with L-parameter $\mu : W_F \to$* $\mathrm{GL}(2,\mathbb{C})$. *Assume ξ is a non-trivial character of $F^\times$ such that $\xi\pi = \pi$. Then there exists a symmetric matrix $S \in$* $\mathrm{GL}(2,\mathbb{C})$ *such that*

$${}^t\mu(w)S\mu(w) = \xi(w)\det(\mu(w))S$$

for all $w \in W_F$.

Proof. For each 2×2 matrix A, we have

$$\det(A)\begin{bmatrix}&1\\-1&\end{bmatrix} = {}^tA\begin{bmatrix}&1\\-1&\end{bmatrix}A.$$

In particular,

$$\det(\mu(w))\begin{bmatrix}&1\\-1&\end{bmatrix} = {}^t\mu(w)\begin{bmatrix}&1\\-1&\end{bmatrix}\mu(w) \qquad \text{for all } w \in W_F. \tag{2.43}$$

Since $\xi\pi = \pi$, the parameter $\xi\mu$ is isomorphic to μ. This implies that there exists an element $C \in \mathrm{GL}(2,\mathbb{C})$ such that

$$\xi(w)\mu(w)C = C\mu(w) \qquad \text{for all } w \in W_F. \tag{2.44}$$

Combining (2.43) and (2.44), we get

$$
{}^t\mu(w)S\mu(w) = \xi(w)\det(\mu(w))S \qquad \text{with} \qquad S = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} C.
$$

Taking transposes, we get the same relation with ${}^tS$ instead of S. Now it is easy to see that, for any $S \in \mathrm{GL}(2,\mathbb{C})$, one of the matrices $S + {}^tS$ or $S - {}^tS$ is invertible. Assume $S - {}^tS$ is invertible. We have

$$
{}^t\mu(w)(S - {}^tS)\mu(w) = \xi(w)\det(\mu(w))(S - {}^tS),
$$

and then also

$$
{}^t\mu(w)\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\mu(w) = \xi(w)\det(\mu(w))\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}
$$

for all $w \in W_F$. Since ξ is non-trivial, this contradicts (2.43). Hence the symmetric matrix $S + {}^tS$ must be invertible. Since

$$
{}^t\mu(w)(S + {}^tS)\mu(w) = \xi(w)\det(\mu(w))(S + {}^tS),
$$

for all $w \in W_F$, the assertion follows. □

Let S be the symmetric matrix from Lemma 2.4.1, and let $B = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} S$. Then

$$
N := \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \tag{2.45}
$$

lies in the Lie algebra of $\mathrm{GSp}(4,\mathbb{C})$. We have

$$
{}^t\mu(w)\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}B\mu(w) = \xi(w)\det(\mu(w))\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}B,
$$

and hence

$$
\xi(w)\det(\mu(w))\mu(w)'B\mu(w)^{-1} = B \qquad \text{for all } w \in W_F.
$$

This implies that

$$
\rho(w)N\rho(w)^{-1} = \nu(w)N \qquad \text{for all } w \in W_F,
$$

where ρ is as in (2.42). Thus we get a representation (ρ, N) of the Weil–Deligne group, and this is the L-parameter we attach to $\delta(\nu\xi, \nu^{-1/2}\pi)$.

Group X

Let π be a supercuspidal representation of $\mathrm{GL}(2,F)$ and σ a character of $F^\times$. We consider the Siegel induced representation $\pi \rtimes \sigma$. Assume that $\omega_\pi \neq \nu^{\pm 1}$, so that $\pi \rtimes \sigma$ is irreducible and of type X. If $\mu:\ W_F \to \mathrm{GL}(2,\mathbb{C})$ is the L-parameter of π, then we attach to $\pi \rtimes \sigma$ the L-parameter

$$
\begin{aligned}
\rho:\ W_F &\longrightarrow \mathrm{GSp}(4,\mathbb{C}), \\
w &\longmapsto \begin{bmatrix} \sigma(w)\det(\mu(w)) & & \\ & \sigma(w)\mu(w) & \\ & & \sigma(w) \end{bmatrix}.
\end{aligned} \tag{2.46}
$$

Group XI

Now let π be a supercuspidal representation of $\mathrm{GL}(2,F)$ with $\omega_\pi = 1$ and σ a character of $F^\times$. The induced representation $\nu^{1/2}\pi \rtimes \nu^{-1/2}\sigma$ decomposes into the XIa type representation $\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ and the XIb type representation $L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$. To the Langlands quotient $L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ we attach the L-parameter with semisimple part

$$\begin{aligned} \rho:\ W_F &\longrightarrow \mathrm{GSp}(4,\mathbb{C}), \\ w &\longmapsto \begin{bmatrix} \sigma(w)\nu^{1/2}(w) & & \\ & \sigma(w)\mu(w) & \\ & & \sigma(w)\nu^{-1/2}(w) \end{bmatrix} \end{aligned} \tag{2.47}$$

and nilpotent part $N = 0$. To $\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ we attach the parameter with the same semisimple part (2.47) but nilpotent part $N = N_2$, where N_2 is defined in (2.36). This parameter cannot be conjugated by a symplectic matrix into a Levi subgroup of $\mathrm{GSp}(4,\mathbb{C})$. The component group $\mathcal{C}(\rho, N_2)$ has two elements, represented by

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix}.$$

Further Properties

This completes the list of L-parameters of the non-supercuspidal representations of $\mathrm{GSp}(4,F)$. In addition, it is easy to check that the following facts hold. Let $\varphi:\ W_F' \to \mathrm{GSp}(4,\mathbb{C})$ be the parameter of π.

i) L-parameters are compatible with twisting by characters: If $\varphi = (\rho, N)$, then $\chi\pi$ has L-parameter $(\chi \otimes \rho, N)$.
ii) The multiplier $\lambda(\rho)$ equals the central character of π under the identification of characters of $F^\times$ and characters of W_F.
iii) π is essentially square-integrable if and only if the image of φ is not contained in any proper Levi subgroup of $\mathrm{GSp}(4,\mathbb{C})$.
iv) Regard φ as a representation of the Weil–Deligne group and write it as a direct sum of indecomposable representations,

$$\varphi = \bigoplus_{i=1}^{t} \rho_i \otimes \mathrm{sp}(n_i),$$

where ρ_i is an irreducible representation of W_F. Then π is tempered if and only if the representations $\nu^{(n_i-1)/2}\rho_i$ are bounded.

Finally, we can describe the L-packets resulting from our assignment of L-parameters to non-supercuspidal representations of $\mathrm{GSp}(4,F)$. It is one of the desiderata of the local Langlands correspondence that if φ is an L-parameter, then the number of elements in a packet $\Pi(\varphi)$ equals the order of the group

$$\mathcal{C}(\varphi)=\mathrm{Cent}(\varphi)/\mathrm{Cent}(\varphi)^0\,\mathbb{C}^\times,$$

where $\mathrm{Cent}(\varphi)$ denotes the centralizer of the image of φ in $\mathrm{GSp}(4,\mathbb{C})$, $\mathrm{Cent}(\varphi)^0$ denotes its identity component, and $\mathbb{C}^\times$ stands for the center of $\mathrm{GSp}(4,\mathbb{C})$. Computations show that for the parameters associated to the non-supercuspidal representations of $\mathrm{GSp}(4,F)$ the order of $\#\mathcal{C}(\varphi)$ equals 1 or 2. This order is 2 for the parameters associated to the representations of type Va, VIa, VIb, VIIIa, VIIIb and XIa. In other words, among the non-supercuspidal, irreducible representations of $\mathrm{GSp}(4,F)$ these are the representations contained in L-packets with more than one element; all the other non-supercuspidal representations form singleton L-packets. In fact, the representations of type VIa,b form an L-packet $\{\tau(S,\nu^{-1/2}\sigma),\tau(T,\nu^{-1/2}\sigma)\}$, and the representations of type VIIIa,b form an L-packet $\{\tau(S,\pi),\tau(T,\pi)\}$. The representations $\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$ of type Va and $\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$ of type XIa should each be members of two-element L-packets whose other members are supercuspidal.

L- and ε-factors of L-parameters

Finally, we recall the definitions of the L- and ε-factors of admissible representations of the Weil–Deligne group. Suppose that $\varphi=(\rho,N)$ is such a representation, acting on the space V. Let $\Phi\in W_F$ be an inverse Frobenius element, and let $I=\mathrm{Gal}(\bar F/F^{\mathrm{un}})\subset W_F$ be the inertia subgroup. Let $V_N=\ker(N)$, $V^I=\{v\in V:\ \rho(g)v=v$ for all $g\in I\}$ and $V_N^I=V^I\cap V_N$. Then the L-factor of φ is defined by

$$L(s,\varphi)=\det(1-q^{-s}\rho(\Phi)\big|V_N^I)^{-1}. \tag{2.48}$$

The ε-factor of φ is defined by

$$\varepsilon(s,\varphi,\psi)=\varepsilon(s,\rho,\psi)\det(-\rho(\Phi)q^{-s}\big|V^I/V_N^I) \tag{2.49}$$

where the factor $\varepsilon(s,\rho,\psi)$ is the one defined in (3.6.4) of [T] (it is denoted by $\varepsilon_L(s,\rho,\psi)$ there). This is the definition in [T] p. 21. Note that formula (4.1.4) of [K] should have $\det(-\Phi\big|V^I/V_N^I)$ replaced by $\det(-\rho(\Phi)q^{-s}\big|V^I/V_N^I)$. For general facts on L- and ε-factors see [T] and [Roh]; note that in the notation of [Roh], our factor $\varepsilon(s,\varphi,\psi)$ is $\epsilon(\varphi\otimes\omega^s,\psi,dx_\psi)$, where ω^s is the character of W_F trivial on I and such that $\omega^s(\Phi)=q^{-s}$, and where dx_ψ is the measure on F that is self-dual with respect to ψ, as on page 144 of [Roh]. The *conductor* of φ is

$$a(\varphi)=a(\rho)+\dim(V^I)-\dim(V_N^I), \tag{2.50}$$

as on page 139 of [Roh]; here, $a(\rho)$ is the conductor of ρ as in section 10 of [Roh]. Note that $a(\varphi)$ is a non-negative integer.

Proposition 2.4.2. *Let φ be an admissible representation of the Weil–Deligne group.*

i) We have $\varepsilon(s,\varphi,\psi)=\varepsilon(1/2,\varphi,\psi)q^{-a(\varphi)(s-1/2)}$.
ii) If the image of φ is contained in $\mathrm{SL}(n,\mathbb{C})$, then $\varepsilon(s,\varphi,\psi)$ does not depend on the particular choice of additive character ψ (recall that we always assume that ψ has conductor $\mathfrak{o}$).
iii) If the image of φ is contained in $\mathrm{Sp}(2n,\mathbb{C})$, then $\varepsilon(1/2,\varphi,\psi)\in\{\pm1\}$.

Proof. i) By (iii) of the proposition on page 143 of [Roh] and the definition (2.49), the factor $\varepsilon(s,\varphi,\psi)$ is of the form $cq^{-a(\varphi)(s-1/2)}$ with a constant c.

ii) The independence of $\varepsilon(s,\varphi,\psi)$ on the character ψ follows from (i) of the proposition on page 143 of [Roh].

iii) This follows from (iii) of the lemma on page 144 of [Roh]. □

Suppose that φ is an L-parameter for GSp(4). Then φ is an admissible representation of the Weil–Deligne group, and $L(s,\varphi)$ and $\varepsilon(s,\varphi,\psi)$ are defined. If the image of φ is contained in $\mathrm{Sp}(4,\mathbb{C})$, we will often write $\varepsilon(s,\varphi)$ instead of $\varepsilon(s,\varphi,\psi)$. If π is a non-supercuspidal, irreducible, admissible representation of $\mathrm{GSp}(4,F)$, and φ_π is the L-parameter of π as defined above, then the image of φ_π is contained in $\mathrm{Sp}(4,\mathbb{C})$ if and only if π has trivial central character.

Proposition 2.4.3. *Let π be a non-supercuspidal, irreducible, admissible representation of $\mathrm{GSp}(4,F)$, and let φ_π be the L-parameter of π as defined above. Then the L-factor $L(s,\varphi_\pi)$ is tabulated in Table A.8. Assume that π has trivial central character. Then the ε-factor $\varepsilon(s,\varphi_\pi)$ is tabulated in Table A.9.*

Proof. This proposition follows by using the involved definitions, and we will omit the case-by-case verification. However, we will make some remarks about the computation of $\varepsilon(s,\varphi_\pi)$ in the case π has trivial central character. First of all, it is useful to note that

$$\varepsilon(1/2,\mu,\psi)\varepsilon(1/2,\mu^\vee,\psi)=\det(\mu)(-1)$$

for any representation μ of the Weil-Deligne group. Second, we will discuss two cases that perhaps require further comment. For $\sigma\mathrm{St}_{\mathrm{GSp}(4)}$ (type IVa) we use the Corollary on page 146 of [Roh]. For $\delta(\nu\xi,\nu^{-1/2}\pi)$ (type IXa), we will do a direct computation of the ε-factor. According to the definition (2.49), we have to compute $\varepsilon(s,\rho)$ and $\det(-\rho(\Phi)q^{-s}\big|V^I/V_N^I)$. By (2.42),

$$\begin{aligned}\rho&\cong\xi\nu^{1/2}\det(\mu)\mu'\oplus\nu^{-1/2}\mu\\&\cong\xi\nu^{1/2}\mu\oplus\nu^{-1/2}\mu\\&=\nu^{1/2}\det(\mu)^{-1}\mu\oplus\nu^{-1/2}\mu\\&=\nu^{1/2}\mu^\vee\oplus\nu^{-1/2}\mu,\end{aligned}$$

where μ is the parameter of the supercuspidal GL(2) representation π. Therefore, $\varepsilon(1/2,\rho)=\xi(-1)$. As for $\det(-\rho(\Phi)q^{-s}\big|V^I/V_N^I)$, we will show that this factor is 1 by proving that V has no non-zero I-invariant vectors under the action of ρ. For this it suffices to show that the two-dimensional irreducible representation μ has no non-zero I-fixed vectors. If the dimension of the space of I-fixed vectors was one, then $\mu(\Phi)$ preserves this space since Φ normalizes I; this is impossible, since μ is irreducible. If the dimension of the space of I-fixed vectors was two, then μ would factor through $W_F/I \cong \mathbb{Z}$; this is impossible, since an abelian group cannot have a two-dimensional irreducible representation. Hence $V^I = 0$. We conclude that $\varepsilon(1/2, \varphi_\pi) = \varepsilon(1/2, \rho) = \xi(-1)$. □

Proposition 2.4.4. *Let π be a non-supercuspidal, generic, irreducible, admissible representation of* GSp(4, F). *Let φ_π be the L-parameter of π as defined above. Then $L(s, \varphi_\pi) = L(s, \pi)$, where $L(s, \pi)$ is as defined in Sect. 2.6.*

Proof. This follows by comparison of Table A.8 with the results of [Tak]; see the proof of Theorem 4.2.1 below. □

2.5 P_3-Theory

In this section we relate admissible representations of GSp(4, F) with trivial central character to smooth representations of the group P_3, which is the subgroup

$$P_3 = \begin{bmatrix} * & * & * \\ * & * & * \\ & & 1 \end{bmatrix}$$

of GL(3, F). The representation theory of the analogous subgroup P_n of GL(n, F) plays an important role in the representation theory of GL(n, F), and there is an extensive theory of P_n smooth representations; see [BZ]. Every irreducible, admissible representation of GL(n, F) defines a smooth representation of P_n of finite length, and this connection can be exploited to prove, for example, the existence of Kirillov models for generic, irreducible, admissible representations of GL(n, F). As it happens, irreducible, admissible representations V of GSp(4, F) with trivial central character also define smooth representations of P_3 of finite length. In the GSp(4, F) case the P_3 representation is not obtained by restriction. Instead, the associated P_3 representation is the quotient V_{Z^J}. It is this need to take a quotient that accounts for the non-existence of naive Kirillov type models for generic, irreducible, admissible representations of GSp(4, F) with trivial central character. Nevertheless, the P_3 representation V_{Z^J} is a useful tool for proving basic results about zeta integrals and studying paramodular vectors.

We begin with some definitions and fundamental facts. Recall from Sect. 2.1 that Z denotes the center of GSp(4, F), and Z^J denotes the center of the Jacobi group. We have $Z(F) \cong F^\times$ and $Z^J(F) \cong F$. For simplicity, we shall write Z for $Z(F)$ and Z^J for $Z^J(F)$, and similarly for other subgroups of

GSp$(4,F)$. The following lemma is the key observation behind the connection to P_3 representations.

Lemma 2.5.1. *The group Z^J is a normal subgroup of Q. Moreover, there is a homomorphism*

$$i : Q \to P_3$$

defined by

$$i(\begin{bmatrix} ad-bc &&& \\ & a & b & \\ & c & d & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 & -y & x & z \\ & 1 && x \\ && 1 & y \\ &&& 1 \end{bmatrix} \begin{bmatrix} u &&& \\ & u && \\ && u & \\ &&& u \end{bmatrix}) = \begin{bmatrix} a & b & \\ c & d & \\ && 1 \end{bmatrix} \begin{bmatrix} 1 && x \\ & 1 & y \\ && 1 \end{bmatrix}.$$

The kernel of i is $Z^J Z$, so that we get an isomorphism $Q/Z^J Z \cong P_3$.

Proof. The existence of an isomorphism is noted in Lemma 6.2 of [PS]. A direct computation verifies that i is a homomorphism. Evidently, i is surjective with kernel $Z^J Z$. □

Let (π, V) be a smooth representation of GSp$(4,F)$ such that the center of GSp$(4,F)$ acts trivially. Let $V(Z^J)$ be the $\mathbb{C}$ vector subspace of V generated by the vectors $v - \pi(z)v$ for $z \in Z^J$ and $v \in V$. Then Q acts on $V(Z^J)$, so that Q acts on $V_{Z^J} = V/V(Z^J)$. Since Z and Z^J act trivially on V_{Z^J}, we obtain an action of $Q/Z^J Z$ on V_{Z^J}. Using the isomorphism induced by i from Lemma 2.5.1, we obtain an action of P_3 on V_{Z^J}. Let $p: V \to V_{Z^J}$ be the projection map. If $q \in Q$ and $v \in V$, then

$$p(\pi(q)v) = i(q)p(v). \tag{2.51}$$

In the remainder of this section we will study V_{Z^J} as a representation of P_3. We need to complete two tasks before we can state the main result about V_{Z^J}.

First, we need to summarize some results about smooth representations of P_3. For details and proofs, the reader should consult [BZ]. Induction from three subgroups is important in the representation theory of P_3. These three subgroups correspond to GL$(0,F) = 1$, GL$(1,F) = F^\times$ and GL$(2,F)$, and the representations to be induced arise from representations of GL$(0,F) = 1$, GL$(1,F) = F^\times$ and GL$(2,F)$. The first subgroup is

$$\begin{bmatrix} 1 & * & * \\ & 1 & * \\ && 1 \end{bmatrix}.$$

Fix a smooth representation of GL$(0,F) = 1$, i.e., a vector space Y, define a unitary character Θ of this group by

$$\Theta(\begin{bmatrix} 1 & u_{12} & * \\ & 1 & u_{23} \\ && 1 \end{bmatrix}) = \psi(u_{12} + u_{23})$$

and let $Y \otimes \Theta$ be the smooth representation of this group defined by $u \cdot y = \Theta(u)y$ for u in the group and $y \in Y$. We consider the smooth representation

$$\tau_{\mathrm{GL}(0)}^{P_3}(Y) = \text{c-Ind}_{\left[\begin{smallmatrix}1&*&*\\&1&*\\&&1\end{smallmatrix}\right]}^{P_3}(Y \otimes \Theta).$$

Evidently,

$$\tau_{\mathrm{GL}(0)}^{P_3}(Y) \cong (\dim Y) \cdot \text{c-Ind}_{\left[\begin{smallmatrix}1&*&*\\&1&*\\&&1\end{smallmatrix}\right]}^{P_3}(\mathbb{C} \otimes \Theta) = (\dim Y) \cdot \text{c-Ind}_{\left[\begin{smallmatrix}1&*&*\\&1&*\\&&1\end{smallmatrix}\right]}^{P_3}\Theta.$$

The representation

$$\tau_{\mathrm{GL}(0)}^{P_3}(\mathbf{1}) = \text{c-Ind}_{\left[\begin{smallmatrix}1&*&*\\&1&*\\&&1\end{smallmatrix}\right]}^{P_3}(\Theta)$$

is irreducible. It is called $\tau_{P_3}^0$ in [BZ]. The next subgroup corresponds to $\mathrm{GL}(1,F) = F^\times$. It is

$$\begin{bmatrix} * & * & * \\ & 1 & * \\ & & 1 \end{bmatrix}.$$

If (χ, X) is a smooth representation of $\mathrm{GL}(1,F) = F^\times$, then we define a smooth representation $\chi \otimes \Theta$ of the above group by letting $\chi \otimes \Theta$ have the same space X as χ and setting

$$(\chi \otimes \Theta)(\begin{bmatrix} a & * & * \\ & 1 & y \\ & & 1 \end{bmatrix}) = \psi(y)\chi(a).$$

We consider the smooth representation

$$\tau_{\mathrm{GL}(1)}^{P_3}(\chi) = \text{c-Ind}_{\left[\begin{smallmatrix}*&*&*\\&1&*\\&&1\end{smallmatrix}\right]}^{P_3}(\chi \otimes \Theta)$$

of P_3. If χ is irreducible, i.e., χ is a character, then this representation is irreducible. Finally, there is a subgroup corresponding to $\mathrm{GL}(2,F)$. This is just P_3. If ρ is a smooth representation of $\mathrm{GL}(2,F)$, then we define a smooth representation

$$\tau_{\mathrm{GL}(2)}^{P_3}(\rho)$$

of P_3 by letting $\tau_{\mathrm{GL}(2)}^{P_3}(\rho)$ have the same space as ρ and action defined by

$$\tau_{\mathrm{GL}(2)}^{P_3}(\rho)(\begin{bmatrix} a & b & * \\ c & d & * \\ & & 1 \end{bmatrix}) = \rho(\begin{bmatrix} a & b \\ c & d \end{bmatrix}).$$

If ρ is irreducible, then $\tau_{\mathrm{GL}(2)}^{P_3}(\rho)$ is irreducible. Every irreducible, smooth representation η of P_3 is isomorphic to

$$\tau^{P_3}_{\mathrm{GL}(0)}(\mathbf{1}), \quad \tau^{P_3}_{\mathrm{GL}(1)}(\chi) \quad \text{or} \quad \tau^{P_3}_{\mathrm{GL}(2)}(\rho)$$

for some irreducible, admissible representation χ of $\mathrm{GL}(1,F) = F^\times$, or some irreducible, admissible representation ρ of $\mathrm{GL}(2,F)$. Moreover, the group $\mathrm{GL}(k,F)$, $k = 0,1,2$ and the irreducible, admissible representation of $\mathrm{GL}(k,F)$ are uniquely determined by η.

Our second task before stating the main theorem of this section is to prove a lemma that will imply V_{Z^J} has finite length as a representation of P_3 when V is an irreducible, admissible representation of P_3 with trivial central character. In the proof of the following lemma M_B, M_P and M_Q are the Levi subgroups of B, P and Q, respectively. Similarly, U, N_P and N_Q are the unipotent radicals of B, P and Q, respectively.

Lemma 2.5.2. *Let (π, V) be an irreducible and admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character. Then*

$$\dim \mathrm{Hom}_U(V, \psi_{c_1,c_2}) < \infty$$

for all $c_1, c_2 \in F$.

Proof. Suppose $c_1 = c_2 = 0$. Then

$$\dim \mathrm{Hom}_U(V, \psi_{c_1,c_2}) = \dim V_U.$$

The Jacquet module V_U is a finitely generated, admissible representation of $M_B \cong F^\times \times F^\times \times F^\times$ by Theorem 3.3.1 of [Ca1]. By 4.1 of [BZ], V_U is of finite length as a representation of M_B. This implies that V_U is finite dimensional.

Suppose $c_1 \neq 0$ and $c_2 = 0$. Then

$$\mathrm{Hom}_U(V, \psi_{c_1,c_2}) = \mathrm{Hom}_{\left[\begin{smallmatrix} 1 & * & & \\ & 1 & & \\ & & 1 & * \\ & & & 1 \end{smallmatrix}\right]}(V_{N_P}, \psi_{c_1,0}).$$

By the same reasoning as in the previous paragraph, V_{N_P} is a representation of finite length of the group $M_P \cong \mathrm{GL}(2,F) \times F^\times$. To prove the claim in this case it thus suffices to prove that

$$\dim \mathrm{Hom}_{\left[\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right] \times 1}(\tau \otimes \chi, \psi_{c_1})$$

is finite for τ an irreducible, admissible representation of $\mathrm{GL}(2,F)$ and χ a character of $F^\times$. This follows from 5.21 of [BZ].

Suppose $c_1 = 0$ and $c_2 \neq 0$. Then

$$\mathrm{Hom}_U(V, \psi_{c_1,c_2}) = \mathrm{Hom}_{\left[\begin{smallmatrix} 1 & & & \\ & 1 & * & \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right]}(V_{N_Q}, \psi_{0,c_2}).$$

Again, V_{N_Q} is a representation of finite length of the group $M_Q \cong \mathrm{GL}(2,F) \times F^\times$, and the same argument as in the previous paragraph applies.

Finally, suppose $c_1 \neq 0$ and $c_2 \neq 0$. Then $\dim \mathrm{Hom}_U(V, \psi_{c_1,c_2}) \leq 1$ by the uniqueness of Whittaker models. $\square$

Theorem 2.5.3. *Let (π, V) be an irreducible, admissible representation of* $\mathrm{GSp}(4, F)$ *with trivial central character. The quotient $V_{Z^J} = V/V(Z^J)$ is a smooth representation of Q/Z^JZ, and hence via Lemma 2.5.1 defines a smooth representation of P_3. As a representation of P_3, V_{Z^J} has finite length. Hence, V_{Z^J} has a finite filtration by P_3 subspaces such that the successive quotients are irreducible and of the form $\tau_{\mathrm{GL}(0)}^{P_3}(\mathbf{1}), \tau_{\mathrm{GL}(1)}^{P_3}(\chi)$ or $\tau_{\mathrm{GL}(2)}^{P_3}(\rho)$ for some character χ of $F^\times$, or some irreducible, admissible representation ρ of* $\mathrm{GL}(2, F)$. *Moreover, the following statements hold:*

i) There exists a chain of P_3 subspaces

$$0 \subset V_2 \subset V_1 \subset V_0 = V_{Z^J}$$

such that:

$$\begin{aligned} V_2 &\cong \tau_{\mathrm{GL}(0)}^{P_3}(V_{U,\psi_{-1,1}}) \cong \dim \mathrm{Hom}_U(V, \psi_{-1,1}) \cdot \tau_{\mathrm{GL}(0)}^{P_3}(\mathbf{1}), \\ V_1/V_2 &\cong \tau_{\mathrm{GL}(1)}^{P_3}(V_{U,\psi_{-1,0}}), \\ V_0/V_1 &\cong \tau_{\mathrm{GL}(2)}^{P_3}(V_{N_Q}). \end{aligned}$$

Here, the complex vector space $V_{U,\psi_{-1,1}}$ defines a smooth representation of $\mathrm{GL}(0, F)$, *the vector space $V_{U,\psi_{-1,0}}$ admits a smooth action of* $\mathrm{GL}(1, F) \cong F^\times$ *induced by the operators*

$$\pi(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix}), \quad a \in F^\times,$$

and V_{N_Q} admits a smooth action of $\mathrm{GL}(2, F)$ *induced by the operators*

$$\pi(\begin{bmatrix} \det g & & \\ & g & \\ & & 1 \end{bmatrix}), \quad g \in \mathrm{GL}(2, F).$$

ii) The representation π is generic if and only if $V_2 \neq 0$, and if π is generic, then $V_2 \cong \tau_{\mathrm{GL}(0)}^{P_3}(\mathbf{1})$.

iii) We have $V_2 = V_{Z^J}$ if and only if π is supercuspidal. If π is supercuspidal and generic, then $V_{Z^J} = V_2 \cong \tau_{\mathrm{GL}(0)}^{P_3}(\mathbf{1})$ is non-zero and irreducible. If π is supercuspidal and non-generic, then $V_{Z^J} = V_2 = 0$.

Proof. First we prove the initial claims of the theorem. Let

$$U_3 = \begin{bmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{bmatrix}.$$

By 5.23 of [BZ], to prove that V_{Z^J} has finite length as a P_3 representation it suffices to prove that $\dim(V_{Z^J})_{U_3,\theta}$ is finite for all characters θ of U_3. This is equivalent to $\dim V_{U,\psi_{c_1,c_2}}$ being finite for all $c_1, c_2 \in F$, and thus follows from Lemma 2.5.2. The remaining initial claims follow from the discussion about irreducible, smooth P_3 representations preceding the theorem.

i) We set $V_i = (V_{Z^J})_i$ for $i = 0, 1, 2$ where $(V_{Z^J})_i$ is as defined as in 5.15 of [BZ]. Then the assertions follow from 5.15 of [BZ] and the involved definitions. Note that there is a misprint in 5.15 of [BZ]; the second sentence should read "We set $\pi_i = (\Phi^+)^i(\Phi^-)^i(\pi)$".

ii) This follows from $V_2 \cong \dim \mathrm{Hom}_U(V, \psi_{-1,1}) \cdot \tau^{P_3}_{\mathrm{GL}(0)}(\mathbf{1})$.

iii) To prove that $V_2 = V_{Z^J}$ if and only if π is supercuspidal we will use the equivalence (1) $\iff$ (2) of the proposition in 5.15 of [BZ]. This equivalence asserts that $V_2 = V_{Z^J}$ if and only if $(V_{Z^J})_{U_\beta} = 0$ for each non-trivial horospherical subgroup U_β of P_3. The non-trivial horospherical subgroups of P_3 are

$$U_3 = U_{(1,1,1)} = \begin{bmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{bmatrix}, \quad U_{(1,2)} = \begin{bmatrix} 1 & * & * \\ & 1 & \\ & & 1 \end{bmatrix}, \quad U_{(2,1)} = \begin{bmatrix} 1 & & * \\ & 1 & * \\ & & 1 \end{bmatrix},$$

and there are isomorphisms of complex vector spaces

$$V_U \cong (V_{Z^J})_{U_3}, \quad V_{N_P} \cong (V_{Z^J})_{U_{(1,2)}}, \quad V_{N_Q} \cong (V_{Z^J})_{U_{(2,1)}}.$$

It follows that $(V_{Z^J})_{U_\beta} = 0$ for each non-trivial horospherical subgroup U_β of P_3 if and only if π is supercuspidal. The remaining claims follow from ii). □

In the tables in Appendix A.4 we have listed the semisimplifications of the P_3 modules V_0/V_1 and V_1/V_2 for each irreducible, admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character.

In the remainder of this section we investigate certain linear functionals on the three types of irreducible representations of P_3. Such linear functionals arise in the consideration of zeta integrals, and these results will be used to prove the functional equation for zeta integrals. Lemma 2.5.5 will also be used to investigate the poles of L-functions of generic representations in Sect. 4.2.

Lemma 2.5.4. *Let χ be a character of $F^\times$. The space of linear functionals*

$$\lambda : \tau^{P_3}_{\mathrm{GL}(0)}(\mathbf{1}) \to \mathbb{C}$$

such that

$$\lambda(\begin{bmatrix} 1 & & \\ & 1 & y \\ & & 1 \end{bmatrix} f) = \psi(y)\lambda(f), \quad \lambda(\begin{bmatrix} 1 & & \\ x & 1 & \\ & & 1 \end{bmatrix} f) = \lambda(f)$$

and

$$\lambda(\begin{bmatrix} a & & \\ & 1 & \\ & & 1 \end{bmatrix} f) = \chi(a)\lambda(f)$$

for $x, y \in F$, $a \in F^\times$ *and* $f \in \tau_{\mathrm{GL}(0)}^{P_3}(\mathbf{1})$ *is one-dimensional. Moreover, if* λ *is such a linear functional, then there exists* $c \in F$ *such that*

$$\lambda(f) = c \int\limits_F \int\limits_{F^\times} f(\begin{bmatrix} a & & \\ x & 1 & \\ & & 1 \end{bmatrix}) \chi(a)^{-1} |a|^{-1} \, d^\times a \, dx$$

for $f \in \tau_{\mathrm{GL}(0)}^{P_3}(\mathbf{1})$.

Proof. We begin by computing the twisted Jacquet module

$$\tau_{\mathrm{GL}(0)}^{P_3}(\mathbf{1})_{\left[\begin{smallmatrix} 1 & & \\ & 1 & * \\ & & 1 \end{smallmatrix}\right], \psi}.$$

As usual, let $\mathcal{S}(F^\times \times F)$ be the space of Schwartz functions on $F^\times \times F$. Define

$$T : \tau_{\mathrm{GL}(0)}^{P_3}(\mathbf{1}) \to \mathcal{S}(F^\times \times F)$$

by

$$T(f)(a, x) = f(\begin{bmatrix} a & & \\ x & 1 & \\ & & 1 \end{bmatrix}).$$

First of all, we claim that T is well-defined. To see this, let $f \in \tau_{\mathrm{GL}(0)}^{P_3}(\mathbf{1})$. By definition, f is left invariant under a compact open subgroup of P_3 and is compactly supported modulo the subgroup

$$\begin{bmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{bmatrix}$$

of P_3. This implies that there exist a positive integer n and $g_i \in \mathrm{GL}(2, F)$, $1 \le i \le t$, such that f is left invariant under

$$\begin{bmatrix} \Gamma(\mathfrak{p}^n) & \\ & 1 \end{bmatrix}$$

and the support of f is contained in a disjoint union

$$\bigsqcup_{i=1}^{t} \begin{bmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{bmatrix} \begin{bmatrix} g_i & \\ & 1 \end{bmatrix} \begin{bmatrix} \Gamma(\mathfrak{p}^n) & \\ & 1 \end{bmatrix}. \tag{2.52}$$

Here, $\Gamma(\mathfrak{p}^n)$ is the subgroup of k in $\mathrm{GL}(2, \mathfrak{o})$ such that $k \equiv 1 \bmod \mathfrak{p}^n$. Using this, it is not hard to see that the locally constant function defined by

$$(a, x) \mapsto f(\begin{bmatrix} a & & \\ x & 1 & \\ & & 1 \end{bmatrix})$$

has compact support, so that T is well-defined.

Second, we claim that the kernel of T is the subspace consisting of all $\mathbb{C}$ linear combinations of the elements

$$\psi(y)h - \begin{bmatrix} 1 & & \\ & 1 & y \\ & & 1 \end{bmatrix} h, \quad h \in \tau_{\mathrm{GL}(0)}^{P_3}(\mathbf{1}),\ y \in F. \tag{2.53}$$

A computation verifies that these elements are in the kernel of T. Conversely, suppose that f is in the kernel of T. Let the support of f be contained in the disjoint union (2.52). For $1 \le i \le t$, let f_i be the restriction of f to the i-th double coset. Then $f_i \in \tau_{\mathrm{GL}(0)}^{P_3}(\mathbf{1})$ for $1 \le i \le t$, and $f = f_1 + \cdots + f_t$. Also, each f_i is in the kernel of T. To prove that f is in the span of the elements (2.53), it suffices to prove that each f_i is in this span. Fix $1 \le i \le t$. If $f_i = 0$, then there is nothing to prove. Assume $f_i \ne 0$. Then f_i is non-zero on every point of the i-th coset. Let $k \in \Gamma(\mathfrak{p}^n)$, and write

$$g_i k = \begin{bmatrix} a(k) & b(k) \\ c(k) & d(k) \end{bmatrix}.$$

We claim that $d(k) \notin 1 + \mathfrak{p}^n$ for all $k \in \Gamma(\mathfrak{p}^n)$. To see this, suppose there exists $k \in \Gamma(\mathfrak{p}^n)$ such that $d(k) \in 1 + \mathfrak{p}^n$. Then $d(k) \ne 0$. Also,

$$\begin{aligned}
f_i(\begin{bmatrix} g_i & \\ & 1 \end{bmatrix}) &= f_i(\begin{bmatrix} g_i & \\ & 1 \end{bmatrix}\begin{bmatrix} k & \\ & 1 \end{bmatrix}) \\
&= f_i(\begin{bmatrix} a(k) & b(k) & \\ c(k) & d(k) & \\ & & 1 \end{bmatrix}) \\
&= \psi(b(k)d(k)^{-1}) f_i(\begin{bmatrix} 1 & -b(k)d(k)^{-1} & \\ & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} a(k) & b(k) & \\ c(k) & d(k) & \\ & & 1 \end{bmatrix}) \\
&= \psi(b(k)d(k)^{-1}) f_i(\begin{bmatrix} a' & & \\ c(k) & d(k) & \\ & & 1 \end{bmatrix})
\end{aligned}$$

for some $a' \in F^\times$. Hence,

$$\begin{aligned}
f_i(\begin{bmatrix} g_i & \\ & 1 \end{bmatrix}) &= \psi(b(k)d(k)^{-1}) f_i(\begin{bmatrix} a' & & \\ c(k) & d(k) & \\ & & 1 \end{bmatrix}) \\
&= \psi(b(k)d(k)^{-1}) f_i(\begin{bmatrix} a' & & \\ c(k) & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & d(k) & \\ & & 1 \end{bmatrix}) \\
&= \psi(b(k)d(k)^{-1}) f_i(\begin{bmatrix} a' & & \\ c(k) & 1 & \\ & & 1 \end{bmatrix})
\end{aligned}$$

$$= 0.$$

The last equality follows from $T(f_i) = 0$. This is a contradiction, since f_i does not vanish at each point of the i-th double coset. Since $d(k) - 1 \notin \mathfrak{p}^n$ for all $k \in \Gamma(\mathfrak{p}^n)$, there exists a positive integer m such that

$$\int_{\mathfrak{p}^{-m}} \psi((d(k)-1)y)\,dy = 0$$

for all $k \in \Gamma(\mathfrak{p}^n)$. Let

$$f' = (\int_{\mathfrak{p}^{-m}} dy)^{-1} \cdot \int_{\mathfrak{p}^{-m}} \psi(y)^{-1}(\psi(y)f_i - \begin{bmatrix}1&&\\&1&y\\&&1\end{bmatrix} f_i)\,dy.$$

Then $f' \in \tau_{\mathrm{GL}(0)}^{P_3}(\mathbf{1})$ and f' is in the span of the elements (2.53). To verify that $f' = f_i$ it suffices to check that

$$f'(\begin{bmatrix}a(k)&b(k)&\\c(k)&d(k)&\\&&1\end{bmatrix}) = f_i(\begin{bmatrix}a(k)&b(k)&\\c(k)&d(k)&\\&&1\end{bmatrix})$$

for $k \in \Gamma(\mathfrak{p}^n)$ and $f'(p) = 0$ for p not in the i-th double coset. Let $p \in P_3$ and write

$$p = \begin{bmatrix}*&*&*\\ *&d&*\\&&1\end{bmatrix}.$$

Then

$$\begin{aligned} f'(p) &= f_i(p) - (\int_{p^{-m}} dy)^{-1} \cdot \int_{\mathfrak{p}^{-m}} \psi(y)^{-1} f_i(p\begin{bmatrix}1&&\\&1&y\\&&1\end{bmatrix})\,dy \\ &= f_i(p) - (\int_{\mathfrak{p}^{-m}} dy)^{-1}(\int_{\mathfrak{p}^{-m}} \psi((d-1)y)\,dy) f_i(p). \end{aligned}$$

This implies that if p is not in the i-th double coset, then $f'(p) = 0$. Also, we see that if

$$p = \begin{bmatrix}a(k)&b(k)&\\c(k)&d(k)&\\&&1\end{bmatrix}, \quad k \in \Gamma(\mathfrak{p}^n),$$

then we also have $f'(p) = f_i(p)$. This proves our claim about the kernel of T.

Next, we prove that T is surjective. Let $\varphi \in \mathcal{S}(F^\times \times F)$. Choose n so large that: $\varphi(a, x+y) = \varphi(a, x)$ for $a \in F^\times$, $x \in F$, $y \in \mathfrak{p}^n$; $\varphi(au, x) = \varphi(a, x)$ for $a \in F^\times$, $x \in F$, $u \in 1 + \mathfrak{p}^n$; and if $\varphi(a, x) \neq 0$ for some $a \in F^\times$ and $x \in F$, then $\varpi^n a \in \mathfrak{o}$. Define $f : P_3 \to \mathbb{C}$ by

$$f(p) = \begin{cases} 0 & \text{if } p \notin \begin{bmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{bmatrix} \begin{bmatrix} * & & \\ * & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} \Gamma(\mathfrak{p}^n) & \\ & 1 \end{bmatrix}, \\ \psi(y+z)\varphi(a,x) & \text{if } p = \begin{bmatrix} 1 & z & * \\ & 1 & y \\ & & 1 \end{bmatrix} \begin{bmatrix} a & & \\ x & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} \Gamma(\mathfrak{p}^n) & \\ & 1 \end{bmatrix}. \end{cases}$$

Computations show that f is a well defined element of $\tau_{\mathrm{GL}(0)}^{P_3}(\mathbf{1})$. By construction, $T(f) = \varphi$, so that T is surjective. Define a group structure on $F^\times \times F$ via the isomorphism of sets

$$F^\times \times F \xrightarrow{\sim} \begin{bmatrix} F^\times & & \\ F & 1 & \\ & & 1 \end{bmatrix}, \qquad (a,x) \mapsto m(a,x) = \begin{bmatrix} a & & \\ x & 1 & \\ & & 1 \end{bmatrix}$$

and transport of structure, where the group law on the second group is multiplication of matrices. Then, in summary, it follows that T induces an isomorphism of $F^\times \times F$ representations

$$\tau_{\mathrm{GL}(0)}^{P_3}(\mathbf{1})_{\left[\begin{smallmatrix} 1 & & \\ & 1 & * \\ & & 1 \end{smallmatrix}\right],\psi} \xrightarrow[\sim]{T} \mathcal{S}(F^\times \times F),$$

where (a,x) acts on the first space by $m(a,x)$ via the P_3 action, and (a,x) acts on the second space by right translation with respect to the just defined group law.

We can now prove the assertions of the lemma. Let λ be a linear functional as in the statement of the lemma. By the first transformation property of λ, λ is trivial on the kernel of T, and thus induces a linear functional on $\mathcal{S}(F^\times \times F)$, which we also call λ. This linear functional satisfies $\lambda((a,x)\varphi) = \chi(a)\lambda(\varphi)$ for $a \in F^\times$, $x \in F$ and $\varphi \in \mathcal{S}(F^\times \times F)$. Let k be a positive integer such that $\chi(1+\mathfrak{p}^k) = 1$. Consider the restriction of λ to the subspace $\mathcal{S}((1+\mathfrak{p}^k) \times F)$. Then $\lambda((a,x)\varphi) = \lambda(\varphi)$ for $a \in 1+\mathfrak{p}^k$, $x \in F$ and $\varphi \in \mathcal{S}((1+\mathfrak{p}^k) \times F)$. Moreover, $(1+\mathfrak{p}^k) \times F$ is a subgroup of $F^\times \times F$. By the proposition in 1.18 of [BZ] , there exists $c \in \mathbb{C}$ such that the restriction of λ to $\mathcal{S}((1+\mathfrak{p}^k) \times F)$ is c times the Haar measure on the unimodular group $(1+\mathfrak{p}^k) \times F$, i.e.,

$$\lambda(\varphi) = c \int\limits_{1+\mathfrak{p}^k} \int\limits_F \varphi(a,x)\, d^\times a\, dx$$

for $\varphi \in \mathcal{S}((1+\mathfrak{p}^k) \times F)$. Let $\varphi \in \mathcal{S}(F^\times \times F)$. We can write $\varphi = (a_1,0)\varphi_1 + \dots + (a_r,0)\varphi_r$, where $\varphi_i \in \mathcal{S}(F^\times \times F)$ has support in $(1+\mathfrak{p}^k) \times F$ and $a_i \in F^\times$ for $1 \le i \le r$. A computation shows that

$$\lambda(\varphi) = c\,\chi(a_1) \int\limits_{1+\mathfrak{p}^k} \int\limits_F \varphi_1(a,x)\, d^\times a\, dx + \dots + c\,\chi(a_r) \int\limits_{1+\mathfrak{p}^k} \int\limits_F \varphi_r(a,x) d^\times a\, dx$$

$$= c \int\limits_{F^\times} \int\limits_{F} \varphi(a,x)\chi(a)^{-1}|a|^{-1}\, d^\times a\, dx.$$

This completes the proof. □

Lemma 2.5.5. *Let χ be a character of $F^\times$. The space of linear functionals*

$$\lambda : \tau_{\mathrm{GL}(1)}^{P_3}(\chi) = \text{c-Ind}_{\left[\begin{smallmatrix} * & * & * \\ & 1 & * \\ & & 1 \end{smallmatrix}\right]}^{P_3}(\chi \otimes \psi) \to \mathbb{C}$$

such that

$$\lambda(\begin{bmatrix} 1 & & \\ & 1 & y \\ & & 1 \end{bmatrix} f) = \psi(y)\lambda(f) \quad \text{and} \quad \lambda(\begin{bmatrix} 1 & & \\ x & 1 & \\ & & 1 \end{bmatrix} f) = \lambda(f)$$

for $x, y \in F$ and $f \in \tau_{\mathrm{GL}(1)}^{P_3}(\chi)$ is one-dimensional, and every such linear functional is a multiple of the linear functional that sends f to

$$\int\limits_F f(\begin{bmatrix} 1 & & \\ x & 1 & \\ & & 1 \end{bmatrix})\, dx.$$

Moreover, if λ is such a linear functional, then

$$\lambda(\begin{bmatrix} a & & \\ & 1 & \\ & & 1 \end{bmatrix} f) = |a|^{-1}\chi(a)\lambda(f)$$

for $a \in F^\times$ and $f \in \tau_{\mathrm{GL}(1)}^{P_3}(\chi)$.

Proof. The proof of this lemma is very similar to the proof of Lemma 2.5.4. The first step is to compute the twisted Jacquet module

$$\tau_{\mathrm{GL}(1)}^{P_3}(\chi)_{\left[\begin{smallmatrix} 1 & & \\ & 1 & * \\ & & 1 \end{smallmatrix}\right], \psi}.$$

Again, we define a map

$$T : \tau_{\mathrm{GL}(1)}^{P_3}(\chi) \to \mathcal{S}(F)$$

by

$$T(f)(x) = f(\begin{bmatrix} 1 & & \\ x & 1 & \\ & & 1 \end{bmatrix}).$$

Proceeding as in the proof of Lemma 2.5.4, we prove that T induces an isomorphism of F spaces

$$\tau_{\mathrm{GL}(1)}^{P_3}(\chi)_{\left[\begin{smallmatrix}1&&\\&1&*\\&&1\end{smallmatrix}\right],\psi} \xrightarrow[\sim]{T} \mathcal{S}(F),$$

where F acts by translation on the second space. Let λ be a linear functional as in the statement of the lemma. By the first transformation property of λ, λ is trivial on the kernel of T, and thus induces a linear functional of $\mathcal{S}(F)$. By the second transformation property of λ this linear functional is invariant under translation by F. By the proposition in 1.18 of [BZ], this induced linear functional is a multiple of the Haar measure on F, i.e., the linear functional on $\mathcal{S}(F)$ which sends $\varphi \in \mathcal{S}(F)$ to

$$\int\limits_F \varphi(x)\,dx.$$

This proves the first assertion of the lemma. The second assertion of the lemma follows by a computation. □

Lemma 2.5.6. *Let ρ be an irreducible, admissible representation of* $\mathrm{GL}(2,F)$*. If $\lambda : \tau_{\mathrm{GL}(2)}^{P_3}(\rho) \to \mathbb{C}$ is a linear functional such that*

$$\lambda(\begin{bmatrix}1&&\\&1&y\\&&1\end{bmatrix} f) = \psi(y)\lambda(f)$$

for $y \in F$ and $f \in \tau_{\mathrm{GL}(2)}^{P_3}(\rho)$, then $\lambda = 0$.

Proof. This follows immediately from the definition of $\tau_{\mathrm{GL}(2)}^{P_3}(\rho)$. □

The following proposition applies the last three lemmas to deduce a result about the uniqueness of certain linear functionals, called *Bessel functionals of split type*, on irreducible admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character. This result will be used to prove the functional equation for zeta integrals.

Proposition 2.5.7. *Let (π, V) be an irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character. Let $C(\pi)$ be the finite set of characters $\alpha : F^\times \to \mathbb{C}^\times$ such that $\tau_{\mathrm{GL}(1)}^{P_3}(\alpha)$ is an irreducible subquotient of V_{Z^J}. Let $\chi : F^\times \to \mathbb{C}^\times$ be a character such that $|\cdot|\chi$ is not contained in $C(\pi)$. Then the $\mathbb{C}$ vector space of linear functionals $L : V \to \mathbb{C}$ such that*

i) $L(\pi(\begin{bmatrix}1&y&&\\&1&&\\&&1&-y\\&&&1\end{bmatrix})v) = \psi(c_1y)L(v)$ *for $y \in F$ and $v \in V$;*

ii) $L(\pi(\begin{bmatrix}1&&&\\&1&&\\&x&1&\\&&&1\end{bmatrix})v) = L(v)$ *for $x \in F$ and $v \in V$;*

iii) $L(\pi(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix})v) = \chi(a)L(v)$ *for* $a \in F^\times$ *and* $v \in V$;

iv) $L(\pi(\begin{bmatrix} 1 &&& z \\ & 1 && \\ && 1 & \\ &&& 1 \end{bmatrix})v) = L(v)$ *for* $z \in F$ *and* $v \in V$

is at most one-dimensional.

Proof. Let L and L' be non-zero linear functionals as in the statement of the proposition. We need to prove that there exists $c \in \mathbb{C}$ such that $L' = cL$. By precomposing L and L' with

$$\pi(\begin{bmatrix} -c_1^{-1} &&& \\ & 1 && \\ && 1 & \\ &&& -c_1 \end{bmatrix})$$

we may assume that $c_1 = -1$. By iv), the linear functionals L and L' induce linear functionals on V_{Z^J}; we will also denote these linear functionals by L and L'. Recall the chain of subspaces

$$0 \subset V_2 \subset V_1 \subset V_0 = V_{Z^J}$$

from Theorem 2.5.3. By i), ii), iii) and Lemma 2.5.4, there exists $c \in \mathbb{C}$ such that $L'(v) = cL(v)$ for $v \in V_2$ (if $L\big|_{V_2} = 0$, then switch the roles of L and L'). Let $J = L' - cL$. We have $J(V_2) = 0$. Suppose that $J(V_1) \neq 0$. Then by i) of Theorem 2.5.3, there exist P_3 subspaces $U, U' \subset V_1$ such that $V_2 \subset U \subset U'$, $J(U) = 0$, $J(U') \neq 0$, and $U'/U \cong \tau_{\mathrm{GL}(1)}^{P_3}(\alpha)$ for some $\alpha \in C(\pi)$. By i), ii), iii) and Lemma 2.5.5, we must have $\chi = |\cdot|^{-1}\alpha$; this contradicts $|\cdot|\chi \notin C(\pi)$. Therefore, $J(V_1) = 0$. A similar argument using Lemma 2.5.6 shows that $J(V_{Z^J}) = 0$, proving the proposition. □

2.6 Zeta Integrals

A certain theory of zeta integrals for generic, irreducible, admissible representations of $\mathrm{GSp}(4, F)$ plays an important role in this work. This theory assigns to every such π an L-factor $L(s, \pi)$ and an ε-factor $\varepsilon(s, \pi, \psi_{c_1,c_2})$. For the convenience of the reader, we will carefully review this theory. This theory was first considered by Novodvorsky. See [N] and the references it contains. Some subsequent important references are Part B of [GPSR], [Bu], [So], [B] and [Tak]. To begin, we need to recall some fundamental results about generic representations. The first result is about the behavior of elements of a Whittaker model on the diagonal subgroup of $\mathrm{GSp}(4, F)$.

Lemma 2.6.1. *Let π be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4,F)$. *There exist a positive integer N, unitary characters $\chi_1,\ldots,\chi_N$, $\chi'_1,\ldots,\chi'_N$, real numbers $u_1,\ldots,u_N$, $u'_1,\ldots,u'_N$, nonnegative integers $n_1\geq 0,\ldots,n_N\geq 0$ and $n'_1\geq 0,\ldots,n'_N\geq 0$ with the following property: For any $W\in\mathcal{W}(\pi,\psi_{c_1,c_2})$ there exist $\varphi_1,\ldots,\varphi_N\in\mathcal{S}(F\times F)$ such that*

$$W(\begin{bmatrix}ab&&&\\&a&&\\&&1&\\&&&b^{-1}\end{bmatrix})=\sum_{k=1}^{N}\varphi_k(a,b)\chi_k(a)\chi'_k(b)|a|^{u_k}|b|^{u'_k}(\log|a|)^{n_k}(\log|b|)^{n'_k}.$$

Proof. This is Proposition 1.1.1 on page 155 of [J]. □

Lemma 2.6.2. *Let π be a generic, irreducible and admissible representation of* $\mathrm{GSp}(4,F)$. *Let $W\in\mathcal{W}(\pi,\psi_{c_1,c_2})$, and let $M\geq 0$ be a nonnegative integer such that $W(gk)=W(g)$ for $g\in\mathrm{GSp}(4,F)$ and $k\in\mathrm{GSp}(4,\mathfrak{o})$ with $k\equiv 1$ mod $\mathfrak{p}^M$.*

i) If $a,b\in F^\times$ and $v(a)<v(b)-M$ or $2v(b)<v(c)-M$, then

$$W(\begin{bmatrix}a&&&\\&b&&\\&&cb^{-1}&\\&&&ca^{-1}\end{bmatrix})=0.$$

ii) If $a\in F^\times$, and $x\notin\mathfrak{p}^{-M}$, then $W(\begin{bmatrix}a&&&\\&a&&\\&x&1&\\&&&1\end{bmatrix})=0.$

Proof. i) Let $a,b\in F^\times$. Let $x,y\in\varpi^M\mathfrak{o}$. Then

$$\begin{aligned}W(\begin{bmatrix}a&&&\\&b&&\\&&cb^{-1}&\\&&&ca^{-1}\end{bmatrix})&=W(\begin{bmatrix}a&&&\\&b&&\\&&cb^{-1}&\\&&&ca^{-1}\end{bmatrix}\begin{bmatrix}1&x&xy&\\&1&y&\\&&1&-x\\&&&1\end{bmatrix})\\&=\psi(c_1ab^{-1}x)\psi(c_2c^{-1}b^2y)W(\begin{bmatrix}a&&&\\&b&&\\&&cb^{-1}&\\&&&ca^{-1}\end{bmatrix}).\end{aligned}$$

Suppose $v(a)<v(b)-M$, i.e., $v(ab^{-1})<-M$. Recalling that $\psi(\mathfrak{p}^{-1})\neq 1$ and $c_1\in\mathfrak{o}^\times$, it follows that $x\mapsto\psi(c_1ab^{-1}x)$ defines a non-trivial character of $\varpi^M\mathfrak{o}$. Letting $y=0$ in the above equation and integrating over $\varpi^M\mathfrak{o}$ gives

$$(\int\limits_{\varpi^M\mathfrak{o}}dx)W(\begin{bmatrix}a&&&\\&b&&\\&&cb^{-1}&\\&&&ca^{-1}\end{bmatrix})=0.$$

The proof for the case $2v(b) < v(c) - M$ is similar.

ii) Let $x \notin \mathfrak{p}^{-M}$, so that $x^{-1} \in \mathfrak{p}^M$ and $v(x) < -M$. Let $a \in F^\times$. Then, by the useful identity (2.8), we have

$$W(\begin{bmatrix} a &&& \\ & a && \\ & x & 1 & \\ &&& 1 \end{bmatrix}) = W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 & x^{-1} & \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & -x^{-1} && \\ && -x & \\ &&& 1 \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 &&& \\ && 1 & \\ & -1 && \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 & x^{-1} & \\ && 1 & \\ &&& 1 \end{bmatrix})$$

$$= \psi(c_2 a x^{-1})(\pi(s_2)W)(\begin{bmatrix} a &&& \\ & -ax^{-1} && \\ && -x & \\ &&& 1 \end{bmatrix}).$$

The element $\pi(s_2)W$ is also invariant under the elements $k \in \mathrm{GSp}(4, \mathfrak{o})$, $k \equiv 1 \mod \mathfrak{p}^M$. Applying i) to $\pi(s_2)W$ we get

$$W(\begin{bmatrix} a &&& \\ & a && \\ & x & 1 & \\ &&& 1 \end{bmatrix}) = 0$$

since $v(a) < v(-ax^{-1}) - M$. $\square$

Proposition 2.6.3. *Let π be a generic, irreducible and admissible representation of $\mathrm{GSp}(4, F)$, and let the notation be as in Lemma 2.6.1. For all $W \in W(\pi, \psi_{c_1,c_2})$ and $s \in \mathbb{C}$ with $\mathrm{Real}(s) > 3/2 - \min(u_1, \ldots, u_N)$ the* zeta integral

$$Z(s, W) := \int\limits_{F^\times} \int\limits_{F} W(\begin{bmatrix} a &&& \\ & a && \\ & x & 1 & \\ &&& 1 \end{bmatrix})|a|^{s-3/2}\, dx\, d^\times a \qquad (2.54)$$

converges absolutely to an element of $\mathbb{C}(q^{-s})$.

Proof. Let $W \in W(\pi, \psi_{c_1,c_2})$. Let M be defined as in Lemma 2.6.2. Then

$$\int\limits_{F^\times} \int\limits_{F} |W(\begin{bmatrix} a &&& \\ & a && \\ & x & 1 & \\ &&& 1 \end{bmatrix})||a|^{\mathrm{Real}(s)-3/2}\, dx\, d^\times a$$

$$= \int\limits_{F^\times} \int\limits_{\mathfrak{p}^{-M}} |W(\begin{bmatrix} a &&& \\ & a && \\ & x & 1 & \\ &&& 1 \end{bmatrix})||a|^{\mathrm{Real}(s)-3/2}\, dx\, d^\times a$$

$$= \int\limits_{F^\times} \int\limits_{\mathfrak{p}^{-M}/\mathfrak{p}^M} \int\limits_{\mathfrak{p}^M} |W(\begin{bmatrix} a & & & \\ & a & & \\ & x+y & 1 & \\ & & & 1 \end{bmatrix})||a|^{\mathrm{Real}(s)-3/2}\, dy\, dx\, d^\times a$$

$$= \sum_{x \in \mathfrak{p}^{-M}/\mathfrak{p}^M} q^{-M} \int\limits_{F^\times} |W_x(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix})||a|^{\mathrm{Real}(s)-3/2}\, d^\times a,$$

where

$$W_x = \pi(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & x & 1 & \\ & & & 1 \end{bmatrix})W$$

for $x \in F$. Fix $x \in \mathfrak{p}^{-M}$. Applying Lemma 2.6.1 to W_x, we have

$$W_x(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix}) = \sum_{\substack{k \in \{1,\dots,N\} \\ n'_k = 0}} \varphi_k(a,1)\chi_k(a)|a|^{u_k}(\log|a|)^{n_k}.$$

Therefore,

$$\int\limits_{F^\times} |W_x(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix})||a|^{\mathrm{Real}(s)-3/2}\, d^\times a$$

$$\leq \sum_{\substack{k \in \{1,\dots,N\} \\ n'_k = 0}} \int\limits_{F^\times} |\varphi_k(a,1)||a|^{\mathrm{Real}(s)+u_k-3/2}|\log|a||^{n_k}\, d^\times a.$$

Let $k \in \{1, \dots, N\}$ with $n'_k = 0$. A computation proves that

$$\int\limits_{F^\times} |\varphi_k(a,1)||a|^{\mathrm{Real}(s)+u_k-3/2}|\log|a||^{n_k}\, d^\times a < \infty$$

for $\mathrm{Real}(s) > 3/2 - u_k$. The claim about convergence from the statement of the proposition follows.

Next, an argument similar to the one above proves that the function $s \mapsto Z(s, W)$, defined on $\mathrm{Real}(s) > 3/2 - \min(u_1, \dots, u_N)$, is a sum of functions of the form

$$\int\limits_{F^\times} \varphi(a)\chi_k(a)|a|^{s+u_k-3/2}(\log|a|)^{n_k}\, d^\times a \tag{2.55}$$

for some $\varphi \in \mathcal{S}(F)$ and k such that $1 \leq k \leq N$ and $n'_k = 0$. Fix such a φ and k. Let $M > 0$ be a positive integer such that φ is constant on $\mathfrak{p}^M$ and has support in $\mathfrak{p}^{-M}$. A computation shows that (2.55) is

$$\varphi(0)(\int\limits_{\mathfrak{o}^\times}\chi_k(u)\,d^\times u)(-\log q)^{n_k}\sum_{i=M}^{\infty} i^{n_k}(\chi_k(\varpi)q^{-(s+u_k-3/2)})^i$$
$$+(-\log q)^{n_k}\sum_{i=-M}^{M+1}\big(\int\limits_{\varpi^i\mathfrak{o}^\times}\varphi(a)\chi_k(a)\,d^\times a\big)i^{n_k}(q^{-(s+u_k-3/2)})^i. \qquad (2.56)$$

This is an element of $\mathbb{C}(q^{-s})$. □

Let π be a generic, irreducible, admissible representation of $\mathrm{GSp}(4,F)$. It is important to understand the dependence of the $Z(s,W)$ on the choice of $c_1,c_2\in\mathfrak{o}^\times$ in the definition of the Whittaker model. Suppose that $c_1,c_2,c_1'c_2'\in\mathfrak{o}^\times$. A straightforward computation verifies that

$$\begin{aligned}W\in&\mathcal{W}(\pi,\psi_{c_1,c_2})\\ &\iff W'=W(\begin{bmatrix}\frac{c_1'}{c_1}&&&\\&1&&\\&&\frac{c_2}{c_2'}&\\&&&\frac{c_1c_2}{c_1'c_2'}\end{bmatrix}\cdot)\in\mathcal{W}(\pi,\psi_{c_1',c_2'}).\end{aligned}$$

Another computation shows that for $W\in\mathcal{W}(\pi,\psi_{c_1,c_2})$,

$$\begin{aligned}&\int\limits_{F^\times}\int\limits_{F}W(\begin{bmatrix}a&&&\\&a&&\\&x&1&\\&&&1\end{bmatrix})|a|^{s-3/2}\,dx\,d^\times a\\ &\quad=\int\limits_{F^\times}\int\limits_{F}W'(\begin{bmatrix}a&&&\\&a&&\\&x&1&\\&&&1\end{bmatrix}\begin{bmatrix}\frac{c_1'}{c_1}&&&\\&1&&\\&&\frac{c_2}{c_2'}&\\&&&\frac{c_1c_2}{c_1'c_2'}\end{bmatrix}^{-1})|a|^{s-3/2}\,dx\,d^\times a.\end{aligned} \qquad (2.57)$$

It follows that $I(\pi)$, the $\mathbb{C}$ vector subspace of $\mathbb{C}(q^{-s})$ spanned by the $Z(s,W)$ for $W\in\mathcal{W}(\pi,\psi_{c_1,c_2})$, does not depend on the choice of $c_1,c_2\in\mathfrak{o}^\times$. We call $I(\pi)$ the *zeta integral ideal* of π. The next result shows that $I(\pi)$ really is a fractional ideal of the appropriate ring.

Proposition 2.6.4. *Let π be a generic, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. Then $I(\pi)$ is a non-zero $\mathbb{C}[q^{-s},q^s]$ module containing $\mathbb{C}$, and there exists $R(X)\in\mathbb{C}[X]$ such that $R(q^{-s})I(\pi)\subset\mathbb{C}[q^{-s},q^s]$, so that $I(\pi)$ is a fractional ideal of the principal ideal domain $\mathbb{C}[q^{-s},q^s]$ whose quotient field is $\mathbb{C}(q^{-s})$. The fractional ideal $I(\pi)$ admits a generator of the form $1/Q(q^{-s})$ with $Q(0)=1$, where $Q(X)\in\mathbb{C}[X]$.*

Proof. To see that $I(\pi)$ is a $\mathbb{C}[q^{-s},q^s]$ submodule of $\mathbb{C}(q^{-s})$ it suffices to show that if $W\in\mathcal{W}(\pi,\psi_{c_1,c_2})$, then $q^{\pm s}Z(s,W)\in I(\pi)$. Let $W\in\mathcal{W}(\pi,\psi_{c_1,c_2})$ and $\varepsilon\in\{\pm1\}$. Then

$$Z(s,\pi(\begin{bmatrix}\varpi^{\varepsilon}&&&\\&\varpi^{\varepsilon}&&\\&&1&\\&&&1\end{bmatrix})W)=q^{\varepsilon s}q^{-\varepsilon/2}Z(s,W).$$

Therefore, $q^{\varepsilon s}Z(s,W)\in I(\pi)$.

Next, we prove that $I(\pi)$ contains $\mathbb{C}$. To do this we will use P_3-theory. Since it is convenient, we will take $c_1=-1$ and $c_2=1$. Let V be a model for π, and let $p:V\to V_{Z^J}=V/V(Z^J)$ be the projection map. Let $V_2\subset V_{Z^J}$ be as in Lemma 2.5.3, and define $X=p^{-1}(V_2)$. Then X is a Q subspace of V. Since π is generic, V_2 is non-zero, and $V_2\cong\tau_{\mathrm{GL}(0)}^{P_3}(\mathbf{1})=\text{c-Ind}_{U_3}^{P_3}\Theta$. Fix an isomorphism

$$V_2\xrightarrow[\sim]{j}\text{c-Ind}_{U_3}^{P_3}\Theta,$$

let

$$\text{c-Ind}_{U_3}^{P_3}\Theta\xrightarrow{e}\mathbb{C}$$

be evaluation at the identity, and let l be the composition

$$X=p^{-1}(V_2)\xrightarrow{p}V_2\xrightarrow{j}\text{c-Ind}_{U_3}^{P_3}\Theta\xrightarrow{e}\mathbb{C}.$$

A computation verifies that

$$l(\pi(\begin{bmatrix}1&x&*&*\\&1&y&*\\&&1&-x\\&&&1\end{bmatrix})W)=\psi(-x+y)l(W)$$

for $W\in X$ and $x,y\in F$. Thus, l is a non-zero element of $\mathrm{Hom}_U(X,\psi_{-1,1})$, and l may be regarded as a non-zero linear map $X_{U,\psi_{-1,1}}\to\mathbb{C}$. By b) of 2.35 of [BZ], the sequence

$$0\longrightarrow X_{U,\psi_{-1,1}}\longrightarrow V_{U,\psi_{-1,1}}$$

is exact. Since π is generic and Whittaker functionals are unique up to scalars, $V_{U,\psi_{-1,1}}$ is one-dimensional. It follows that l admits an extension to V that is an element of $\mathrm{Hom}_U(V,\psi_{-1,1})$. Let $V=\mathcal{W}(\pi,\psi_{-1,1})$. The evaluation linear functional on V which sends $W\in V$ to $W(1)$ also defines a non-zero element of $\mathrm{Hom}_U(V,\psi_{-1,1})$. Therefore, there exists $c\in\mathbb{C}^\times$ such that $l(W)=cW(1)$ for $W\in X$. By absorbing c into j we may assume $c=1$. Let W be in X, and set $f=j(p(W))\in\text{c-Ind}_{U_3}^{P_3}(\Theta)$. Then, for $q\in Q$,

$$\begin{aligned}W(q)&=(\pi(q)W)(1)\\&=(j\circ p)(\pi(q)W)(1)\\&=j(p(\pi(q)W))(1)\\&\overset{(2.51)}{=}j(i(q)p(W))(1)\end{aligned}$$

$$= (i(q)j(p(W)))(1)$$
$$= j(p(W))(i(q))$$
$$= f(i(q)).$$

Hence

$$Z(s,W) = \int\limits_{F^\times}\int\limits_{F} W(\begin{bmatrix} a & & & \\ & a & & \\ & x & 1 & \\ & & & 1 \end{bmatrix})|a|^{s-3/2}\,dx\,d^\times a$$
$$= \int\limits_{F^\times}\int\limits_{F} f(\begin{bmatrix} a & & \\ x & 1 & \\ & & 1 \end{bmatrix})|a|^{s-3/2}\,dx\,d^\times a.$$

Therefore, $I(\pi)$ contains all the elements of $\mathbb{C}(q^{-s})$ of the form

$$\int\limits_{F^\times}\int\limits_{F} f(\begin{bmatrix} a & & \\ x & 1 & \\ & & 1 \end{bmatrix})|a|^{s-3/2}\,dx\,d^\times a \tag{2.58}$$

for $f \in \text{c-Ind}_{U_3}^{P_3}\Theta$. If $q \in P_3$, we can write

$$q = \begin{bmatrix} 1 & u_{12} & u_{13} \\ & 1 & u_{23} \\ & & 1 \end{bmatrix} \begin{bmatrix} t_1 & & \\ & t_2 & \\ & & 1 \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} & \\ k_{21} & k_{22} & \\ & & 1 \end{bmatrix}$$

where $u_{12}, u_{13}, u_{23} \in F$, $t_1, t_2 \in F^\times$ and

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \in \mathrm{GL}(2, \mathfrak{o}).$$

Define $f_0 \in \text{c-Ind}_{U_3}^{P_3}\Theta$ by

$$f_0(q) = \psi(u_{12} + u_{23})\chi_{\mathfrak{o}^\times}(t_1)\chi_{\mathfrak{o}^\times}(t_2).$$

A computation verifies that f_0 is well-defined and gives an element of $\text{c-Ind}_{U_3}^{P_3}\Theta$. Using (2.8) it is easily checked that the value of the integral (2.58) (with f_0 in place of f) is constant. This proves that $\mathbb{C} \subset I(\pi)$.

To prove the existence of $R(X)$ as in the statement of the proposition we note that in the proof of Proposition 2.6.3 we showed that the elements of $I(\pi)$ are linear combinations of functions of the form (2.56) with $\varphi \in \mathcal{S}(F)$ and k such that $1 \le k \le N$ and $n_k' = 0$. Since M, the n_k, the χ_k and the u_k are determined by π, there exists $R(X) \in \mathbb{C}[X]$ such that $R(q^{-s})$ multiplied by the functions in (2.56) is contained in $\mathbb{C}[q^{-s}, q^s]$.

Finally, since $I(\pi) \neq 0$ is a fractional ideal of $\mathbb{C}[q^{-s}, q^s]$, and since $\mathbb{C}[q^{-s}, q^s]$ is a principal ideal domain, there exist $P_1(X), P_2(X) \in \mathbb{C}[X]$ such that $I(\pi) =$

$\mathbb{C}[q^{-s},q^s](P_1(q^{-s})/P_2(q^{-s}))$. We may assume that $P_1(X)$ and $P_2(X)$ are relatively prime. Since $1 \in I(\pi)$, there exists a polynomial $A(X,Y) \in \mathbb{C}[X,Y]$ such that $1 = A(q^{-s},q^s)(P_1(q^{-s})/P_2(q^{-s}))$, so that $P_2(q^{-s}) = A(q^{-s},q^s)P_1(q^{-s})$. Since $P_1(X)$ and $P_2(X)$ are relatively prime, and any element z of $\mathbb{C}^\times$ can be written as $z = q^{-s}$ for some $s \in \mathbb{C}$, it follows that $P_1(X)$ has no non-zero roots. Write $P_1(X) = aX^t$ for $t \geq 0$ a nonnegative integer and $a \in \mathbb{C}^\times$. Then $I(\pi) = \mathbb{C}[q^{-s},q^s](a(q^{-s})^t/P_2(q^{-s})) = \mathbb{C}[q^{-s},q^s](1/P_2(q^{-s}))$. Write $P_2(X) = bX^rQ(X)$ for $r \geq 0$ a nonnegative integer, $b \in \mathbb{C}^\times$, and $Q(X) \in \mathbb{C}[X]$ with $Q(0) = 1$. Then $I(\pi) = \mathbb{C}[q^{-s},q^s](1/Q(q^{-s}))$. □

With Q being as in the proposition, the function

$$L(s,\pi) = \frac{1}{Q(q^{-s})}$$

is called the *L-function* of the generic, irreducible, admissible representation π. These functions have been explicitly computed in [Tak].

The zeta integrals (2.54) satisfy a local functional equation. We require it only for representations π with trivial central character, so, for simplicity, we shall make this assumption in the statement of the result. Let

$$w = u_0 = \begin{bmatrix} & & 1 & \\ & & & -1 \\ 1 & & & \\ & -1 & & \end{bmatrix} = s_2^{-1}s_1s_2. \tag{2.59}$$

Proposition 2.6.5. *Let π be a generic, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. Then there exists an element $\gamma(s,\pi,\psi_{c_1,c_2})$ of $\mathbb{C}(q^{-s})$ such that*

$$Z(1-s,\pi(w)W) = \gamma(s,\pi,\psi_{c_1,c_2})Z(s,W) \tag{2.60}$$

for W in $\mathcal{W}(\pi,\psi_{c_1,c_2})$. The γ-factor $\gamma(s,\pi,\psi_{c_1,c_2})$ does not depend on the choice of $c_1,c_2 \in \mathfrak{o}^\times$.

Proof. By Proposition 2.6.3 there exists a positive real number σ such that for $\mathrm{Real}(s) > \sigma$ and $W \in \mathcal{W}(\pi,\psi_{c_1,c_2})$ the integral defining $Z(s,W)$ converges absolutely. Let $I(\pi) = (1/Q(q^{-s}))\mathbb{C}[q^{-s},q^s]$ as in Proposition 2.6.4. We may assume that σ is sufficiently large so that $Q(q^{-s}) \neq 0$ and $Q(q^{-(1-s)}) \neq 0$ for $\mathrm{Real}(s) > \sigma$. It follows that $Z(s,W)$, regarded as a rational functional in q^{-s}, can be evaluated at s and $1-s$ for $\mathrm{Real}(s) > \sigma$ and for all $W \in \mathcal{W}(\pi,\psi_{c_1,c_2})$. Fix $s \in \mathbb{C}$ such that $\mathrm{Real}(s) > \sigma$. We consider the linear functionals

$$L : \mathcal{W}(\pi,\psi_{c_1,c_2}) \to \mathbb{C}, \quad L' : \mathcal{W}(\pi,\psi_{c_1,c_2}) \to \mathbb{C}$$

defined by

$$L(W) = Z(s,W), \quad L'(W) = Z(1-s,\pi(w)W)$$

for $W \in \mathcal{W}(\pi, \psi_{c_1,c_2})$. Computations show that L and L' satisfy i), ii), iii) and iv) of Proposition 2.5.7 with $\chi = |\cdot|^{1/2-s}$. We may assume that σ is sufficiently large so that $|\cdot||\cdot|^{1/2-s}$ is not contained in $C(\pi)$; see Proposition 2.5.7 for the notation. Also, since $I(\pi)$ contains $\mathbb{C}$, we have $L \neq 0$. By Proposition 2.5.7 there exists a complex number $\gamma(s, \pi, \psi_{c_1,c_2})$ such that $L' = \gamma(s, \pi, \psi_{c_1,c_2})L$, i.e.,

$$Z(1-s, \pi(w)W) = \gamma(s, \pi, \psi_{c_1,c_2})Z(s, W)$$

for $W \in \mathcal{W}(\pi, \psi_{c_1,c_2})$. Choose $W \in \mathcal{W}(\pi, \psi_{c_1,c_2})$ such that $Z(s, W) = 1$. Then $\gamma(s, \pi, \psi_{c_1,c_2}) = Z(1-s, \pi(w)W)$ for $\mathrm{Real}(s) > \sigma$, so that the function $\gamma(s, \pi, \psi_{c_1,c_2})$ is a rational functional in q^{-s}. The first claim of the proposition follows. The second assertion of the proposition is proved using the functional equation and the equality (2.57). □

We note that several references incorrectly state that the functional equation (2.60) holds with w replaced with certain other elements. The paper [N] asserts in Theorem 1 that (2.60) holds with w replaced by

$$\begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}.$$

This is false, since

$$\begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix} \begin{bmatrix} 1 & y & & \\ & 1 & & \\ & & 1 & -y \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -y & & \\ & 1 & & \\ & & 1 & y \\ & & & 1 \end{bmatrix}.$$

This error also appears in [B], pages 49-50. Similarly, the reference [Bu], page 93, asserts that the functional equation holds with w replaced by

$$\begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix}.$$

This is also false, for the same reason. The same error appears in [Tak], Theorem 2.1.

Let π be a generic, irreducible, admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character. Another form of the functional equation involves the Atkin–Lehner element u_n defined in (2.2). Easy algebra shows that

$$Z(1-s, \pi(u_n)W) = q^{n(s-1/2)}\gamma(s, \pi, \psi_{c_1,c_2})Z(s, W) \tag{2.61}$$

for all $W \in \mathcal{W}(\pi, \psi_{c_1,c_2})$. The *ε-factor* of the representation π with trivial central character is defined as

$$\varepsilon(s,\pi,\psi_{c_1,c_2}) = \gamma(s,\pi,\psi_{c_1,c_2})\frac{L(s,\pi)}{L(1-s,\pi)}.$$

Using ε, the functional equation takes the form

$$\frac{Z(1-s,\pi(w)W)}{L(1-s,\pi)} = \varepsilon(s,\pi,\psi_{c_1,c_2})\frac{Z(s,W)}{L(s,\pi)}. \tag{2.62}$$

Proposition 2.6.6. *Let π be a generic, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. The factor $\varepsilon(s,\pi,\psi_{c_1,c_2})$ does not depend on the choice of $c_1,c_2 \in \mathfrak{o}^\times$. There exists $\varepsilon = \pm 1$ and an integer N such that $\varepsilon(s,\pi,\psi_{c_1,c_2}) = \varepsilon q^{-N(s-1/2)}$.*

Proof. The factor $\varepsilon(s,\pi,\psi_{c_1,c_2})$ does not depend on the choice of $c_1,c_2 \in \mathfrak{o}^\times$ since $\gamma(s,\pi,\psi_{c_1,c_2})$ does not depend on $c_1,c_2 \in \mathfrak{o}^\times$. A computation using the definition of $\varepsilon(s,\pi,\psi_{c_1,c_2})$ shows that

$$\varepsilon(s,\pi,\psi_{c_1,c_2})\varepsilon(1-s,\pi,\psi_{c_1,c_2}) = \gamma(s,\pi,\psi_{c_1,c_2})\gamma(1-s,\pi,\psi_{c_1,c_2}).$$

Two applications of the functional equation (2.60) yield

$$Z(1-s,\pi(w)W) = \gamma(s,\pi,\psi_{c_1,c_2})\gamma(1-s,\pi,\psi_{c_1,c_2})Z(1-s,\pi(w)W)$$

for $W \in \mathcal{W}(\pi,\psi_{c_1,c_2})$. By Proposition 2.6.4, there exists $W \in \mathcal{W}(\pi,\psi_{c_1,c_2})$ such that $Z(1-s,\pi(w)W) \neq 0$. Hence, $\gamma(s,\pi,\psi_{c_1,c_2})\gamma(1-s,\pi,\psi_{c_1,c_2}) = 1$, so that $\varepsilon(s,\pi,\psi_{c_1,c_2})\varepsilon(1-s,\pi,\psi_{c_1,c_2}) = 1$. Next, let $W_0 \in \mathcal{W}(\pi,\psi_{c_1,c_2})$ such that $Z(s,W_0) = L(s,\pi)$. By the ε-factor form (2.62) of the functional equation, we have

$$\varepsilon(s,\pi,\psi_{c_1,c_2}) = \frac{Z(1-s,\pi(w)W_0)}{L(1-s,\pi)}.$$

By Proposition 2.6.4, this is a polynomial in q^{-s} and q^s. Hence, there exists an integer M such that $q^{-Ms}\varepsilon(s,\pi,\psi_{c_1,c_2}) = R(q^{-s})$ for some $R(X) \in \mathbb{C}[X]$. Using $\varepsilon(s,\pi,\psi_{c_1,c_2})\varepsilon(1-s,\pi,\psi_{c_1,c_2}) = 1$ we get $q^M R(q^{-s})R(q^{-(1-s)}) = 1$. It follows that R has no zeros in $\mathbb{C}^\times$; hence, R is of the form aq^{-bs} for some $a \in \mathbb{C}^\times$ and integer b. Therefore, we may write $\varepsilon(s,\pi,\psi_{c_1,c_2}) = \varepsilon q^{-N(s-1/2)}$ for some $\varepsilon \in \mathbb{C}^\times$ and integer N. Evaluating at $1/2$, we obtain $\varepsilon(1/2,\pi,\psi_{c_1,c_2}) = \varepsilon$; on the other hand, $1 = \varepsilon(1/2,\pi,\psi_{c_1,c_2})\varepsilon(1-1/2,\pi,\psi_{c_1,c_2}) = \varepsilon^2$, so that $\varepsilon = \pm 1$. □

Our work will result in more precise information about N and the sign ε from Proposition 2.6.6.

3

Paramodular Vectors

In this chapter we define and begin the analysis of paramodular vectors in representations of $\mathrm{GSp}(4,F)$ with trivial central character. In the first section we prove that paramodular vectors at different levels are linearly independent provided that the subspace of vectors fixed by $\mathrm{Sp}(4,F)$ is trivial. In the second section we introduce three level raising operators and prove that, except for one explicit non-generic Iwahori-spherical representation, these level raising operators are injective. Thus, except for this representation, the spaces of paramodular vectors in irreducible, admissible representations grow as the level increases. These level raising operators are integral to many of the proofs of the main results of this work, and are precisely the operators needed to obtain oldforms from newforms. The third section defines certain level lowering operators and gives explicit formulas. These level lowering operators are less important for our purposes than the level raising operators, but they appear in certain relations involving Hecke operators in Chap. 6. The methods used to prove the results of the first three sections are algebraic, and do not require explicit models or realizations. The final section of this chapter gives the first indication that P_3-theory is useful in the study of paramodular vectors. The basic reason for this is that non-zero paramodular vectors do not vanish when projected to the P_3-filtration. As a first consequence of P_3-theory we determine a list of non-generic, irreducible, admissible representations with trivial central characters that do not admit non-zero paramodular vectors. This list includes all non-generic supercuspidal representations; in fact, this list turns out to be exactly the list of all non-paramodular representations.

3.1 Linear Independence

To begin, let (π, V) be a smooth representation of $\mathrm{GSp}(4,F)$ for which the center of $\mathrm{GSp}(4,F)$ acts trivially. For a non-negative integer $n \geq 0$, we denote by $V(n)$ the subspace of V consisting of all vectors $v \in V$ such that $\pi(k)v = v$ for all $k \in \mathrm{K}(\mathfrak{p}^n)$. Here, $\mathrm{K}(\mathfrak{p}^n)$ is the paramodular group of level $\mathfrak{p}^n$ as defined in Sect. 2.1. A non-zero vector in any of the $V(n)$ is called a *paramodular vector*. By the definition of admissibility, if π is admissible, then $V(n)$ is finite-dimensional for all n. Another initial observation involves the operator $\pi(u_n)$ where u_n is the Atkin–Lehner element as in (2.2). Since $\mathrm{K}(\mathfrak{p}^n)$ is normalized by u_n, the operator $\pi(u_n)$ induces an endomorphism of $V(n)$. Since we assumed that the center of π acts trivially, and since u_n^2 is a scalar, $\pi(u_n^2)$ is the identity on V. Consequently, the endomorphism $\pi(u_n)$ of $V(n)$ is an involution and splits $V(n)$ into ± 1 eigenspaces $V_\pm(n)$.

Our first result in this chapter asserts that paramodular vectors at different levels are linearly independent. A consequence of this is that no $V(n)$ is contained in a $V(m)$ for $m \neq n$, and even $V(n) \cap V(m) = 0$ for $m \neq n$. To prove linear independence we need a lemma which asserts that with one extra particular element, the intersection of a finite collection of paramodular groups generates a group containing $\mathrm{Sp}(4,F)$. This lemma in turn requires the following analogous result about $\mathrm{SL}(2,F)$.

Lemma 3.1.1. *Let $N > 0$ be a positive integer. The subgroup J of* $\mathrm{SL}(2,F)$ *generated by*

$$\begin{bmatrix} 1 & \mathfrak{p}^{-N} \\ & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$$

is $\mathrm{SL}(2,F)$.

Proof. The subgroup J contains

$$\begin{bmatrix} 1 & \mathfrak{o} \\ & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & \\ \mathfrak{o} & 1 \end{bmatrix}$$

and hence $\mathrm{SL}(2,\mathfrak{o})$. The identity

$$\begin{bmatrix} \varpi^{-N} & \\ & \varpi^{N} \end{bmatrix} = \begin{bmatrix} (1+\varpi^N)^{-1} & \\ & 1+\varpi^N \end{bmatrix} \begin{bmatrix} 1 & \\ -(1+\varpi^N)^{-1}\varpi^N & 1 \end{bmatrix}$$
$$\cdot \begin{bmatrix} 1 & \varpi^{-N} \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -(1+\varpi^N)^{-1} \\ & 1 \end{bmatrix}$$

proves that

$$\begin{bmatrix} \varpi^{-N} & \\ & \varpi^{N} \end{bmatrix} \in J.$$

The identities

$$\begin{bmatrix} \varpi^{-N} & \\ & \varpi^{N} \end{bmatrix}^{j} \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} \varpi^{-N} & \\ & \varpi^{N} \end{bmatrix}^{-j} = \begin{bmatrix} 1 & \varpi^{-2Nj}x \\ & 1 \end{bmatrix},$$
$$\begin{bmatrix} \varpi^{-N} & \\ & \varpi^{N} \end{bmatrix}^{-j} \begin{bmatrix} 1 & \\ y & 1 \end{bmatrix} \begin{bmatrix} \varpi^{-N} & \\ & \varpi^{N} \end{bmatrix}^{j} = \begin{bmatrix} 1 & \\ \varpi^{-2Nj}y & 1 \end{bmatrix}$$

for $x, y \in F$ prove that the elements of $\mathrm{SL}(2,F)$ of the form

$$\begin{bmatrix} 1 & * \\ & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & \\ * & 1 \end{bmatrix}$$

are contained in J. It is known that such elements generate $\mathrm{SL}(2,F)$. Hence, $J = \mathrm{SL}(2,F)$. □

The last lemma can be used to prove an analogous result for $\mathrm{GSp}(4,F)$ involving the paramodular group. In direct analogy to Lemma 3.1.1, this statement asserts that a fixed paramodular group, or even the intersection of finitely many paramodular groups, along with a certain extra group element, generates a subgroup that contains at least $\mathrm{Sp}(4,F)$.

Lemma 3.1.2. *Let $k \geq 1$ be an integer, and let $0 \leq n_1 < \cdots < n_k$ be non-negative integers. Let $m \geq 0$ be a non-negative integer such that $m < n_1$ or $n_k < m$. Then the subgroup of $\mathrm{GSp}(4,F)$ generated by*

$$\mathrm{K}(\mathfrak{p}^{n_1}) \cap \cdots \cap \mathrm{K}(\mathfrak{p}^{n_k}) \quad \text{and} \quad t_m = \begin{bmatrix} & & & -\varpi^{-m} \\ & 1 & & \\ & & 1 & \\ \varpi^m & & & \end{bmatrix} \in \mathrm{K}(\mathfrak{p}^m)$$

contains $\mathrm{Sp}(4,F)$.

Proof. Let H be the subgroup generated by $\mathrm{K}(\mathfrak{p}^{n_1}) \cap \cdots \cap \mathrm{K}(\mathfrak{p}^{n_k})$ and t_m. We first prove the claim that H contains the subgroup

$$\begin{bmatrix} a & & & b \\ & 1 & & \\ & & 1 & \\ c & & & d \end{bmatrix}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,F).$$

By definition the group $\mathrm{K}(\mathfrak{p}^{n_1}) \cap \cdots \cap \mathrm{K}(\mathfrak{p}^{n_k})$, and hence H, contains the elements

$$\begin{bmatrix} a & & & b\varpi^{-n_1} \\ & 1 & & \\ & & 1 & \\ c\varpi^{n_k} & & & d \end{bmatrix}$$

where $a,b,c,d \in \mathfrak{o}$ and

$$\begin{bmatrix} a & b\varpi^{-n_1} \\ c\varpi^{n_k} & d \end{bmatrix} \in \mathrm{SL}(2,F).$$

By definition, H also contains t_m. Therefore, to prove our initial claim it will suffice to show that the subgroup H' of $\mathrm{SL}(2,F)$ generated by

$$\begin{bmatrix} & -\varpi^{-m} \\ \varpi^m & \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b\varpi^{-n_1} \\ c\varpi^{n_k} & d \end{bmatrix}, \quad a,b,c,d \in \mathfrak{o}, \quad ad - bc\varpi^{n_k-n_1} = 1$$

is $\mathrm{SL}(2,F)$. To prove $H' = \mathrm{SL}(2,F)$ it will suffice to prove that

$$\begin{bmatrix} \varpi^m & \\ & 1 \end{bmatrix} H' \begin{bmatrix} \varpi^{-m} & \\ & 1 \end{bmatrix} = \mathrm{SL}(2,F).$$

This subgroup H'' is generated by

$$\begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b\varpi^{m-n_1} \\ c\varpi^{n_k-m} & d \end{bmatrix}, \quad a,b,c,d \in \mathfrak{o}, \quad ad - bc\varpi^{n_k-n_1} = 1.$$

Assume $m < n_1$. Then H'' contains

$$\begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & \mathfrak{p}^{m-n_1} \\ & 1 \end{bmatrix}.$$

By Lemma 3.1.1 we have $H'' = \mathrm{SL}(2,F)$. Assume $m > n_k$. Then H'' contains

$$\begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & \\ \mathfrak{p}^{n_k-m} & 1 \end{bmatrix},$$

and again by Lemma 3.1.1, $H'' = \mathrm{SL}(2,F)$. Our claim follows.

Finally, we complete the proof of the lemma. By what we have already shown, H contains all the elements of $\mathrm{Sp}(4,F)$ of the form

$$\begin{bmatrix} 1 & & & * \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

By definition, H contains $\mathrm{K}(\mathfrak{p}^{n_1}) \cap \dots \cap \mathrm{K}(\mathfrak{p}^{n_k})$ and hence

$$\begin{bmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & & 1 & \\ & 1 & & 1 \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Conjugating these elements by the elements

$$\begin{bmatrix} a & & & \\ & 1 & & \\ & & 1 & \\ & & & a^{-1} \end{bmatrix}, \quad a \in F^\times$$

of H proves that H also contains all the elements of $\mathrm{Sp}(4,F)$ of the form

$$\begin{bmatrix} 1 & * & & \\ & 1 & & \\ & & 1 & * \\ & & & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & & * & \\ & 1 & & * \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Therefore, H contains the Heisenberg subgroup of $\mathrm{Sp}(4,F)$, i.e., all the elements of $\mathrm{Sp}(4,F)$ of the form

$$\begin{bmatrix} 1 & * & * & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 \end{bmatrix}.$$

A similar argument proves that H contains the transpose of this subgroup. Since $\mathrm{SL}(2,F)$ is generated by elements of the form

$$\begin{bmatrix} 1 & * \\ & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & \\ * & 1 \end{bmatrix}$$

it follows that H contains the Levi subgroup of the Siegel parabolic subgroup of $\mathrm{Sp}(4,F)$, and in particular the element s_1. Conjugating elements that we already know lie in H by s_1 proves that H contains the elements of $\mathrm{Sp}(4,F)$ of the form

$$\begin{bmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & & & \\ & 1 & & \\ * & * & 1 & \\ * & * & & 1 \end{bmatrix}.$$

These elements generate $\mathrm{Sp}(4,F)$, so that the group generated by H contains $\mathrm{Sp}(4,F)$. □

Using the last generating result it is straightforward to prove the linear independence of paramodular vectors at different levels.

Theorem 3.1.3. *Let (π,V) a smooth representation of $\mathrm{GSp}(4,F)$ such that the center of $\mathrm{GSp}(4,F)$ acts trivially. Assume that the subspace of vectors of V fixed by $\mathrm{Sp}(4,F)$ is trivial. Then paramodular vectors at different levels are linearly independent. More precisely, for $i=1,\ldots,r$ let $v_i \in V(n_i)$, where $n_i \neq n_j$ for $i \neq j$. Then $v_1+\cdots+v_r=0$ implies $v_1=\cdots=v_r=0$.*

Proof. We may assume that $n_1<\cdots<n_r$. The element $-v_1=v_2+\cdots+v_r$ is invariant under t_{n_1} (see (2.3)) and under $\mathrm{K}(\mathfrak{p}^{n_2})\cap\cdots\cap\mathrm{K}(\mathfrak{p}^{n_r})$. Since $n_1<n_2$, it is invariant under $\mathrm{Sp}(4,F)$ by Lemma 3.1.2. Hence, $v_2+\cdots+v_r=0$. Applying the same argument successively gives $v_1=\cdots=v_r=0$. □

Suppose that (π,V) is a smooth representation of $\mathrm{GSp}(4,F)$ such that the center of $\mathrm{GSp}(4,F)$ acts trivially. Let

$$V_{\mathrm{para}}=\bigoplus_{n\geq 0} V(n). \tag{3.1}$$

If the subspace of vectors in V fixed by $\mathrm{Sp}(4,F)$ is trivial, then by Theorem 3.1.3 the space V_{para} can be identified with the subspace spanned by the spaces $V(n)$, $n\geq 0$. One of our goals is to describe the structure of this vector space.

3.2 The Level Raising Operators θ, θ' and η

Let (π, V) be a smooth representation of $\mathrm{GSp}(4, F)$ such that the center of $\mathrm{GSp}(4, F)$ acts trivially. In this section we relate paramodular vectors in V at different levels by defining and studying three level raising operators $\theta : V(n) \to V(n+1)$, $\theta' : V(n) \to V(n+1)$, and $\eta : V(n) \to V(n+2)$ for non-negative integers $n \geq 0$. These operators are examples of a general way to define operators between spaces of paramodular vectors at two levels. We now describe this method.

Level Changing Operators

Again let (π, V) be a smooth representation of $\mathrm{GSp}(4, F)$ such that the center of $\mathrm{GSp}(4, F)$ acts trivially, and fix the Haar measure on $\mathrm{GSp}(4, F)$ that gives $\mathrm{GSp}(4, \mathfrak{o})$ measure one. Let $g \in \mathrm{GSp}(4, F)$, and let $m \geq 0$ be a non-negative integer. Let $V \to V(m)$ be projection, i.e., the map that sends $v \in V$ to

$$\int\limits_{\mathrm{K}(\mathfrak{p}^m)} \pi(k)v\, dk.$$

The composition of $\pi(g) : V \to V$ with projection $V \to V(m)$ defines a linear map $V \to V(m)$, and linear combinations of such maps are the most general linear maps from V to $V(m)$ that can be constructed using the operators defined by the representation. If $n \geq 0$ is another non-negative integer, then we can consider the restriction of this map to $V(n)$, divide it by the volume of $J = \mathrm{K}(\mathfrak{p}^m) \cap \pi(g)\mathrm{K}(\mathfrak{p}^n)\pi(g)^{-1}$, and thus obtain a map

$$T : V(n) \to V(m).$$

We call such a map a *level changing operator*. If $n \leq m$, then we refer to T as a *level raising operator*. In Sect. 3.3 we will also consider *level lowering operators*. By its method of definition, the map T has an extension to V, which is given by

$$Tv = \frac{1}{\mathrm{vol}(J)} \int\limits_{\mathrm{K}(\mathfrak{p}^m)} \pi(kg)v\, dk, \quad v \in V.$$

Note that we also denote the extension to V by T. It should be clear from the context whether we mean $T : V(n) \to V(m)$ or its extension to V. It will be convenient to write $T : V(n) \to V(m)$ as a finite sum. Let

$$\mathrm{K}(\mathfrak{p}^m) = \sqcup_{i \in I} h_i J$$

be a disjoint decomposition. Then a computation shows that

$$Tv = \sum_{i \in I} \pi(h_i g)v, \quad v \in V(n). \tag{3.2}$$

Of course, even though T has an extension to V, this formula is only valid on $V(n)$. Since $V(n)$ and $V(m)$ admit the Atkin–Lehner involutions $\pi(u_n)$ and $\pi(u_m)$, respectively, we can also consider the operator $\pi(u_m)\circ T\circ\pi(u_n)$ from $V(n)$ to $V(m)$. We call this operator the *dual* of T. Another computation verifies that this dual is the level changing operator associated to $u_m g u_n$, and that

$$(\pi(u_m)\circ T\circ\pi(u_n))v=\sum_{i\in I}\pi(u_m h_i u_m)\pi(u_m g u_n)v,\quad v\in V(n). \tag{3.3}$$

The extension of $\pi(u_m)\circ T\circ\pi(u_n)$ to V is given by

$$(\pi(u_m)\circ T\circ\pi(u_n))v=\frac{1}{\mathrm{vol}(J)}\int\limits_{\mathrm{K}(\mathfrak{p}^m)}\pi(k u_m g u_n)v,\quad v\in V.$$

Definitions of θ, θ' and η

Continue to let (π,V) be a smooth representation of $\mathrm{GSp}(4,F)$ such that the center of $\mathrm{GSp}(4,F)$ acts trivially, and let $n\geq 0$ be a non-negative integer. The first level raising operator $\theta' : V(n)\to V(n+1)$ that we define is obtained by letting $m=n+1$ and $g=1$ in the definitions of the last subsection. The formula for the extension of θ' to V is

$$\theta' v=\frac{1}{\mathrm{vol}(\mathrm{K}(\mathfrak{p}^{n+1})\cap\mathrm{K}(\mathfrak{p}^n))}\int\limits_{\mathrm{K}(\mathfrak{p}^{n+1})}\pi(k)v\,dk,\quad v\in V. \tag{3.4}$$

The second level raising operator $\theta : V(n)\to V(n+1)$ is defined to be the dual $\pi(u_{n+1})\circ\theta'\circ\pi(u_n)$. By the last subsection, this is the level raising operator from $V(n)$ to $V(n+1)$ associated to $u_{n+1}u_n$. This element is ϖ^n times

$$\begin{bmatrix}1&&&\\&1&&\\&&\varpi&\\&&&\varpi\end{bmatrix}.$$

Since the center of $\mathrm{GSp}(4,F)$ acts trivially on V, θ is also the level raising operator from $V(n)$ to $V(n+1)$ defined by the last displayed element of $\mathrm{GSp}(4,F)$. The extension of θ to V is given by the formula

$$\theta v=\frac{1}{\mathrm{vol}(\mathrm{K}(\mathfrak{p}^{n+1})\cap\mathrm{K}(\mathfrak{p}^n))}\int\limits_{\mathrm{K}(\mathfrak{p}^{n+1})}\pi(k\begin{bmatrix}1&&&\\&1&&\\&&\varpi&\\&&&\varpi\end{bmatrix})v\,dk,\quad v\in V.$$

For most irreducible, admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character we have $\theta\neq\theta'$ for some n. We will later characterize the few representations for which $\theta = \theta'$; see Proposition 5.5.13. Finally, we define $\eta : V(n) \to V(n+2)$ to be the level raising operator associated to

$$\eta = \begin{bmatrix} \varpi^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix} \tag{3.5}$$

Since $\mathrm{K}(\mathfrak{p}^{n+2}) = \mathrm{K}(\mathfrak{p}^{n+2}) \cap \eta \mathrm{K}(\mathfrak{p}^n)\eta^{-1}$, on V this operator is just given by

$$\eta v = \pi(\eta) v, \quad v \in V.$$

The matrix identity

$$\varpi \begin{bmatrix} \varpi^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} & & 1 & \\ & & & -1 \\ \varpi^n & & & \\ & -\varpi^n & & \end{bmatrix} = \begin{bmatrix} & & 1 & \\ & & & -1 \\ \varpi^{n+2} & & & \\ & -\varpi^{n+2} & & \end{bmatrix} \begin{bmatrix} \varpi^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}$$

shows that $\eta \circ \pi(u_n) = \pi(u_{n+2}) \circ \eta$. In other words, η is compatible with Atkin–Lehner involutions (or self-dual).

Our next task is to obtain summation formulas for θ and θ'. For this we require the following lemma.

Lemma 3.2.1. *Let $n \geq 0$ be a non-negative integer. A complete system of representatives for $\mathrm{K}(\mathfrak{p}^{n+1})/(\mathrm{K}(\mathfrak{p}^{n+1}) \cap \mathrm{K}(\mathfrak{p}^n))$ is given by the $q+1$ elements*

$$t_{n+1} = \begin{bmatrix} & & & -\varpi^{-(n+1)} \\ & 1 & & \\ & & 1 & \\ \varpi^{n+1} & & & \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} 1 & & & c\varpi^{-(n+1)} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \quad c \in \mathfrak{o}/\mathfrak{p}.$$

Proof. The group $\mathrm{K}(\mathfrak{p}^{n+1}) \cap \mathrm{K}(\mathfrak{p}^n)$ is the subgroup of $k \in \mathrm{GSp}(4, F)$ such that $\lambda(k) \in \mathfrak{o}^\times$ and

$$k \in \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-n} \\ \mathfrak{p}^{n+1} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^{n+1} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^{n+1} & \mathfrak{p}^{n+1} & \mathfrak{p}^{n+1} & \mathfrak{o} \end{bmatrix}.$$

It is easy to see that the cosets represented by the elements from the statement of the lemma are pairwise disjoint. To prove that they exhaust $\mathrm{K}(\mathfrak{p}^{n+1})$, let $k \in \mathrm{K}(\mathfrak{p}^{n+1})$. Write

$$k = \begin{bmatrix} a_1 & a_2 & b_1 & b_2\varpi^{-(n+1)} \\ a_3\varpi^{n+1} & a_4 & b_3 & b_4 \\ c_1\varpi^{n+1} & c_2 & d_1 & d_2 \\ c_3\varpi^{n+1} & c_4\varpi^{n+1} & d_3\varpi^{n+1} & d_4 \end{bmatrix}$$

where $a_i, b_i, c_i, d_i \in \mathfrak{o}$, $1 \leq i \leq 4$. If $d_4 \in \mathfrak{o}^\times$, then

$$\begin{bmatrix} 1 & & & -d_4^{-1}b_2\varpi^{-(n+1)} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} k \in \mathrm{K}(\mathfrak{p}^{n+1}) \cap \mathrm{K}(\mathfrak{p}^n).$$

If $d_4 \notin \mathfrak{o}^\times$, then

$$t_{n+1}^{-1}k = \begin{bmatrix} c_3 & c_4 & d_3 & d_4\varpi^{-(n+1)} \\ a_3\varpi^{n+1} & a_4 & b_3 & b_4 \\ c_1\varpi^{n+1} & c_2 & d_1 & d_2 \\ -a_1\varpi^{n+1} & -a_2\varpi^{n+1} & -b_1\varpi^{n+1} & -b_2 \end{bmatrix}$$

is contained in $\mathrm{K}(\mathfrak{p}^{n+1}) \cap \mathrm{K}(\mathfrak{p}^n)$. $\square$

Lemma 3.2.2. *Let (π, V) be a smooth representation of* $\mathrm{GSp}(4, F)$ *such that the center of* $\mathrm{GSp}(4, F)$ *acts trivially, and let $n \geq 0$ be a non-negative integer. Then the operators θ and θ' from $V(n)$ to $V(n+1)$ have the following explicit formulas.*

i)

$$\theta v = \pi(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix})v + \sum_{c \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & c\varpi^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix})v, \quad v \in V(n). \tag{3.6}$$

ii)

$$\theta' v = \eta v + \sum_{c \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & & & c\varpi^{-n-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v, \quad v \in V(n). \tag{3.7}$$

Proof. i) By the definition of θ, (3.3), Lemma 3.2.1, and the matrix identity

$$u_{n+1} \begin{bmatrix} d & & & -c\varpi^{-(n+1)} \\ & 1 & & \\ & & 1 & \\ -b\varpi^{n+1} & & & a \end{bmatrix} u_{n+1} = \varpi^{n+1} \begin{bmatrix} 1 & & & \\ & a & b & \\ & c & d & \\ & & & 1 \end{bmatrix}$$

we have

$$\theta v = \pi(\begin{bmatrix} 1 & & & \\ & & -1 & \\ & 1 & & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix})v + \sum_{c \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & c & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix})v$$

for $v \in V(n)$. Using (2.8) we calculate:

$$
\begin{aligned}
\theta v &= \pi(\begin{bmatrix}1&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})v + \sum_{c\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix}1&&&\\&1&&\\&&\varpi&\\&&&\varpi\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&c\varpi^{-1}&1&\\&&&1\end{bmatrix})v\\
&= \pi(\begin{bmatrix}1&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})v + \pi(\begin{bmatrix}1&&&\\&1&&\\&&\varpi&\\&&&\varpi\end{bmatrix})v\\
&\quad + \sum_{c\in(\mathfrak{o}/\mathfrak{p})^\times} \pi(\begin{bmatrix}1&&&\\&1&&\\&&\varpi&\\&&&\varpi\end{bmatrix}\begin{bmatrix}1&&&\\&-c^{-1}\varpi&&\\&&-c\varpi^{-1}&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&c\varpi^{-1}&\\&&1&\\&&&1\end{bmatrix})v\\
&= \pi(\begin{bmatrix}1&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})v + \pi(\begin{bmatrix}1&&&\\&1&&\\&&\varpi&\\&&&\varpi\end{bmatrix})v\\
&\qquad\qquad + \sum_{c\in(\mathfrak{o}/\mathfrak{p})^\times} \pi(\begin{bmatrix}1&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix}\begin{bmatrix}1&&&\\&1&c^{-1}\varpi^{-1}&\\&&1&\\&&&1\end{bmatrix})v\\
&= \pi(\begin{bmatrix}1&&&\\&1&&\\&&\varpi&\\&&&\varpi\end{bmatrix})v + \sum_{c\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix}1&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix}\begin{bmatrix}1&&&\\&1&c\varpi^{-1}&\\&&1&\\&&&1\end{bmatrix})v.
\end{aligned}
$$

This proves i). To prove ii) we note that by (3.2) and Lemma 3.2.1, we have for $v \in V(n)$

$$
\theta' v = \pi(t_{n+1})v + \sum_{c\in\mathfrak{o}/p} \pi(\begin{bmatrix}1&&&c\varpi^{-(n+1)}\\&1&&\\&&1&\\&&&1\end{bmatrix})v.
$$

Now $t_{n+1} = \eta t_n$. Therefore, $\pi(t_{n+1})v = \pi(\eta)\pi(t_n)v = \pi(\eta)v$ for $v \in V(n)$. The formula in ii) follows. $\square$

In particular, we note that the linear maps θ and θ' from $V(n)$ to $V(n+1)$ can be defined using elements of $B(F)$ only.

Oldforms and Newforms

Let (π, V) be a smooth representation of $\mathrm{GSp}(4, F)$ such that the center of $\mathrm{GSp}(4, F)$ acts trivially. By piecing together the level raising operators θ, θ' and η at all levels, we obtain endomorphisms of the space V_{para} defined in (3.1). We denote these endomorphisms again by θ, θ' and η, respectively.

Lemma 3.2.3. *Let* (π, V) *be a smooth representation of* $\mathrm{GSp}(4, F)$ *such that the center of* $\mathrm{GSp}(4, F)$ *acts trivially. The endomorphisms* θ, θ' *and* η *of* V_{para} *commute pairwise.*

Proof. The matrix η in (3.5) commutes with the matrices occurring in (3.6). Hence η and θ commute. We already noticed that η commutes with Atkin–Lehner involutions. Since $\theta' = \pi(u_{n+1}) \circ \theta \circ \pi(u_n)$ it follows that η and θ' commute. Finally, it is an easy calculation using the formulas in Lemma 3.2.2 to check that θ and θ' commute. □

Let (π, V) be a smooth representation of $\mathrm{GSp}(4, F)$ such that the center of $\mathrm{GSp}(4, F)$ acts trivially. Let $\mathcal{A}$ be the commutative subalgebra of $\mathrm{End}(V_{\mathrm{para}})$ generated by the endomorphisms θ, θ' and η. Let $\mathcal{I} \subset \mathcal{A}$ be the ideal generated by θ, θ' and η. Assume that π has non-zero paramodular vectors, and let N_π be the minimal paramodular level, i.e., the smallest integer n such that $V(n) \neq 0$. By definition,

$$V_{\mathrm{para}} = V(N_\pi) \oplus \bigoplus_{n > N_\pi} V(n).$$

We call $V(N_\pi)$ the space of *newforms* and $\bigoplus_{n>N_\pi} V(n)$ the space of *oldforms* of the representation π. The Oldforms Principle, which will be proved in Theorem 7.5.7, is the statement that $\mathcal{I}V(N_\pi) = \bigoplus_{n>N_\pi} V(n)$. In other words, it says that any paramodular vector of level higher than the minimal level can be obtained by repeatedly applying level raising operators to the newforms and taking linear combinations.

Theta Injectivity

Let (π, V) be a smooth representation of $\mathrm{GSp}(4, F)$ such that the center of $\mathrm{GSp}(4, F)$ acts trivially. In this subsection we compute the kernels of our level raising operators. Since η is invertible, we need to just consider θ and θ'. We will prove that the kernel of $\theta_n : V(n) \to V(n+1)$ is $\eta^l \ker \theta_0$ if $n = 2l$ and $\eta^l \ker \theta_1$ if $n = 2l + 1$. A similar result holds for θ'. As a corollary, we obtain that $\theta, \theta' : V(n) \to V(n+1)$ are injective for all non-negative integers $n \geq 0$ if and only if $\ker \theta_0 = \ker \theta_1 = 0$. In the next subsection we will determine all the irreducible, admissible representations of $\mathrm{GSp}(4, F)$ with trivial central character such that $\ker \theta_0 \neq 0$ or $\ker \theta_1 \neq 0$, that is, all the irreducible, admissible representations such that θ or θ' are not injective for some n. It turns out there is only one such representation.

The computations of $\ker \theta$ and $\ker \theta'$ will be a consequence of a stronger result. Namely, suppose that n is a positive integer $n \geq 2$, and $v \in V(n)$ and $v_1 \in V(n-1)$ are such that $\theta' v = \eta v_1$. We will prove that v is divisible by η, that is, that there exists $v_2 \in V(n-2)$ such that $\eta v_2 = v$. To apply this result to the computation of the kernels we take $v_1 = 0$ and use induction. The first step toward proving the stronger result is to determine under what conditions elements of $V(n)$ are divisible by η.

Lemma 3.2.4. *Let (π, V) be a smooth representation of* GSp(4, F) *such that the center of* GSp(4, F) *acts trivially. Let $n \geq 0$ be a non-negative integer, and let $v \in V(n)$.*

i) Assume $n \geq 2$. Then $v = \eta v_1$ for some $v_1 \in V(n-2)$ if and only if v is invariant under the group

$$\begin{bmatrix} 1 & \mathfrak{p}^{-1} & & \\ & 1 & & \\ & & 1 & \mathfrak{p}^{-1} \\ & & & 1 \end{bmatrix}. \tag{3.8}$$

ii) Assume $n = 0$ or $n = 1$, and assume that the subspace of vectors of V fixed by Sp(4, F) *is trivial. Then v is invariant under the group (3.8) if and only if $v = 0$.*

Proof. i) It is immediate that v is invariant under elements of the form (3.8) if $v = \eta v_1$ for some $v_1 \in V(n-2)$. To show the converse, assume that $v \in V(n)$ is invariant under the group (3.8), and define

$$v_1 := \eta^{-1} v = \pi(\begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi^{-1} \end{bmatrix})v.$$

We need to prove that $v_1 \in V(n-2)$, i.e., that v_1 is invariant under $\mathrm{K}(\mathfrak{p}^{n-2})$. Now by Lemma 3.3.1 below, $\mathrm{K}(\mathfrak{p}^{n-2})$ is generated by the elements in

$$\mathrm{Kl}(\mathfrak{p}^{n-2}), \qquad \begin{bmatrix} & & & \varpi^{-(n-2)} \\ & 1 & & \\ & & 1 & \\ \varpi^{n-2} & & & \end{bmatrix}, \qquad \begin{bmatrix} 1 & & \mathfrak{p}^{-(n-2)} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Since v is $\mathrm{K}(\mathfrak{p}^n)$ invariant, v_1 is invariant under $\eta^{-1}\mathrm{K}(\mathfrak{p}^n)\eta$, i.e., the group of $k \in \mathrm{GSp}(4, F)$ such that $\lambda(k) \in \mathfrak{o}^\times$ and

$$k \in \begin{bmatrix} \mathfrak{o} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p}^{-(n-2)} \\ \mathfrak{p}^{n-1} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p}^{n-1} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p} \\ \mathfrak{p}^{n-2} & \mathfrak{p}^{n-1} & \mathfrak{p}^{n-1} & \mathfrak{o} \end{bmatrix}.$$

As the second two types of generators are in $\eta^{-1}\mathrm{K}(\mathfrak{p}^n)\eta$, it follows that we are reduced to showing that v_1 is invariant under $\mathrm{Kl}(\mathfrak{p}^{n-2})$. Now $\mathrm{Kl}(\mathfrak{p}^{n-2})$ is generated by the elements

$$\begin{bmatrix} u & & & \\ & a & b & \\ & c & d & \\ & & & \lambda u^{-1} \end{bmatrix}, \quad \begin{bmatrix} 1 & x & y & z \\ & 1 & & y \\ & & 1 & -x \\ & & & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & & & \\ x\varpi^{n-2} & 1 & & \\ y\varpi^{n-2} & & 1 & \\ z\varpi^{n-2} & y\varpi^{n-2} & -x\varpi^{n-2} & 1 \end{bmatrix},$$

where

$$u, \lambda \in \mathfrak{o}^\times, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathfrak{o}), \ ad - bc = \lambda, \ x, y, z \in \mathfrak{o}.$$

The identities

$$\begin{bmatrix} 1 & & & \\ x\varpi^{n-2} & 1 & & \\ y\varpi^{n-2} & & 1 & \\ z\varpi^{n-2} & y\varpi^{n-2} & -x\varpi^{n-2} & 1 \end{bmatrix} = t_{n-2} \begin{bmatrix} 1 & y & -x & z\varpi^{-(n-2)} \\ & 1 & & -x \\ & & 1 & -y \\ & & & 1 \end{bmatrix} t_{n-2}^{-1},$$

$$\begin{bmatrix} 1 & y & & \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{bmatrix} = s_2 \begin{bmatrix} 1 & -y & & \\ & 1 & & \\ & & 1 & y \\ & & & 1 \end{bmatrix} s_2^{-1},$$

along with the fact that $\eta^{-1}\mathrm{K}(\mathfrak{p}^n)\eta$ contains t_{n-2}, s_2 and the elements

$$\begin{bmatrix} u & & & \\ & a & b & \\ & c & d & \\ & & & \lambda u^{-1} \end{bmatrix}, \quad \begin{bmatrix} 1 & & & z\varpi^{-(n-2)} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

where

$$u, \lambda \in \mathfrak{o}^\times, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathfrak{o}), \quad z \in \mathfrak{o},$$

imply that we are reduced to showing that v_1 is invariant under the elements

$$\begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix}, \quad x \in \mathfrak{o}.$$

This is true since v is invariant under the elements (3.8).

ii) Suppose $n = 1$ and v is invariant under the group (3.8). We need to show that $v = 0$. By the assumption on π, it will suffice to show that v is invariant under $\mathrm{Sp}(4, F)$. By Lemma 3.1.2, to prove that v is invariant under $\mathrm{Sp}(4, F)$ it suffices to show that v is invariant under, say, t_3. The identity

$$\begin{bmatrix} 1 & & & \\ x & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} = s_2 t_1 \begin{bmatrix} 1 & x\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -x\varpi^{-1} \\ & & & 1 \end{bmatrix} (s_2 t_1)^{-1}$$

for $x \in \mathfrak{o}$, along with the fact that v is invariant under all the elements on the right, implies that v is invariant under this element. Since $\mathrm{SL}(2, \mathfrak{o})$ is generated by the subgroups

$$\begin{bmatrix} 1 & \mathfrak{o} \\ & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & \\ \mathfrak{o} & 1 \end{bmatrix},$$

it follows that v is invariant under the elements

$$\begin{bmatrix} A & \\ & {}^tA^{-1} \end{bmatrix}, \qquad A \in \mathrm{SL}(2, \mathfrak{o}).$$

In particular, v is invariant under s_1. It follows from Lemma 3.1.1 that v is invariant under

$$\begin{bmatrix} A & \\ & {}^tA^{-1} \end{bmatrix}, \qquad A \in \mathrm{SL}(2, F). \tag{3.9}$$

The identity

$$t_3 = \begin{bmatrix} \varpi^{-1} &&& \\ & \varpi && \\ && \varpi^{-1} & \\ &&& \varpi \end{bmatrix} \begin{bmatrix} &&& -\varpi^{-1} \\ & 1 && \\ && 1 & \\ \varpi &&& \end{bmatrix} \begin{bmatrix} \varpi &&& \\ & \varpi^{-1} && \\ && \varpi & \\ &&& \varpi^{-1} \end{bmatrix}$$

now implies that v is invariant under t_3; note that the middle element on the right is t_1.

Finally, suppose $n = 0$ and v is invariant under the group (3.8). By an argument similar to the last paragraph, it will suffice to show that v is invariant under t_2. Again, Lemma 3.1.1 shows that v is invariant under the group (3.9). The identity

$$t_2 = \begin{bmatrix} \varpi^{-1} &&& \\ & \varpi && \\ && \varpi^{-1} & \\ &&& \varpi \end{bmatrix} \begin{bmatrix} &&& -1 \\ & 1 && \\ && 1 & \\ 1 &&& \end{bmatrix} \begin{bmatrix} \varpi &&& \\ & \varpi^{-1} && \\ && \varpi & \\ &&& \varpi^{-1} \end{bmatrix}$$

implies that v is invariant under t_2. This completes the proof. □

Next, we present a sufficient condition for an element of V to be invariant under the group in (3.8).

Lemma 3.2.5. *Let (π, V) be a smooth representation of* $\mathrm{GSp}(4, F)$ *such that the center of* $\mathrm{GSp}(4, F)$ *acts trivially. Let $n \geq 0$ be a non-negative integer, and let $v' \in V$. Assume v' is invariant under the following groups:*

$$\text{(i)} \begin{bmatrix} 1 &&& \mathfrak{p}^{-n} \\ & 1 && \\ && 1 & \\ &&& 1 \end{bmatrix}; \qquad \text{(ii)} \begin{bmatrix} 1 &&& \\ && 1 & \\ &&& 1 \\ \mathfrak{p}^{n+1} &&& 1 \end{bmatrix};$$

$$\text{(iii)} \begin{bmatrix} u &&& \\ & 1 && \\ && 1 & \\ &&& u^{-1} \end{bmatrix}, \quad u \in \mathfrak{o}^\times; \qquad \text{(iv)} \begin{bmatrix} 1 &&& \\ & 1 && \\ \mathfrak{p}^n && 1 & \\ & \mathfrak{p}^n && 1 \end{bmatrix}.$$

If

$$0 = \sum_{x \in \mathfrak{p}^{-(n+1)}/\mathfrak{p}^{-n}} \pi(\begin{bmatrix} 1 & & & x \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v',$$

then v' is invariant under t_{n+1} and under

$$\begin{bmatrix} 1 & \mathfrak{p}^{-1} & & \\ & 1 & & \\ & & 1 & \mathfrak{p}^{-1} \\ & & & 1 \end{bmatrix}.$$

Proof. The element

$$\sum_{x \in \mathfrak{p}^{-(n+1)}/\mathfrak{p}^{-n}} \pi(\begin{bmatrix} 1 & & & x \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v'$$

is well-defined by the invariance of v' under type (i) elements; it is zero by assumption. Therefore,

$$\begin{aligned} -v' &= \sum_{\substack{x \in \mathfrak{p}^{-(n+1)}/\mathfrak{p}^{-n} \\ x \neq 0}} \pi(\begin{bmatrix} 1 & & & x \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v' \\ &= \sum_{u \in (\mathfrak{o}/\mathfrak{p})^\times} \pi(\begin{bmatrix} 1 & & & u\varpi^{-(n+1)} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v'. \end{aligned}$$

Hence, using the invariance of v' under (ii) and (iii),

$$\begin{aligned} -\pi(t_{n+1})v' &= \sum_{u \in (\mathfrak{o}/\mathfrak{p})^\times} \pi(t_{n+1} \begin{bmatrix} 1 & & & u\varpi^{-(n+1)} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v' \\ &= \sum_{u \in (\mathfrak{o}/\mathfrak{p})^\times} \pi(\begin{bmatrix} 1 & & & -u^{-1}\varpi^{-(n+1)} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \\ &\qquad \cdot \begin{bmatrix} 1 & & & u^{-1}\varpi^{-(n+1)} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} t_{n+1} \begin{bmatrix} 1 & & & u\varpi^{-(n+1)} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v' \\ &= \sum_{u \in (\mathfrak{o}/\mathfrak{p})^\times} \pi(\begin{bmatrix} 1 & & & -u^{-1}\varpi^{-(n+1)} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})\pi(\begin{bmatrix} u^{-1} & & & \\ & 1 & & \\ & & 1 & \\ \varpi^{n+1} & & & u \end{bmatrix})v' \end{aligned}$$

$$= \sum_{u \in (\mathfrak{o}/\mathfrak{p})^\times} \pi(\begin{bmatrix} 1 & & -u^{-1}\varpi^{-(n+1)} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v'$$
$$= -v'.$$

Therefore, $\pi(t_{n+1})v' = v'$. Now for $a \in \mathfrak{o}$,

$$\begin{bmatrix} 1 & a\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -a\varpi^{-1} \\ & & & 1 \end{bmatrix} = t_{n+1} \begin{bmatrix} 1 & & & \\ & 1 & & \\ -a\varpi^n & & 1 & \\ & -a\varpi^n & & 1 \end{bmatrix} t_{n+1}^{-1}.$$

As v' is invariant under the three elements on the right – the middle element is of type (iv) – it is invariant under the element on the left. □

We are now ready to prove the stronger result mentioned at the beginning of this subsection. For the proof it will be convenient to introduce some notation. Let (π, V) a smooth representation of $\mathrm{GSp}(4, F)$ such that the center of $\mathrm{GSp}(4, F)$ acts trivially. Assume $v \in V$ is invariant under the elements of $\mathrm{GSp}(4, F)$ in

$$\begin{bmatrix} 1 & \mathfrak{o} & & \\ & 1 & & \\ & & 1 & \mathfrak{o} \\ & & & 1 \end{bmatrix}.$$

We define

$$Sv = \sum_{x \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & x\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -x\varpi^{-1} \\ & & & 1 \end{bmatrix})v. \tag{3.10}$$

Evidently, Sv is invariant under all the elements in the group

$$\begin{bmatrix} 1 & \mathfrak{p}^{-1} & & \\ & 1 & & \\ & & 1 & \mathfrak{p}^{-1} \\ & & & 1 \end{bmatrix}.$$

Clearly, v is invariant under this group if and only if $Sv = qv$; and $Sv = qv$ if and only if $Sv - qv$ is invariant under this group.

Theorem 3.2.6. *Let (π, V) be smooth representation of* $\mathrm{GSp}(4, F)$ *such that the center of* $\mathrm{GSp}(4, F)$ *acts trivially. Let n be a non-negative integer such that $n \geq 0$, and let $v \in V(n)$.*

i) Assume that $n \geq 2$. If $\theta' v = \eta v_1$ for some $v_1 \in V(n-1)$, then there exists $v_2 \in V(n-2)$ such that $\eta v_2 = v$.

ii) Assume that $n=1$ and that the subspace of vectors of V fixed by $\mathrm{Sp}(4,F)$ is trivial. If $\theta' v = 0$, then $v=0$.

Proof. i) Assume that $n \geq 2$ and that $\theta' v = \eta v_1$ for some $v_1 \in V(n-1)$. By Lemma 3.2.4, since $\theta' v = \eta v_1$, the vector $\theta' v$ is invariant under the group in (3.8). By the remarks preceding the lemma, we have $S\theta' v = q\theta' v$. To start, we will compute $S\theta' v$ and $q\theta' v$ and then deduce a consequence of the equality $S\theta' v = q\theta' v$. By the formula in Lemma 3.2.2 we have

$$\begin{aligned}
S\theta' v &= \sum_{x \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & x\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -x\varpi^{-1} \\ & & & 1 \end{bmatrix})\theta' v \\
&= \sum_{x \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & x\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -x\varpi^{-1} \\ & & & 1 \end{bmatrix})(\eta v + \sum_{y \in \mathfrak{p}^{-(n+1)}/\mathfrak{p}^{-n}} \pi(\begin{bmatrix} 1 & & & y \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v) \\
&= \sum_{x \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & x\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -x\varpi^{-1} \\ & & & 1 \end{bmatrix})\eta v \\
&\quad + \sum_{\substack{x \in \mathfrak{o}/\mathfrak{p} \\ y \in \mathfrak{p}^{-(n+1)}/\mathfrak{p}^{-n}}} \pi(\begin{bmatrix} 1 & x\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -x\varpi^{-1} \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & y \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v \\
&= q\eta v + \sum_{y \in \mathfrak{p}^{-(n+1)}/\mathfrak{p}^{-n}} \pi(\begin{bmatrix} 1 & & & y \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})(Sv). \qquad (3.11)
\end{aligned}$$

On the other hand, by another application of the formula in Lemma 3.2.2,

$$q\theta' v = q\eta v + q \sum_{y \in \mathfrak{p}^{-(n+1)}/\mathfrak{p}^{-n}} \pi(\begin{bmatrix} 1 & & & y \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v. \qquad (3.12)$$

Comparing (3.11) and (3.12), we get

$$\sum_{y \in \mathfrak{p}^{-(n+1)}/\mathfrak{p}^{-n}} \pi(\begin{bmatrix} 1 & & & y \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v' = 0,$$

where

$$v' = Sv - qv.$$

Next, we verify that v' satisfies the assumptions of Lemma 3.2.5, i.e., that v' is invariant under the elements in (i), (ii), (iii) and (iv) of Lemma 3.2.5. Because type (i) elements lie in the center of the Jacobi group and v is in $V(n)$, v' is invariant under type (i) elements. To prove v' is invariant under type (ii), (iii) and (iv) elements it will suffice to prove that Sv is invariant under these elements, as v is already invariant under these elements. We have for $a, b \in \mathfrak{o}$,

$$\begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ a\varpi^{n+1} &&& 1 \end{bmatrix} \begin{bmatrix} 1 & b\varpi^{-1} && \\ & 1 && \\ && 1 & -b\varpi^{-1} \\ &&& 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & b\varpi^{-1} && \\ & 1 && \\ && 1 & -b\varpi^{-1} \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ && 1 & \\ ab\varpi^{n} & ab^2\varpi^{n-1} & 1 & \\ a\varpi^{n+1} & ab\varpi^{n} && 1 \end{bmatrix}.$$

This implies that Sv is invariant under type (ii) elements. It is easy to see that Sv is invariant under type (iii) elements. Finally, we have for $a, b \in \mathfrak{o}$,

$$\begin{bmatrix} 1 &&& \\ & 1 && \\ a\varpi^{n} && 1 & \\ & a\varpi^{n} && 1 \end{bmatrix} \begin{bmatrix} 1 & b\varpi^{-1} && \\ & 1 && \\ && 1 & -b\varpi^{-1} \\ &&& 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & b\varpi^{-1} && \\ & 1 && \\ && 1 & -b\varpi^{-1} \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ && 1 & \\ a\varpi^{n} & 2ab\varpi^{n-1} & 1 & \\ & a\varpi^{n} && 1 \end{bmatrix}.$$

This implies that Sv is invariant under type (iv) elements. Applying now Lemma 3.2.5, we find that v' is invariant under

$$\begin{bmatrix} 1 & \mathfrak{p}^{-1} && \\ & 1 && \\ && 1 & \mathfrak{p}^{-1} \\ &&& 1 \end{bmatrix}.$$

Since Sv is also invariant under this group, so is v. By Lemma 3.2.4, this implies, finally, $v = \eta v_2$ for some $v_2 \in V(n-2)$.

ii) Assume that $n = 1$ and that the subspace of vectors of V fixed by $\mathrm{Sp}(4, F)$ is trivial. Suppose that $\theta' v = 0$. Then certainly $S\theta' v = q\theta' v$. Arguing exactly as in the previous case, we get that v is invariant under the group (3.8). By Lemma 3.2.4 we have $v = 0$. □

We now compute the kernels of θ and θ'.

Corollary 3.2.7 (Injectivity of θ and θ'). *Let (π, V) be a smooth representation of* GSp$(4,F)$ *such that the center of* GSp$(4,F)$ *acts trivially. If n is a non-negative integer, write θ_n and θ'_n for the θ and θ' operators $V(n) \to V(n+1)$, respectively. Then for all $n \geq 0$,*

$$\ker\theta_n = \begin{cases} \eta^l \ker\theta_0 & \text{if } n = 2l, \\ \eta^l \ker\theta_1 & \text{if } n = 2l+1, \end{cases}$$

and

$$\ker\theta'_n = \begin{cases} \eta^l \ker\theta'_0 & \text{if } n = 2l, \\ \eta^l \ker\theta'_1 & \text{if } n = 2l+1. \end{cases}$$

If θ_0 and θ_1, or equivalently, θ'_0 and θ'_1, are injective, then θ_n and θ'_n are injective for all $n \geq 0$.

Proof. First we prove the statement about θ'. We prove this by induction on n. If $n = 0$ or $n = 1$ the statement is clear. Suppose $n \geq 2$ and the statement holds for all $k \leq n$. We will prove that it holds for $n+1$. Write $n+1 = 2l+\delta$, where l is a positive integer and $\delta = 0$ or 1. Let $v \in \ker\theta'_{n+1}$. Then $\theta'_{n+1}v = \eta 0$. Hence, by Theorem 3.2.6, there exists $v_2 \in V(n-1)$ such that $v = \eta v_2$. Now $0 = \theta'_{n+1}v = \theta'_{n+1}\eta v_2 = \eta\theta'_{n-1}v_2$ since θ' and η commute by Lemma 3.2.3. Hence, $\theta'_{n-1}v_2 = 0$. By the induction hypothesis, $v_2 \in \eta^{l-1}\ker\theta'_\delta$. Therefore, $v = \eta v_2 \in \eta^l \ker\theta'_\delta$. Conversely, suppose $v \in \eta^l\ker\theta'_\delta$. Write $v = \eta^l v'$ for $v' \in \ker\theta'_\delta$. We have $\theta'_{n+1}v = \theta'_{n+1}\eta^l v' = \eta\theta'_{n-1}\eta^{l-1}v'$, again because θ' and η commute by Lemma 3.2.3. By the induction hypothesis, $\eta^{l-1}v' \in \ker\theta'_{n-1}$. Hence, $\theta'_{n-1}\eta^{l-1}v' = 0$, so that $\theta'_{n+1}v = 0$. Therefore, $\ker\theta'_{n+1} = \eta^l\ker\theta_\delta$, so that the statement about $\ker\theta'_n$ follows by induction.

To prove the statement about $\ker\theta_n$, write $n = 2l+\delta$ for l a non-negative integer and $\delta = 0$ or 1. We have

$$\begin{aligned} v \in \ker\theta_n &\iff \theta_n v = 0 \\ &\iff (\pi(u_{n+1}) \circ \theta'_n \circ \pi(u_n))v = 0 \\ &\iff \theta'_n(\pi(u_n)v) = 0 \\ &\iff \pi(u_n)v \in \eta^l\ker\theta'_\delta \\ &\iff v \in \pi(u_n)\eta^l\ker\theta'_\delta \\ &\iff v \in \eta^l\pi(u_\delta)\ker\theta'_\delta \\ &\iff v \in \eta^l\pi(u_\delta)\ker(\pi(u_{\delta+1}) \circ \theta_\delta \circ \pi(u_\delta)) \\ &\iff v \in \eta^l\ker\theta_\delta. \end{aligned}$$

The statements about the injectivity of θ and θ' follow from the computations of their kernels. □

For future use it will also be useful to observe the following corollary of Theorem 3.2.6.

Corollary 3.2.8. *Let (π, V) be a smooth representation of* $\mathrm{GSp}(4, F)$ *such that the center of* $\mathrm{GSp}(4, F)$ *acts trivially. Let $n \geq 0$ be a non-negative integer, and let $v \in V(n)$. Let $k \geq 0$ be a non-negative integer, and assume that $S\theta'^k v = q\theta'^k v$.*

i) Assume that $n \geq 2$. Then there exists $v_2 \in V(n-2)$ such that $\eta v_2 = v$.
ii) Assume that $n = 1$ and that the subspace of vectors of V fixed by $\mathrm{Sp}(4, F)$ *is trivial. Then either $v = 0$, or $\theta' v \neq 0$ and there exists $v_2 \in V(0)$, $v_2 \neq 0$, such that $\theta' v = \eta v_2$.*

Proof. i) Assume that $n \geq 2$. We prove the claim by induction on k. The statement is clear if $k = 0$. Assume the statement hold for k. We will prove that it holds for $k+1$. Suppose that $S\theta'^{k+1} v = q\theta'^{k+1} v$ for $v \in V(n)$. Since $S\theta'^k(\theta' v) = q\theta'^k(\theta' v)$, the induction hypothesis implies that $\theta' v = \eta v_1$ for some $v_1 \in V(n-1)$. By Theorem 3.2.6, $v = \eta v_2$ for some $v_2 \in V(n-2)$.

ii) Assume that $n = 1$ and that the subspace of vectors of V fixed by $\mathrm{Sp}(4, F)$ is trivial. If $k = 0$, then $Sv = qv$, so that W is invariant under the subgroup (3.8); by Lemma 3.2.4 we have $v = 0$. Assume $k \geq 1$. By i) applied to the equation $S\theta'^{k-1}\theta' v = q\theta'^{k-1}\theta' v$ we have $\theta' v = \eta v_2$ for some $v_2 \in V(n-1) = V(0)$. If $v \neq 0$, then $\theta' v \neq 0$ by Theorem 3.2.6. This completes the proof. □

Iwahori-Spherical Representations

In this subsection we determine the irreducible, admissible representations (π, V) of $\mathrm{GSp}(4, F)$ with trivial central character for which the endomorphisms θ and θ' of V_{para} are not injective. By Corollary 3.2.7, if (π, V) is such a representation, then necessarily $V(0) \neq 0$ or $V(1) \neq 0$. Now $V(0)$ is the space of $\mathrm{GSp}(4, \mathfrak{o})$ fixed vectors, and $V(1)$ is the space of vectors fixed under $\mathrm{K}(\mathfrak{p})$, which is also a parahoric subgroup. It follows that any counterexample to θ injectivity necessarily contains non-zero fixed vectors under the Iwahori subgroup

$$I = \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} \end{bmatrix} \subset \mathrm{GSp}(4, \mathfrak{o}).$$

It is well known that such Iwahori-spherical representations are exactly the constituents of the representations parabolically induced from an unramified character of the Borel subgroup $B(F)$; see [Bo1]. Table A.13 in Appendix A.8 contains the complete list of all such representations; all the inducing characters are understood to be unramified. The following theorem determines these representations and provides information about their paramodular vectors. In this theorem we mention the Langlands parameter of an Iwahori-spherical, irreducible, admissible representation π of $\mathrm{GSp}(4, F)$: by this we mean the admissible representation $\varphi_\pi : W_F' \to \mathrm{GSp}(4, \mathbb{C})$ assigned to π by the desiderata of the local Langlands conjecture; see 11.3 of [Bo2] and our Sect. 2.4.

Theorem 3.2.9. *Table A.13 contains the complete list of Iwahori-spherical, irreducible, admissible representations* (π, V) *of* $\mathrm{GSp}(4, F)$ *with trivial central character. The table also lists the the dimensions of the spaces* $V(0), V(1), V(2)$ *and* $V(3)$*, and under each dimension the eigenvalues of the Atkin–Lehner involution* u_n*. These eigenvalues are correct if one assumes that*

- *in group II, where the central character is* $\chi^2\sigma^2$*, the character* $\chi\sigma$ *is trivial.*
- *in groups IV, V and VI, where the central character is* σ^2*, the character* σ *itself is trivial.*

If these assumptions are not met, then one has to interchange the plus and minus signs in the $V(1)$ *and the* $V(3)$ *column. Finally, the table lists the conductor* a *and* ε*-factor* $\varepsilon(1/2, \varphi_\pi)$ *of the Langlands parameter* φ_π *of* π.

Proof. By [Bo1], each Iwahori-spherical representation of $\mathrm{GSp}(4, F)$ can be realized as a subrepresentation of an induced representation $\chi_1 \times \chi_2 \rtimes \sigma$ with unramified characters χ_1, χ_2 and σ of $F^\times$. In Proposition 5.1.2 we will determine representatives for the double cosets $B(F)\backslash\mathrm{GSp}(4, F)/\mathrm{K}(\mathfrak{p}^n)$, for each $n \geq 0$. For $n = 0, 1, 2, 3$ the number of elements of this double coset space is $1, 2, 4, 6$, respectively, and the general formula is $[(n+2)^2/4]$. These are also the dimensions of the spaces of $\mathrm{K}(\mathfrak{p}^n)$ invariant vectors in $\chi_1 \times \chi_2 \rtimes \sigma$, since such vectors can be given as functions in the induced model taking arbitrary values on the double coset representatives. This explains the dimensions for group I representations in Table A.13. The Atkin–Lehner eigenvalues can be verified by direct calculations in the induced model.

If the induced representation $\chi_1 \times \chi_2 \rtimes \sigma$ is reducible, then one has to determine how the paramodular vectors are distributed among the irreducible constituents. As an example, we treat group II representations. The full induced representation is $\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$ with unramified characters χ and σ such that $\chi^2 \neq \nu^{\pm 1}$ and $\chi \neq \nu^{\pm 3/2}$; see Table A.1. The IIb constituent is given as the Siegel induced representation $\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$. Similarly as above, the dimension of the space of $\mathrm{K}(\mathfrak{p}^n)$ invariant vectors for IIb is given by the number of elements of $P(F)\backslash\mathrm{GSp}(4, F)/\mathrm{K}(\mathfrak{p}^n)$. By Proposition 5.1.2, this cardinality is $[(n+2)/2]$. The dimensions for IIa are then obtained by subtracting the IIb dimensions from the dimensions $[(n+2)^2/4]$ for the full induced representation. Similarly, the Atkin–Lehner eigenvalues for IIb can be found by direct calculation, and together with the eigenvalues for the full induced representation determine the eigenvalues for IIa.

The dimensions and eigenvalues for group III representations can be found in a similar way, using the fact that the number of elements of $Q(F)\backslash\mathrm{GSp}(4, F)/\mathrm{K}(\mathfrak{p}^n)$ is $n + 1$; see Proposition 5.1.2.

For group IV representations we use the fact that IVd is one-dimensional and observe table (2.9). The dimensions for the Siegel induced $\nu^{3/2}\mathbf{1}_{\mathrm{GL}(2)} \rtimes \nu^{-3/2}\sigma$ are $[(n+2)/2]$, as above. Consequently, the dimensions for IVb are one less. The dimensions for the Klingen induced $\nu^2 \rtimes \nu^{-1}\sigma\mathbf{1}_{\mathrm{GSp}(2)}$ are $n+1$, and the dimensions for IVc are one less. Subtracting everything from the dimensions $[(n+2)^2/4]$ for the full induced representation, we get the dimensions for IVa.

In Lemma 5.5.7 below we will prove that the dimensions for the unramified representation of type Vd are $1,0,1,0,\ldots$. Using this information and table (2.10), which shows how the full induced representation decomposes, we can determine all the dimensions for group V representations.

Finally, for group VI, it is easiest to use table (2.11) and the fact, to be proved in Theorem 3.4.3 below, that VIb has no paramodular vectors. □

Returning to the problem of the injectivity of θ and θ', there is an obvious counterexample to θ and θ' injectivity in Table A.13, namely the representation $L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$ of type Vd, where σ and ξ are unramified quadratic characters and ξ is non-trivial. Note that

$$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)=L(\nu\xi,\xi\rtimes\nu^{-1/2}\xi\sigma), \tag{3.13}$$

i.e., the Vd type representation is invariant under twisting with ξ. This follows from table (2.10), which shows that $L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$ can be characterized as the common constituent of $\nu^{1/2}\xi\mathbf{1}_{\mathrm{GL}(2)}\rtimes\nu^{-1/2}\sigma$ and $\nu^{1/2}\xi\mathbf{1}_{\mathrm{GL}(2)}\rtimes\nu^{-1/2}\xi\sigma$. Hence there is only *one* unramified representation of type Vd.

Corollary 3.2.10. *Let (π,V) be an irreducible, admissible representation of $\mathrm{GSp}(4,F)$ for which the center acts trivially. Assume that π is different from the Vd type representation $L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$ with unramified, quadratic characters $\xi\neq 1_{F^\times}$ and σ. Then the θ and θ' operators on each of the spaces $V(n)$ are injective.*

Proof. In view of Corollary 3.2.7, we have to show that $\theta_0:\ V(0)\to V(1)$ and $\theta_1:\ V(1)\to V(2)$ are injective. By Theorem 3.2.6 ii), θ_1 is injective for any irreducible π. It remains to show that θ_0 is injective for any irreducible representation other than Vd with non-zero $\mathrm{GSp}(4,\mathfrak{o})$ fixed vectors. These are the representations of type I, IIb, IIIb, IVd and VId. In each case the verification is easily accomplished by realizing π as a subrepresentation of an appropriate full induced representation $\chi_1\times\chi_2\rtimes\sigma$ with unramified characters χ_1, χ_2 and σ. □

It is worth noting that the last result implies that with the one exception of the Vd type representation, the dimensions of the spaces $V(n)$ are non-decreasing, even though $V(n)$ is not a subspace of $V(n+1)$.

To close this section, we note that several of the themes of this work are already present in Table A.13. Namely, we observe that, with the exception of the VIb type representations,

- the conductor of the local parameter coincides with the minimal paramodular level;
- the dimension of $V(n)$ at the minimal level is 1;
- the (unique) Atkin–Lehner eigenvalue at the minimal level coincides with $\varepsilon(1/2,\varphi_\pi)$.

As for VIb, this representation shares an L-parameter with VIa. All the other representations in this table have pairwise distinct L-parameters; see Sect. 2.4. Hence, the above statements are true without exception if read at the level of L-packets. In this work we prove statements similar to the above for each irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character.

3.3 Level Lowering Operators

In this section (π, V) is a smooth representation of $\mathrm{GSp}(4,F)$ for which the center acts trivially. By its definition in Sect. 3.2, the level raising operator $\theta' : V(n) \to V(n+1)$ is nothing but the natural summation or "trace" operator from $\mathrm{K}(\mathfrak{p}^n)$- to $\mathrm{K}(\mathfrak{p}^{n+1})$-invariant vectors. In the present section we study trace operators in the other direction, from $V(n)$ to $V(n-1)$. Our goal is to give explicit formulas for these level lowering operators, which will be used in later calculations involving Hecke operators. For each operator we study, the formulas for $n=1$ turn out to be different from the formulas for $n \geq 2$.

Some Coset Decompositions and Volumes

We start with some useful coset decompositions. The following lemma is fundamental.

Lemma 3.3.1. *Let n be a non-negative integer. Then there is a disjoint decomposition*

$$\mathrm{K}(\mathfrak{p}^n) = \bigsqcup_{u \in \mathfrak{o}/\mathfrak{p}^n} \begin{bmatrix} 1 & & & u\varpi^{-n} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{Kl}(\mathfrak{p}^n)$$
$$\sqcup \bigsqcup_{v \in \mathfrak{o}/\mathfrak{p}^{n-1}} t_n \begin{bmatrix} 1 & & & v\varpi^{-n+1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{Kl}(\mathfrak{p}^n). \tag{3.14}$$

Here, t_n is the element defined in (2.3). (If $n=0$, the second union is not present and the first union is $\mathrm{Kl}(\mathfrak{p}^n) = \mathrm{K}(\mathfrak{p}^n) = \mathrm{GSp}(4,\mathfrak{o})$.*)*

Proof. We will assume $n \geq 1$, since the assertion is trivial for $n = 0$. It is easy to see that the cosets of the first type are pairwise disjoint, and that the cosets of the second type are also pairwise disjoint. Moreover, the lower right coefficient of an element in a coset of the first type is a unit, while the lower right coefficient of an element in a coset of the second type lies in $\mathfrak{p}$. Hence all the cosets in (3.14) are indeed disjoint.

Next, we prove that $\mathrm{K}(\mathfrak{p}^n)$ is contained in the complete union. Let $k \in \mathrm{K}(\mathfrak{p}^n)$, and write

$$k = \begin{bmatrix} a_1 & a_2 & b_1 & b_2\varpi^{-n} \\ a_3\varpi^n & a_4 & b_3 & b_4 \\ c_1\varpi^n & c_2 & d_1 & d_2 \\ c_3\varpi^n & c_4\varpi^n & d_3\varpi^n & d_4 \end{bmatrix},$$

with $a_i, b_i, c_i, d_i \in \mathfrak{o}$ for $1 \le i \le 4$. If $d_4 \in \mathfrak{o}^\times$, then

$$\begin{bmatrix} 1 & & & -b_2d_4^{-1}\varpi^{-n} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} k \in \mathrm{Kl}(\mathfrak{p}^n),$$

because the upper-right entry of this matrix is 0. This implies that k is contained in one of the sets from the first union. Assume d_4 is not a unit. We have

$$\det(k) = (a_1d_4 - b_2c_3)(a_4d_1 - b_3c_2) + a\varpi^n$$

for some $a \in \mathfrak{o}$. Since $d_4 \notin \mathfrak{o}^\times$ and $\det(k) \in \mathfrak{o}^\times$ by the definition of the paramodular group, we obtain that $b_2 \in \mathfrak{o}^\times$. We have

$$t_n^{-1}k = \begin{bmatrix} c_3 & c_4 & d_3 & d_4\varpi^{-n} \\ a_3\varpi^n & a_4 & b_3 & b_4 \\ c_1\varpi^n & c_2 & d_1 & d_2 \\ -a_1\varpi^n & -a_2\varpi^n & -b_1\varpi^n & -b_2 \end{bmatrix},$$

and hence

$$\begin{bmatrix} 1 & & & d_4b_2^{-1}\varpi^{-n} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} t_n^{-1}k \in \mathrm{Kl}(\mathfrak{p}^n).$$

Since $v(d_4) > 0$, we see that k is contained in one of the sets from the second union. This completes the proof. $\square$

Lemma 3.3.2. *Let $n \ge 2$. There is a disjoint decomposition*

$$\mathrm{Kl}(\mathfrak{p}^{n-1}) = \bigsqcup_{\lambda,\mu,\kappa \in \mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & & & \\ \lambda\varpi^{n-1} & 1 & & \\ \mu\varpi^{n-1} & & 1 & \\ \kappa\varpi^{n-1} & \mu\varpi^{n-1} & -\lambda\varpi^{n-1} & 1 \end{bmatrix} \mathrm{Kl}(\mathfrak{p}^n).$$

Proof. This follows immediately from the Iwahori factorization (2.7) for the Klingen congruence subgroup. $\square$

Lemma 3.3.3. *If the Haar measure on* $\mathrm{GSp}(4, F)$ *is normalized so that* $\mathrm{vol}(\mathrm{GSp}(4, \mathfrak{o})) = 1$*, then*

$$\mathrm{vol}(\mathrm{Kl}(\mathfrak{p}^n)) = \frac{1}{(1+q^{-1})(1+q^{-2})q^{3n}}, \qquad \mathrm{vol}(\mathrm{K}(\mathfrak{p}^n)) = \frac{1}{(1+q^{-2})q^{2n}}$$

for any $n \geq 1$.

Proof. Let Q be the Klingen parabolic subgroup. By the Bruhat decomposition we have over any field k

$$\mathrm{GSp}(4,k) = Q \sqcup Q s_1 \begin{bmatrix} 1 & * & & \\ & 1 & & \\ & & 1 & * \\ & & & 1 \end{bmatrix} \sqcup Q s_1 s_2 \begin{bmatrix} 1 & & * & \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{bmatrix} \sqcup Q s_1 s_2 s_1 \begin{bmatrix} 1 & * & * & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 \end{bmatrix}. \quad (3.15)$$

If k is the finite field with q elements, it follows that $\#(\mathrm{GSp}(4,k)/Q) = 1+q+q^2+q^3$. This is the index of $\mathrm{Kl}(\mathfrak{p})$ in $K = \mathrm{GSp}(4,\mathfrak{o})$, hence

$$\mathrm{vol}(\mathrm{Kl}(\mathfrak{p})) = \frac{1}{(1+q)(1+q^2)}.$$

Using Lemma 3.3.2, this proves the formula for $\mathrm{vol}(\mathrm{Kl}(\mathfrak{p}^n))$. The formula for $\mathrm{vol}(\mathrm{K}(\mathfrak{p}^n))$ is then easily obtained from Lemma 3.3.1. □

Lemma 3.3.4. *Let* $n \geq 2$. *Define*

$$J = \mathrm{K}(\mathfrak{p}^{n-1}) \cap \mathrm{K}(\mathfrak{p}^n).$$

Then there is a disjoint decomposition

$$\mathrm{K}(\mathfrak{p}^{n-1}) = \bigsqcup_{\lambda,\mu,\kappa\in\mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & & & \\ \lambda\varpi^{n-1} & 1 & & \\ \mu\varpi^{n-1} & & 1 & \\ \kappa\varpi^{n-1} & \mu\varpi^{n-1} & -\lambda\varpi^{n-1} & 1 \end{bmatrix} J$$
$$\sqcup \bigsqcup_{\lambda,\mu\in\mathfrak{o}/\mathfrak{p}} t_{n-1} \begin{bmatrix} 1 & & & \\ \lambda\varpi^{n-1} & 1 & & \\ \mu\varpi^{n-1} & & 1 & \\ & \mu\varpi^{n-1} & -\lambda\varpi^{n-1} & 1 \end{bmatrix} J.$$

(For the definition of t_{n-1} *see (2.3).)*

Proof. It follows from Lemmas 3.2.1 and 3.3.3 that

$$\mathrm{vol}(J) = \frac{1}{(1+q)(1+q^{-2})q^{2n}}.$$

Another application of Lemma 3.3.3 shows that the index of J in $\mathrm{K}(\mathfrak{p}^{n-1})$ is q^3+q^2. Hence we need only show that the cosets given in the lemma are pairwise disjoint. This is a straightforward verification. □

Finally, we shall need the following coset decomposition involving $\mathrm{Kl}(\mathfrak{p})$, the Klingen congruence subgroup of level $\mathfrak{p}$, which is a parahoric subgroup.

Lemma 3.3.5. *We have the disjoint decomposition*

$$\mathrm{GSp}(4,\mathfrak{o}) = s_1\mathrm{Kl}(\mathfrak{p}) \sqcup \bigsqcup_{x\in\mathfrak{o}/\mathfrak{p}} \begin{bmatrix}1&x&&\\&1&&x\\&&1&\\&&&1\end{bmatrix} s_1s_2s_1\mathrm{Kl}(\mathfrak{p})$$
$$\sqcup \bigsqcup_{x,y\in\mathfrak{o}/\mathfrak{p}} \begin{bmatrix}1&&&\\x&1&&y\\&&1&\\&&-x&1\end{bmatrix} s_2s_1\mathrm{Kl}(\mathfrak{p}) \sqcup \bigsqcup_{x,y,z\in\mathfrak{o}/\mathfrak{p}} \begin{bmatrix}1&&&\\x&1&&\\y&&1&\\z&y&-x&1\end{bmatrix}\mathrm{Kl}(\mathfrak{p}).$$

Proof. Let Q be the Klingen parabolic subgroup. By the Bruhat decomposition we have

$$\mathrm{GSp}(4,k) = Q \sqcup \begin{bmatrix}1&*&&\\&1&&\\&&1&*\\&&&1\end{bmatrix} s_1Q \sqcup \begin{bmatrix}1&&*&\\&1&*&*\\&&1&\\&&&1\end{bmatrix} s_2s_1Q \sqcup \begin{bmatrix}1&*&*&*\\&1&&*\\&&1&*\\&&&1\end{bmatrix} s_1s_2s_1Q \tag{3.16}$$

over any field k. Multiplying from the left with $s_1s_2s_1$ gives

$$\mathrm{GSp}(4,k) = s_1s_2s_1Q \sqcup \begin{bmatrix}1&&&\\&1&&\\*&&1&\\&*&&1\end{bmatrix} s_1Q \sqcup \begin{bmatrix}1&&&\\*&1&*&\\&&1&\\&&*&1\end{bmatrix} s_2s_1Q \sqcup \begin{bmatrix}1&&&\\*&1&&\\*&&1&\\*&*&*&1\end{bmatrix} Q. \tag{3.17}$$

Taking for k the residue field $\mathfrak{o}/\mathfrak{p}$, it follows that

$$\mathrm{GSp}(4,\mathfrak{o}) = s_1s_2s_1\mathrm{Kl}(\mathfrak{p}) \sqcup \bigsqcup_{x\in\mathfrak{o}/\mathfrak{p}} \begin{bmatrix}1&&&\\&1&&\\x&&1&\\&x&&1\end{bmatrix} s_1\mathrm{Kl}(\mathfrak{p})$$
$$\sqcup \bigsqcup_{x,y\in\mathfrak{o}/\mathfrak{p}} \begin{bmatrix}1&&&\\x&1&&y\\&&1&\\&&-x&1\end{bmatrix} s_2s_1\mathrm{Kl}(\mathfrak{p}) \sqcup \bigsqcup_{x,y,z\in\mathfrak{o}/\mathfrak{p}} \begin{bmatrix}1&&&\\x&1&&\\y&&1&\\z&y&-x&1\end{bmatrix}\mathrm{Kl}(\mathfrak{p}). \tag{3.18}$$

Splitting the second union into $x \in \mathfrak{o}^\times$ and $x = 0$, we obtain the desired decomposition after some straightforward manipulations involving the "useful identity" (2.8). □

The Operator δ_1

Now let (π, V) be a smooth representation of $\mathrm{GSp}(4,F)$ for which the center acts trivially. We shall define level lowering operators δ_1, δ_2 and δ_3, starting with the natural summation operator $\delta_1 : V(n) \to V(n-1)$ given by

$$\delta_1 v = \sum_{g \in \mathrm{K}(\mathfrak{p}^{n-1})/(\mathrm{K}(\mathfrak{p}^{n-1}) \cap \mathrm{K}(\mathfrak{p}^n))} \pi(g)v. \tag{3.19}$$

As mentioned above, explicit formulas for δ_1 look different for $n = 1$ and for $n \geq 2$.

Lemma 3.3.6. *The level lowering operator $\delta_1 : V(1) \to V(0)$ has the explicit formula*

$$\begin{aligned}\delta_1 v = \pi(s_1)v &+ \sum_{x \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & x & & \\ & 1 & x & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1 s_2 s_1)v \\ &+ \sum_{x,y \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & & & \\ x & 1 & y & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} s_2 s_1)v + \sum_{x,y,z \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & & & \\ x & 1 & & \\ y & & 1 & \\ z & y & -x & 1 \end{bmatrix})v. \end{aligned} \tag{3.20}$$

Alternatively,

$$\begin{aligned}\delta_1 v = v &+ \sum_{x,y \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi^{-1} & \\ & & & 1 \end{bmatrix} s_1)v \\ &+ \sum_{x \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} s_1)v + \sum_{x,y,z \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & x & y & z \\ & 1 & & y \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi^{-1} \end{bmatrix})v \end{aligned} \tag{3.21}$$

for $v \in V(1)$.

Proof. By Lemma 3.3.5, the formula given in (3.20) is in fact the formula for the natural trace operator $V^{\mathrm{Kl}(\mathfrak{p})} \to V(0)$. But $\mathrm{GSp}(4, \mathfrak{o}) \cap \mathrm{K}(\mathfrak{p}) = \mathrm{Kl}(\mathfrak{p})$, hence it is also the formula for $\delta_1 : V(1) \to V(0)$. The alternative formula follows from (3.16), using the fact that $v \in V(1)$ is invariant under $\mathrm{diag}(\varpi^{-1}, 1, 1, \varpi)s_1 s_2 s_1$. □

Lemma 3.3.7. *Let $n \geq 2$. Let $v \in V(n)$. Then:*

$$\begin{aligned}\delta_1 v = &\sum_{\lambda,\mu,\kappa \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & & & \\ \lambda\varpi^{n-1} & 1 & & \\ \mu\varpi^{n-1} & & 1 & \\ \kappa\varpi^{n-1} & \mu\varpi^{n-1} & -\lambda\varpi^{n-1} & 1 \end{bmatrix})v \\ &+ \sum_{\lambda,\mu \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & \lambda & \mu & \\ & 1 & & \mu \\ & & 1 & -\lambda \\ & & & 1 \end{bmatrix} \eta^{-1})v. \end{aligned} \tag{3.22}$$

Alternatively,

$$\delta_1 v = \sum_{\lambda,\mu,\kappa\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & \lambda & \mu & \kappa\varpi^{-(n-1)} \\ & 1 & & \mu \\ & & 1 & -\lambda \\ & & & 1 \end{bmatrix} \eta^{-1})v$$
$$+ \sum_{\lambda,\mu\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & & & \\ \lambda\varpi^{n-1} & 1 & & \\ \mu\varpi^{n-1} & & 1 & \\ & \mu\varpi^{n-1} & -\lambda\varpi^{n-1} & 1 \end{bmatrix})v. \qquad (3.23)$$

Here η is as in (3.5).

Proof. By Lemma 3.3.4,

$$\delta_1 v = \sum_{\lambda,\mu,\kappa\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & & & \\ \lambda\varpi^{n-1} & 1 & & \\ \mu\varpi^{n-1} & & 1 & \\ \kappa\varpi^{n-1} & \mu\varpi^{n-1} & -\lambda\varpi^{n-1} & 1 \end{bmatrix})v$$
$$+ \sum_{\lambda,\mu\in\mathfrak{o}/\mathfrak{p}} \pi(t_{n-1})\pi(\begin{bmatrix} 1 & & & \\ \lambda\varpi^{n-1} & & & \\ \mu\varpi^{n-1} & & 1 & \\ & \mu\varpi^{n-1} & -\lambda\varpi^{n-1} & 1 \end{bmatrix})v$$
$$= \sum_{\lambda,\mu,\kappa\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & & & \\ \lambda\varpi^{n-1} & 1 & & \\ \mu\varpi^{n-1} & & 1 & \\ \kappa\varpi^{n-1} & \mu\varpi^{n-1} & -\lambda\varpi^{n-1} & 1 \end{bmatrix})v$$
$$+ \sum_{\lambda,\mu\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & \mu & \lambda & \\ & 1 & & \lambda \\ & & 1 & -\mu \\ & & & 1 \end{bmatrix})\pi(t_{n-1})v.$$

Since $t_{n-1} = \eta^{-1}t_n$, we obtain (3.22). The second formula follows by applying t_{n-1} to the first one. □

The Operator δ_2

We next study the dual operator $\delta_2 := u_{n-1} \circ \delta_1 \circ u_n$, which is also a linear operator from $V(n)$ to $V(n-1)$. We claim that

$$\delta_2 v = \sum_{g\in \mathrm{K}(\mathfrak{p}^{n-1})/J'} \pi(g) \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} v,$$

where

$$J' = \mathrm{K}(\mathfrak{p}^{n-1}) \cap \begin{bmatrix} \varpi &&& \\ & \varpi && \\ && 1 & \\ &&& 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n) \begin{bmatrix} \varpi^{-1} &&& \\ & \varpi^{-1} && \\ && 1 & \\ &&& 1 \end{bmatrix}.$$

Indeed, this follows easily from the facts that

$$u_{n-1}^{-1} \mathrm{K}(\mathfrak{p}^n) u_{n-1} = \begin{bmatrix} \varpi &&& \\ & \varpi && \\ && 1 & \\ &&& 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n) \begin{bmatrix} \varpi^{-1} &&& \\ & \varpi^{-1} && \\ && 1 & \\ &&& 1 \end{bmatrix}$$

and that

$$\begin{bmatrix} \varpi &&& \\ & \varpi && \\ && 1 & \\ &&& 1 \end{bmatrix} u_n^{-1} = u_{n-1}^{-1}.$$

Lemma 3.3.8. *The operator $\delta_2 : V(1) \to V(0)$ has the explicit formula*

$$\delta_2 v = \eta^{-1}\theta' u_1 v + \sum_{x,y,z \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 && y & \\ x & 1 & z & y \\ && 1 & \\ && -x & 1 \end{bmatrix} \begin{bmatrix} \varpi &&& \\ & \varpi && \\ && 1 & \\ &&& 1 \end{bmatrix})v$$
$$+ \sum_{x,y \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & x && \\ & 1 && \\ y && 1 & -x \\ & y && 1 \end{bmatrix} \begin{bmatrix} \varpi &&& \\ & 1 && \\ && \varpi & \\ &&& 1 \end{bmatrix})v.$$

Here, u_1 is the Atkin–Lehner involution at level $\mathfrak{p}$. An explicit formula for the level raising operator θ' is given in Lemma 3.2.2.

Proof. Dualizing the formula (3.20), we obtain

$$\delta_2 v = \sum_{x,y,z \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 && y & \\ x & 1 & z & y \\ && 1 & \\ && -x & 1 \end{bmatrix} \begin{bmatrix} \varpi &&& \\ & \varpi && \\ && 1 & \\ &&& 1 \end{bmatrix})v$$
$$+ \sum_{x,y \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 &&& \\ x & 1 && \\ && 1 & \\ y && -x & 1 \end{bmatrix} s_1 \begin{bmatrix} \varpi &&& \\ & 1 && \\ && \varpi & \\ &&& 1 \end{bmatrix})v$$
$$+ \sum_{x \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 &&& \\ & 1 && \\ x && 1 & \\ & x && 1 \end{bmatrix} \begin{bmatrix} \varpi &&& \\ & 1 && \\ && \varpi & \\ &&& 1 \end{bmatrix})v + \pi(s_1 \begin{bmatrix} \varpi &&& \\ & \varpi && \\ && 1 & \\ &&& 1 \end{bmatrix})v.$$

Some standard manipulations show that

$$\sum_{y\in\mathfrak{o}/\mathfrak{p}}\sum_{x\in(\mathfrak{o}/\mathfrak{p})^\times}\pi(\begin{bmatrix}1&&&\\x&1&&\\&&1&\\y&&-x&1\end{bmatrix}s_1\begin{bmatrix}\varpi&&&\\&1&&\\&&\varpi&\\&&&1\end{bmatrix})v$$
$$=\sum_{y\in\mathfrak{o}/\mathfrak{p}}\sum_{x\in(\mathfrak{o}/\mathfrak{p})^\times}\pi(\begin{bmatrix}1&x&&\\&1&&\\y&&1&-x\\&y&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&1&&\\&&\varpi&\\&&&1\end{bmatrix})v,$$

and that

$$\sum_{y\in(\mathfrak{o}/\mathfrak{p})^\times}\pi(\begin{bmatrix}1&&&\\&1&&\\&&1&\\y&&&1\end{bmatrix}s_1\begin{bmatrix}\varpi&&&\\&1&&\\&&\varpi&\\&&&1\end{bmatrix})v=\sum_{y\in(\mathfrak{o}/\mathfrak{p})^\times}\pi(\begin{bmatrix}1&&&y\\&1&&\\&&1&\\&&&1\end{bmatrix}s_1\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})v.$$

Using Lemma 3.2.2 ii) and collecting all the pieces, the assertion follows easily. □

Lemma 3.3.9. *For $n\geq 2$ the operator $\delta_2:\ V(n)\to V(n-1)$ has the explicit formula*

$$\delta_2 v=\sum_{\lambda,\mu,\kappa\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&&&\\\lambda\varpi^{n-1}&1&&\\&&1&\\&&-\lambda\varpi^{n-1}&1\end{bmatrix}\begin{bmatrix}1&&\mu&\\&1&\kappa&\mu\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})v$$
$$+\sum_{\lambda,\mu\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&&&\\&1&&\\\lambda\varpi^{n-1}&&1&\\&\lambda\varpi^{n-1}&&1\end{bmatrix}\begin{bmatrix}1&\mu&&\\&1&&\\&&1&-\mu\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&1&&\\&&\varpi&\\&&&1\end{bmatrix})v.\qquad(3.24)$$

Proof. This follows by conjugating the elements in Lemma 3.3.4 with u_{n-1}. □

The Operator δ_3

We define yet another operator $\delta_3:\ V(n)\to V(n-1)$ by

$$\delta_3 v=\sum_{g\in\mathrm{K}(\mathfrak{p}^{n-1})/(\mathrm{K}(\mathfrak{p}^{n-1})\cap\eta^{-1}\mathrm{K}(\mathfrak{p}^n)\eta)}\pi(g\eta^{-1})v.$$

Lemma 3.3.10. *The operator $\delta_3:\ V(1)\to V(0)$ has the explicit formula*

$$\delta_3 v=\eta^{-1}\theta u_1 v+\sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&x&y&z\\&1&&y\\&&1&-x\\&&&1\end{bmatrix}\eta^{-1})v$$

$$+\sum_{x,y\in\mathfrak{o}/\mathfrak{p}}\pi(s_1s_2s_1\begin{bmatrix}1&x&y&\\&1&&y\\&&1&-x\\&&&1\end{bmatrix}\eta^{-1})v. \tag{3.25}$$

Here, u_1 is the Atkin–Lehner involution at level $\mathfrak{p}$. An explicit formula for the level raising operator θ is given in Lemma 3.2.2.

Proof. We have

$$\eta^{-1}\mathrm{K}(\mathfrak{p})\eta = t_0\mathrm{K}(\mathfrak{p})t_0^{-1} = \begin{bmatrix}\mathfrak{o}&\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\\\mathfrak{o}&\mathfrak{o}&\mathfrak{o}&\mathfrak{p}\\\mathfrak{o}&\mathfrak{o}&\mathfrak{o}&\mathfrak{p}\\\mathfrak{p}^{-1}&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\end{bmatrix}.$$

Hence $J := \mathrm{GSp}(4,\mathfrak{o})\cap\eta^{-1}\mathrm{K}(\mathfrak{p})\eta$ is the $s_1s_2s_1s_2$ conjugate of $\mathrm{Kl}(\mathfrak{p})$. Conjugating (3.18) with the longest Weyl group element, we obtain

$$\mathrm{GSp}(4,\mathfrak{o}) = s_1s_2s_1J\sqcup\bigsqcup_{x\in\mathfrak{o}/\mathfrak{p}}\begin{bmatrix}1&&x&\\&1&&x\\&&1&\\&&&1\end{bmatrix}s_1J$$

$$\sqcup\bigsqcup_{x,y\in\mathfrak{o}/\mathfrak{p}}\begin{bmatrix}1&x&&\\&1&&\\&y&1&-x\\&&&1\end{bmatrix}s_2s_1J\sqcup\bigsqcup_{x,y,z\in\mathfrak{o}/\mathfrak{p}}\begin{bmatrix}1&x&y&z\\&1&&y\\&&1&-x\\&&&1\end{bmatrix}J.$$

It follows that

$$\delta_3v = \pi(s_1s_2s_1\eta^{-1})v + \sum_{x\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&&x&\\&1&&x\\&&1&\\&&&1\end{bmatrix}s_1\eta^{-1})v$$

$$+\sum_{x,y\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&x&&\\&1&&\\&y&1&-x\\&&&1\end{bmatrix}s_2s_1\eta^{-1})v + \sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&x&y&z\\&1&&y\\&&1&-x\\&&&1\end{bmatrix}\eta^{-1})v.$$

The first term is equal to $\pi(t_1)v = v$. Standard manipulations show that

$$\sum_{x\in(\mathfrak{o}/\mathfrak{p})^\times}\sum_{y\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&x&&\\&1&&\\&y&1&-x\\&&&1\end{bmatrix}s_2s_1\eta^{-1})v$$

$$=\sum_{x\in(\mathfrak{o}/\mathfrak{p})^\times}\sum_{y\in\mathfrak{o}/\mathfrak{p}}\pi(s_1s_2s_1\begin{bmatrix}1&y&x&\\&1&&x\\&&1&-y\\&&&1\end{bmatrix}\eta^{-1})v$$

and that

$$\sum_{y\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&&&\\&1&&\\&y&1&\\&&&1\end{bmatrix}s_2s_1\eta^{-1})v=\pi(s_1)v+\sum_{y\in(\mathfrak{o}/\mathfrak{p})^\times}\pi(s_1\eta^{-1}\begin{bmatrix}1&&&y\varpi^{-2}\\&1&&\\&&1&\\&&&1\end{bmatrix})v.$$

Furthermore,

$$\sum_{x\in(\mathfrak{o}/\mathfrak{p})^\times}\pi(\begin{bmatrix}1&x&&\\&1&&x\\&&1&\\&&&1\end{bmatrix}s_1\eta^{-1})v=\sum_{x\in(\mathfrak{o}/\mathfrak{p})^\times}\pi(s_1s_2s_1\begin{bmatrix}1&x&&\\&1&&\\&&1&-x\\&&&1\end{bmatrix}\eta^{-1})v.$$

Hence, using Lemma 3.2.2 ii),

$$\begin{aligned}\delta_3v&=\pi(s_1\eta^{-1})\theta'v\\&\quad+\sum_{x,y\in\mathfrak{o}/\mathfrak{p}}\pi(s_1s_2s_1\begin{bmatrix}1&x&y&\\&1&&y\\&&1&-x\\&&&1\end{bmatrix}\eta^{-1})v+\sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&x&y&z\\&1&&y\\&&1&-x\\&&&1\end{bmatrix}\eta^{-1})v.\end{aligned}$$

Since

$$\pi(s_1\eta^{-1})\theta'v=\pi(s_1\eta^{-1})\theta'u_1^2v=\pi(s_1\eta^{-1}u_2)\theta u_1v=\pi(\eta^{-1}t_2)\theta u_1v,$$

the assertion follows. □

Lemma 3.3.11. *For $n\geq 2$ the operator $\delta_3:V(n)\to V(n-1)$ has the explicit formula*

$$\begin{aligned}\delta_3v=&\sum_{\lambda,\mu,\kappa\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&\mu&&\\&1&&\\&&1&-\mu\\&&&1\end{bmatrix}\begin{bmatrix}1&&\lambda&\kappa\varpi^{-n+1}\\&1&&\lambda\\&&1&\\&&&1\end{bmatrix}\eta^{-1})v\\&+\sum_{\lambda,\mu\in\mathfrak{o}/\mathfrak{p}}\pi(t_{n-1}\begin{bmatrix}1&\mu&&\\&1&&\\&&1&-\mu\\&&&1\end{bmatrix}\begin{bmatrix}1&\lambda&&\\&1&&\lambda\\&&1&\\&&&1\end{bmatrix}\eta^{-1})v.\end{aligned}\tag{3.26}$$

Proof. Note that $\eta^{-1}\mathrm{K}(\mathfrak{p}^n)\eta=t_{n-1}\mathrm{K}(\mathfrak{p}^n)t_{n-1}^{-1}$. Conjugating the representatives given in Lemma 3.3.4 with t_{n-1}, we obtain the asserted formula. □

Relations Between Level Lowering and Level Raising Operators

Lemma 3.3.12. *Let $n\geq 0$. If $v\in V(n)$, then*

$$\delta_1\eta v=q^2\theta' v.$$

Proof. Using Lemma 3.3.7, we compute

$$
\begin{aligned}
(\delta_1\eta)v &= \sum_{\lambda,\mu,\kappa\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix}1&\lambda&\mu&\kappa\varpi^{-(n+1)}\\&1&&\mu\\&&1&-\lambda\\&&&1\end{bmatrix}\eta^{-1})\pi(\eta)v\\
&\quad+\sum_{\lambda,\mu\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&&&\\ \lambda\varpi^{n+1}&1&&\\ \mu\varpi^{n+1}&&1&\\ &\mu\varpi^{n+1}&-\lambda\varpi^{n+1}&1\end{bmatrix})\pi(\eta)v\\
&= \sum_{\lambda,\mu,\kappa\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix}1&\lambda&\mu&\kappa\varpi^{-(n+1)}\\&1&&\mu\\&&1&-\lambda\\&&&1\end{bmatrix})v\\
&\quad+\pi(\eta)\sum_{\lambda,\mu\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&&&\\ \lambda\varpi^{n}&1&&\\ \mu\varpi^{n}&&1&\\ &\mu\varpi^{n}&-\lambda\varpi^{n}&1\end{bmatrix})v\\
&= q^2\sum_{\kappa\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&&&\kappa\varpi^{-(n+1)}\\&1&&\\&&1&\\&&&1\end{bmatrix})v+q^2\pi(\eta)v.
\end{aligned}
$$

The last expression equals $q^2\theta' v$ by Lemma 3.2.2. □

3.4 Paramodular Vectors and P_3-Theory

As we mentioned in Sect. 2.5, P_3-theory is a useful tool for the investigation of paramodular vectors. In this section, using the linear independence of paramodular vectors and properties of level raising operators, we will prove that if (π, V) is any smooth representation of $\mathrm{GSp}(4, F)$ such that the center of $\mathrm{GSp}(4, F)$ acts trivially, and the subspace of vectors of V fixed by $\mathrm{Sp}(4, F)$ is trivial, then the restriction of the projection map $p : V \to V_{Z^J}$ to $V(n)$ is injective for all n. As a corollary we will deduce that some representations do not admit non-zero paramodular vectors. Thus, we will prove that, for example, nongeneric, supercuspidal, irreducible, admissible representations of $\mathrm{GSp}(4, F)$ with trivial central character have no non-zero paramodular vectors.

To start, let (π, V) be a smooth representation of $\mathrm{GSp}(4, F)$ such that the center of $\mathrm{GSp}(4, F)$ acts trivially. It is easy to see that if n is a non-negative integer and W is in $V(n)$, then $p(W) = 0$ if and only if

$$\int\limits_{\mathfrak{p}^{-(n+k)}/p^{-n}} \pi(\begin{bmatrix} 1 & & & x \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v\,dx = 0$$

for some non-negative integer k. The next lemma analyzes this integral.

Lemma 3.4.1. *Let (π, V) be a smooth representation of* $\mathrm{GSp}(4, F)$ *such that the center of* $\mathrm{GSp}(4, F)$ *acts trivially. Let $n \geq 0$ and $k \geq 0$ be non-negative integers and $v \in V(n)$. Then*

$$\int\limits_{\mathfrak{p}^{-(n+k)}/\mathfrak{p}^{-n}} \pi(\begin{bmatrix} 1 & & & x \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v\,dx = \theta'^k v + \eta v_1' + \eta v_2' \tag{3.27}$$

where $v_1' \in V(n+k-2)$ and $v_2' \in V(n+k-1)$. Here, $v_1' = 0$ if $n+k-2 < 0$ and $v_2' = 0$ if $n+k-1 < 0$.

Proof. We prove this assertion by induction on k. If $k = 0$, then the assertion is true with $v_1' = v_2' = 0$. Assume it is true for k, so that (3.27) holds with $v_1' \in V(n+k-2)$ and $v_2' \in V(n+k-1)$; we will prove it for $k+1$. We have by (3.7)

$$\begin{aligned}
&\int\limits_{\mathfrak{p}^{-(n+k+1)}/\mathfrak{p}^{-n}} \pi(\begin{bmatrix} 1 & & & y \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v\,dy \\
&= \int\limits_{\mathfrak{p}^{-(n+k+1)}/\mathfrak{p}^{-(n+k)}} \pi(\begin{bmatrix} 1 & & & y \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})(\int\limits_{\pi^{-(n+k)}/\mathfrak{p}^{-n}} \pi(\begin{bmatrix} 1 & & & x \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v\,dx)\,dy \\
&= \int\limits_{\mathfrak{p}^{-(n+k+1)}/\mathfrak{p}^{-(n+k)}} \pi(\begin{bmatrix} 1 & & & y \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})(\theta'^k v + \eta v_1' + \eta v_2')\,dy \\
&= (\theta'(\theta'^k v) - \eta(\theta'^k v)) + (\theta'(\eta v_1') - \eta(\eta v_1')) + q\eta v_2' \\
&= \theta'^{k+1} v + \eta(\theta' v_1' + q v_2') + \eta(-\theta'^k v - \eta v_1').
\end{aligned}$$

As $\theta' v_1' + q v_2' \in V(n+(k+1)-2)$ and $-\theta'^k v - \eta v_1' \in V(n+(k+1)-1)$, the assertion follows by induction. □

Using the last lemma, we can prove the important fact that non-zero paramodular vectors have non-zero projections to V_{Z^J}. In fact, we can determine the kernel of the restriction of the projection to V_{para}.

Proposition 3.4.2. *Let (π, V) be a smooth representation of* $\mathrm{GSp}(4,F)$ *such that the center of* $\mathrm{GSp}(4,F)$ *acts trivially. Assume that the subspace of vectors of V fixed by* $\mathrm{Sp}(4,F)$ *is trivial. Let $p: V \to V_{Z^J}$ be the projection map. If $v \in V(n)$ for some non-negative integer $n \geq 0$ and $p(v) = 0$, then $v = 0$. More generally, if $v \in V_{\mathrm{para}}$ and $p(v) = 0$, then v is a linear combination of vectors of the form*

$$qw - \theta' w + \eta w, \tag{3.28}$$

where $w \in V(m)$ for some non-negative integer $m \geq 0$.

Proof. To prove the first assertion of the lemma, let $n \geq 0$ be a non-negative integer and $v \in V(n)$, and assume $p(v) = 0$. We need to prove $v = 0$. Since $p(v) = 0$, there exists a non-negative integer $k \geq 0$ such that

$$\int\limits_{\mathfrak{p}^{-(n+k)}/\mathfrak{p}^{-n}} \pi(\begin{bmatrix} 1 & & & x \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v\,dx = 0. \tag{3.29}$$

Using induction on k, we will prove that if (3.29) holds, then $v = 0$. If $k = 0$, this is clear. Suppose it holds for k; we will prove that it holds for $k+1$. By Lemma 3.4.1, there exist $v_1' \in V(n+k-2)$ and $v_2' \in V(n+k-1)$ such that

$$\int\limits_{\mathfrak{p}^{-(n+k)}/\mathfrak{p}^{-n}} \pi(\begin{bmatrix} 1 & & & x \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v\,dx = \theta'^k v + \eta v_1' + \eta v_2'. \tag{3.30}$$

As in the proof of Lemma 3.4.1,

$$\begin{aligned} &\int\limits_{\mathfrak{p}^{-(n+k+1)}/\mathfrak{p}^{-n}} \pi(\begin{bmatrix} 1 & & & y \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v\,dy \\ &\qquad = \underbrace{\theta'^{k+1} v}_{\in V(n+k+1)} + \underbrace{\eta(\theta' v_1' + q v_2')}_{\in V(n+k+1)} + \underbrace{\eta(-\theta'^k v - \eta v_1')}_{\in V(n+k+2)}. \end{aligned}$$

By assumption, this is zero. By Theorem 3.1.3, we have

$$\begin{aligned} \theta'^{k+1} v + \eta\theta' v_1' + q\eta v_2' &= 0, \\ \eta(\theta'^k v + \eta v_1') &= 0. \end{aligned}$$

Since η is invertible, the second equation implies that $\theta'^k v = -\eta v_1'$. Substituting this into the first equation, we obtain $q\eta v_2' = 0$. This implies $v_2' = 0$. Since $\theta'^k v = -\eta v_1'$ and $v_2' = 0$, by (3.30),

$$\int_{\mathfrak{p}^{-(n+k)}/\mathfrak{p}^{-n}} \pi(\begin{bmatrix}1&&&x\\&1&&\\&&1&\\&&&1\end{bmatrix})v\,dx = 0.$$

By the induction hypothesis, this implies $v = 0$.

To prove the second assertion of the proposition we require a new concept. If $v \in V_{\mathrm{para}}$, then v can be written in the form $v = v_1 + \cdots + v_t$, where $v_i \in V(n_i)$, v_i is non-zero, and $0 \le n_1 < \cdots < n_t$. By Theorem 3.1.3, this representation is unique. We define the *length* of v to be $l(v) = n_t - n_1$; if $t = 1$, then this is defined to be zero. Suppose that $v \in V_{\mathrm{para}}$ and $p(v) = 0$. We will prove that v is a linear combination of elements of the form (3.28) by induction on $l(v)$. If $l(v) = 0$, then this follows from the last paragraph. Suppose that if $v \in V_{\mathrm{para}}$, $p(v) = 0$ and $l(v) \le m$, then v is a linear combination of elements of the form (3.28). We will prove that if $v \in V_{\mathrm{para}}$, $p(v) = 0$ and $l(v) = m+1$, then v is a linear combination of elements of the form (3.28). Write $v = v_1 + \cdots + v_t$ as in the definition of length. Set

$$v' = \int_{\mathfrak{p}^{-(n_1+1)}/\mathfrak{p}^{-n_1}} \pi(\begin{bmatrix}1&&&x\\&1&&\\&&1&\\&&&1\end{bmatrix})v\,dx.$$

Then $p(v') = 0$. Also, by (3.7),

$$\begin{aligned} v' &= \int_{\mathfrak{p}^{-(n_1+1)}/\mathfrak{p}^{-n_1}} \pi(\begin{bmatrix}1&&&x\\&1&&\\&&1&\\&&&1\end{bmatrix})v\,dx + qv_2 + \cdots + qv_t \\ &= \theta' v_1 - \eta v_1 + qv_2 + \cdots + qv_t. \end{aligned}$$

This formula implies that $l(v') \le n_t - (n_1 + 1) = l(v) - 1 = m$. By the induction hypothesis, v' is the sum of elements of the form (3.28). Since

$$v = q^{-1}(v' + qv_1 - \theta' v_1 + \eta v_1),$$

it follows that v is a linear combination of elements of the form (3.28). □

The final result of this section uses Proposition 3.4.2 to prove that some representations, including all non-generic, supercuspidal, irreducible, admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character, are not paramodular. The idea of the proof is to consider the projections of paramodular vectors into the P_3-filtration of V_{Z^J}.

Theorem 3.4.3. *The following non-generic, irreducible, admissible representations of* $\mathrm{GSp}(4,F)$ *with trivial central character do not admit non-zero paramodular vectors of any level:*

		representation	condition on defining data
II	b	$\chi 1_{\mathrm{GL}(2)} \rtimes \sigma$	$\chi\sigma$ ramified
III	b	$\chi \rtimes \sigma 1_{\mathrm{GSp}(2)}$	σ ramified
IV	b	$L(\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GSp}(2)})$	σ ramified
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$	σ ramified
	d	$\sigma 1_{\mathrm{GSp}(4)}$	σ ramified
V	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	σ ramified
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$	$\xi\sigma$ ramified
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	σ or ξ ramified
VI	b	$\tau(T, \nu^{-1/2}\sigma)$	none
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	σ ramified
	d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$	σ ramified
VIII	b	$\tau(T, \pi)$	none
IX	b	$L(\nu\xi, \nu^{-1/2}\pi)$	none
XI	b	$L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	σ ramified
		π supercuspidal	non-generic

Proof. Let (π, V) be one of the representations in the table in the statement of the theorem. Let $n \geq 0$ be a non-negative integer, and let $v \in V(n)$. We need to prove $v = 0$. Let $p : V \to V_{Z^J} = V/V(Z^J)$ be the projection. Let

$$0 = V_2 \subset V_1 \subset V_0 = V_{Z^J}$$

be the filtration of P_3-subspaces from Theorem 2.5.3. Note that, as stated in Theorem 2.5.3, $V_2 = 0$ because π is not generic. By Proposition 3.4.2, it suffices to prove that $p(v) = 0$. The vector $p(v)$ is invariant under $P_3(\mathfrak{o})$. Using the assumptions on π from the table above, and the tables A.6 and A.5, which list the semi-simplifications of V_1/V_2 and V_0/V_1, one can verify that each of the non-zero irreducible P_3-subquotients of V_{Z^J} are either of the form $\tau_{\mathrm{GL}(1)}^{P_3}(\chi)$ for χ a ramified character of $F^\times$ or of the form $\tau_{\mathrm{GL}(2)}^{P_3}(\rho)$ for ρ a ramified, irreducible, admissible representation of $\mathrm{GL}(2, F)$. By Lemma 3.4.4 below we have $p(v) = 0$. □

Lemma 3.4.4. *Let χ be a character of $F^\times$, and let ρ be an irreducible, admissible representation of* $\mathrm{GL}(2, F)$*. If $\tau_{\mathrm{GL}(1)}^{P_3}(\chi)$ contains a non-zero vector fixed by $P_3(\mathfrak{o})$ then χ is unramified, and if $\tau_{\mathrm{GL}(2)}^{P_3}(\rho)$ contains a non-zero vector fixed by $P_3(\mathfrak{o})$ then ρ is unramified.*

Proof. The assertion about $\rho_{\mathrm{GL}(2)}^{P_3}(\rho)$ follows immediately from the definition of $\tau_{\mathrm{GL}(2)}^{P_3}(\rho)$. Suppose $\tau_{\mathrm{GL}(1)}^{P_3}(\chi)$ contains a non-zero vector f fixed by $P_3(\mathfrak{o})$. Let $p \in P_3$ be such that $f(p) \neq 0$. We can write

$$p = \begin{bmatrix} 1 & & z \\ & 1 & y \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} a & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & b & \\ & & 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & \\ k_3 & k_4 & \\ & & 1 \end{bmatrix}$$

for some $x, y, z \in F$, $a, b \in F^\times$ and $\left[\begin{smallmatrix} k_1 & k_2 \\ k_3 & k_4 \end{smallmatrix}\right] \in \mathrm{GL}(2, \mathfrak{o})$. Let $u \in \mathfrak{o}^\times$. Then

$$p \begin{bmatrix} u & & \\ & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & z \\ & 1 & y \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} au & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & b & \\ & & 1 \end{bmatrix} \begin{bmatrix} k_1 & u^{-1}k_2 & \\ uk_3 & k_4 & \\ & & 1 \end{bmatrix}.$$

Hence,

$$f(p) = f(p \begin{bmatrix} u & & \\ & 1 & \\ & & 1 \end{bmatrix}),$$

$$\psi(y)\chi(a)f(\begin{bmatrix} 1 & & \\ & b & \\ & & 1 \end{bmatrix}) = \psi(y)\chi(au)f(\begin{bmatrix} 1 & & \\ & b & \\ & & 1 \end{bmatrix}).$$

Since this is non-zero for some b, we obtain $\chi(u) = 1$. $\square$

4

Zeta Integrals

As we saw in the first chapter, if (π, V) is a generic, irreducible, admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character, then a theory of zeta integrals for π exists. This theory is used to define the L- and ε-factors for π. In this chapter we consider zeta integrals of paramodular vectors and prove central results required to fully exploit zeta integrals as a tool for investigating paramodular vectors. A major obstruction is the existence of *degenerate vectors*, i.e., paramodular vectors with vanishing zeta integrals; this phenomenon does not occur in the $\mathrm{GL}(2)$ theory. We prove the important *η Principle*, which fully accounts for degenerate vectors via the level raising operator η. The η Principle is proved using P_3-theory. To apply P_3-theory, we will prove a result that relates poles of the L-functions of generic representations to certain irreducible subquotients in the associated P_3-filtration; this is a general result that has nothing to do with paramodular vectors. In the last section of this chapter we also use P_3-theory to prove the existence of non-zero paramodular vectors in any generic representation. These results are proved after some basic observations about the zeta integrals of paramodular vectors.

4.1 Paramodular Vectors and Zeta Integrals

In this section we work with a *generic*, irreducible, admissible representation (π, V) of $\mathrm{GSp}(4, F)$ with trivial central character, where $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$ is the Whittaker model of π with respect to ψ_{c_1,c_2}. As usual when working with Whittaker models, we shall make the following assumptions: i) ψ has conductor $\mathfrak{o}$; ii) $c_1, c_2 \in \mathfrak{o}^\times$.

First Observations

We begin with some lemmas that show that the zeta integrals of paramodular vectors simplify considerably.

Lemma 4.1.1. *Assume that $W \in \mathcal{W}(\pi, \psi_{c_1,c_2})$ is right invariant under the following elements:*

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & \mathfrak{o} & 1 & \\ & & & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & & & \\ & 1 & \mathfrak{p} & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & & \mathfrak{o} & \\ & 1 & & \mathfrak{o} \\ & & 1 & \\ & & & 1 \end{bmatrix}. \tag{4.1}$$

Then the zeta integral of W as defined in (2.54) is given by the simplified formula

$$Z(s,W) = \int\limits_{F^\times} W(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix})|a|^{s-3/2}\,d^\times a. \tag{4.2}$$

In particular, the zeta integral of any paramodular vector W is given by (4.2).

Proof. Write $Z(s,W) = Z_1 + Z_2$ with $Z_1 = \int_{F^\times}\int_{\mathfrak{o}} \ldots dx\,d^\times a$ and $Z_2 = \int_{F^\times}\int_{F\setminus\mathfrak{o}} \ldots dx\,d^\times a$. By the invariance properties of W we can omit the x–variable in Z_1, so that Z_1 is equal to the right side of (4.2). We shall show that Z_2 vanishes. Using the fundamental identity (2.8) and the invariance properties of W, we find

$$Z_2 = \int\limits_{F^\times}\int\limits_{F\setminus\mathfrak{o}} W(\begin{bmatrix} a & & & \\ & ax^{-1} & & \\ & & x & \\ & & & 1 \end{bmatrix} s_2)\psi(c_2ax^{-1})|a|^{s-3/2}\,dx\,d^\times a. \tag{4.3}$$

But now, for any $y \in \mathfrak{o}$,

$$\begin{aligned} W(\begin{bmatrix} a & & & \\ & ax^{-1} & & \\ & & x & \\ & & & 1 \end{bmatrix} s_2) &= W(\begin{bmatrix} a & & & \\ & ax^{-1} & & \\ & & x & \\ & & & 1 \end{bmatrix} s_2 \begin{bmatrix} 1 & & y & \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix}) \\ &= \psi(c_1xy)W(\begin{bmatrix} a & & & \\ & ax^{-1} & & \\ & & x & \\ & & & 1 \end{bmatrix} s_2). \end{aligned}$$

Since $x \notin \mathfrak{o}$ and the character $\psi(c_1\,\cdot\,)$ has conductor $\mathfrak{o}$, this implies that the integrand in (4.3) is zero. □

Lemma 4.1.2. *Assume that $W \in \mathcal{W}(\pi, \psi_{c_1,c_2})$ is $\mathrm{K}(\mathfrak{p}^n)$-invariant for some $n \geq 0$. Then*

$$W(\begin{bmatrix} a & & & \\ & b & & \\ & & cb^{-1} & \\ & & & ca^{-1} \end{bmatrix}) = 0 \qquad \text{if } v(a) < v(b) \ \text{ or } \ 2v(b) < v(c).$$

Here, v is the normalized valuation on F.

Proof. The argument is similar to the one at the end of the proof of Lemma 4.1.1. For each $x, y \in \mathfrak{o}$ we have

$$W(\begin{bmatrix} a &&& \\ & b && \\ && cb^{-1} & \\ &&& ca^{-1} \end{bmatrix}) = W(\begin{bmatrix} a &&& \\ & b && \\ && cb^{-1} & \\ &&& ca^{-1} \end{bmatrix} \begin{bmatrix} 1 & x & xy & \\ & 1 & y & \\ && 1 & -x \\ &&& 1 \end{bmatrix})$$

$$= \psi(c_1 ab^{-1} x)\, \psi(c_2 b^2 c^{-1} y)\, W(\begin{bmatrix} a &&& \\ & b && \\ && cb^{-1} & \\ &&& ca^{-1} \end{bmatrix}).$$

Since $\psi(c_1\,\cdot\,)$ and $\psi(c_2\,\cdot\,)$ have conductor $\mathfrak{o}$, the assertion follows. □

Proposition 4.1.3. *Let $n \geq 0$ be a non-negative integer. Then, for any paramodular vector $W \in V(n)$, we have*

$$Z(s, \theta W) = q^{-s+3/2} Z(s, W),$$
$$Z(s, \theta' W) = q Z(s, W),$$
$$Z(s, \eta W) = 0.$$

Proof. These are easy computations using the previous two lemmas and the explicit formulas from Lemma 3.2.2. □

If $W \in V(n)$ is a paramodular vector for which $Z(s, W) = 0$, then we say that W is *degenerate*. Hence the η operator produces degenerate vectors. The η *Principle*, which we shall prove in Sect. 4.3, is the converse of this statement: If $W \in V(n)$ is non-zero and degenerate, then $n \geq 2$ and there exists a $W_1 \in V(n-2)$ such that $W = \eta W_1$.

Zeta Polynomials

The elements W of $V(n)$ are such that

$$Z(s, W) = \sum_{k=0}^{\infty} a(k) X^k, \qquad X = q^{-s},$$

the point being that no negative k occur. This follows immediately from Lemmas 4.1.1 and 4.1.2. The formula for $a(k)$ is

$$a(k) = W(\begin{bmatrix} \varpi^k &&& \\ & \varpi^k && \\ && 1 & \\ &&& 1 \end{bmatrix}) q^{3k/2} (1 - q^{-1}). \qquad (4.4)$$

Proposition 4.1.4 (Zeta Polynomials). *Let (π, V) be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4, F)$ *with trivial central character, where $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. Write the ε-factor of π as $\varepsilon(s, \pi) = \varepsilon q^{-N(s-1/2)}$ with an integer N and a sign $\varepsilon \in \{\pm 1\}$, as in Proposition 2.6.6. Let $n \geq 0$ be a non-negative integer. Then we have the following statements for any $W \in V(n)$.*

i) There exists a polynomial $P_W(X) \in \mathbb{C}[X]$ such that

$$\frac{Z(s, W)}{L(s, \pi)} = P_W(q^{-s})$$

We call P_W the zeta polynomial *of W.*

ii) The Atkin–Lehner involution $\pi(u_n)$ has the following effect on zeta polynomials,

$$P_{\pi(u_n)W}(X) = \varepsilon q^{(n-N)/2} X^{n-N} P_W(q^{-1}X^{-1}).$$

iii) The degree of $P_W \in \mathbb{C}[X]$ is at most $n - N$ (if $n - N < 0$ this means that $P_W = 0$).

iv) The θ and θ' operators have the following effect on zeta polynomials,

$$P_{\theta W}(X) = q^{3/2} X\, P_W(X), \qquad P_{\theta' W}(X) = q P_W(X).$$

Proof. i) follows from the three facts that a) $Z(s, W)$ is a power series in q^{-s}, b) $Z(s, W)/L(s, \pi) \in \mathbb{C}[q^s, q^{-s}]$, and c) $L(s, \pi) = 1/Q(q^{-s})$ for some $Q \in \mathbb{C}[X]$. ii) follows from a straightforward computation using the functional equation (2.61). iii) is a consequence of ii). iv) is immediate from Proposition 4.1.3. □

We will eventually prove that N as in Proposition 4.1.4 coincides with N_π, the minimal paramodular level. It follows from Proposition 4.1.4 that if $N_\pi < N$, then all the vectors in $V(N_\pi)$ are degenerate.

Klingen Vectors and θ'

One of the basic assertions of this work is the existence of certain paramodular vectors with prescribed zeta integrals. One approach to proving this assertion is to fix a non-negative integer $n \geq 0$ and consider the extension $\theta' : V \to V(n+1)$ of $\theta' : V(n) \to V(n+1)$; see (3.4). Since θ' takes values in $V(n+1)$, to prove the assertion it suffices to find $W \in V$ such that $Z(s, \theta' W)$ has the desired form. Via the next lemma, which generalizes one of the formulas from Proposition 4.1.3, we will use this idea in the proof of Theorem 4.4.1.

Lemma 4.1.5. *Let (π, V) be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4, F)$ *with trivial central character. Let $n \geq 0$ be a non-negative integer. Then*

$$Z(s, \theta' W) = q Z(s, W) \quad \text{for} \quad W \in V^{\mathrm{Kl}(\mathfrak{p}^n)}.$$

Proof. In this proof we use the abbreviation

$$z^J(x) = \begin{bmatrix} 1 & & & x \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Let $W \in V^{\mathrm{Kl}(\mathfrak{p}^n)}$. We have by (3.4), the decomposition from Lemma 3.3.1, the volume computation from Lemma 3.3.3 and the volume computation from the proof of Lemma 3.3.4,

$$\begin{aligned} \theta' W &= \frac{1}{\mathrm{vol}(\mathrm{K}(\mathfrak{p}^{n+1}) \cap \mathrm{K}(\mathfrak{p}^n))} \int\limits_{\mathrm{K}(\mathfrak{p}^{n+1})} \pi(g) W\,dg \\ &= \frac{\mathrm{vol}(\mathrm{Kl}(\mathfrak{p}^{n+1}))}{\mathrm{vol}(\mathrm{K}(\mathfrak{p}^{n+1}) \cap \mathrm{K}(\mathfrak{p}^n))} \Big(\sum_{u \in \mathfrak{o}/\mathfrak{p}^{n+1}} \pi(z^J(u\varpi^{-n-1}))W \\ &\qquad + \sum_{v \in \mathfrak{o}/\mathfrak{p}^n} \pi(t_{n+1} z^J(v\varpi^{-n}))W \Big) \\ &= q^{-n} \Big(\sum_{u \in \mathfrak{o}/\mathfrak{p}^{n+1}} \pi(z^J(u\varpi^{-n-1}))W \\ &\qquad + \sum_{v \in \mathfrak{o}/\mathfrak{p}^n} \pi(t_{n+1} z^J(v\varpi^{-n}))W \Big). \end{aligned} \tag{4.5}$$

Since $z^J(x)$ is in the center of the Jacobi group for any $x \in F$, we have $Z(s, \pi(z^J(u\varpi^{-n-1}))W) = Z(s, W)$. It follows that we are done if we can show that $Z(s, \pi(t_{n+1} z^J(v\varpi^{-n}))W) = 0$ for $v \in \mathfrak{o}$. Let $v \in \mathfrak{o}$ and set $W' = \pi(t_{n+1} z^J(v\varpi^{-n}))W$. Then W' is clearly invariant under

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & \mathfrak{o} & 1 & \\ & & & 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 & & & \\ & 1 & \mathfrak{p} & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Let $b \in \mathfrak{o}$. Then

$$\begin{aligned} &\begin{bmatrix} 1 & b & & \\ & 1 & & b \\ & & 1 & \\ & & & 1 \end{bmatrix} t_{n+1} \begin{bmatrix} 1 & & & v\varpi^{-n} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \\ &= t_{n+1} \begin{bmatrix} 1 & & & v\varpi^{-n} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & v b\varpi & \\ \varpi^{n+1} b & 1 & & v b\varpi \\ & & 1 & \\ & & -\varpi^{n+1} b & 1 \end{bmatrix}. \end{aligned}$$

This identity proves that W' is also invariant under

$$\begin{bmatrix} 1 & & \mathfrak{o} & \\ & 1 & & \mathfrak{o} \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

By Lemma 4.1.1,

$$Z(s,\pi(t_{n+1}z^J(v\varpi^{-n}))W) = \int_{F^\times} W(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} t_{n+1} \begin{bmatrix} 1 & & & v\varpi^{-n} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})|a|^{s-3/2}\, d^\times a.$$

Let $a \in F^\times$ and $x \in \mathfrak{p}^{-1}$. Then

$$\begin{aligned}
&\psi(c_1 x)W(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} t_{n+1} \begin{bmatrix} 1 & & & v\varpi^{-n} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})\\
&= W(\begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} t_{n+1} \begin{bmatrix} 1 & & & v\varpi^{-n} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})\\
&= W(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} t_{n+1} \begin{bmatrix} 1 & & & v\varpi^{-n} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & v x\varpi & \\ & & 1 & & \\ -\varpi^{n+1}x & & & 1 & -vx\varpi \\ & -\varpi^{n+1}x & & 1 \end{bmatrix})\\
&= W(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} t_{n+1} \begin{bmatrix} 1 & & & v\varpi^{-n} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}).
\end{aligned}$$

Since ψ has conductor $\mathfrak{o}$ and $c_1 \in \mathfrak{o}^\times$, this implies that

$$W(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} t_{n+1} \begin{bmatrix} 1 & & & v\varpi^{-n} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}) = 0,$$

so that $Z(s,\pi(t_{n+1}z^J(v\varpi^{-n}))W) = 0$. This completes the proof. □

4.2 Poles and P_3-Theory

Let π be a generic, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. To prove the η Principle in Section 4.3 it is necessary to relate certain irreducible subquotients of the P_3-filtration of V_{Z^J} to zeta integrals. We will explain this connection in this section. Since it is convenient in working with the P_3-filtration, throughout this section we will take $c_1 = -1$ and $c_2 = 1$ when working with the Whittaker model $\mathcal{W}(\pi, \psi_{c_1,c_2})$.

To begin, we will state the relationship between the P_3-filtration and $L(s,\pi)$. This theorem depends strongly on the computation of $L(s,\pi)$ due to Takloo-Bighash; see [Tak].

Theorem 4.2.1. *Let π be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character, let $V = \mathcal{W}(\pi, \psi_{-1,1})$, and let*

$$0 \subset V_2 \subset V_1 \subset V_0 = V_{Z^J}$$

be the chain of P_3-subspaces from Theorem 2.5.3. If $V_2 = V_1$, then $L(s,\pi) = 1$. Assume $V_2 \neq V_1$. Let

$$V_2 = U_1 \subsetneqq \cdots \subsetneqq U_M \subsetneqq U_{M+1} = V_1$$

be a filtration by P_3-subspaces such that U_{l+1}/U_l is irreducible for $1 \leq l \leq M$, and write

$$U_{l+1}/U_l = \tau_{\mathrm{GL}(1)}^{P_3}(\chi_l)$$

where χ_l is a character of $F^\times$, for $1 \leq l \leq M$. Then

$$L(s,\pi) = \prod_{l=1}^{M} L(s-3/2, \chi_l).$$

Proof. Thanks to [Tak], $L(s,\pi)$ has been computed for all π. Note that the statement of Theorem 5.1 of [Tak] has a misprint. In b) of that theorem "quotient" should be replaced by "subrepresentation"; statement c) can be omitted. Also, Theorem 4.1 of [Tak] does not cover $\sigma\mathrm{St}_{\mathrm{GSp}(4)}$ for non-trivial σ; however, $L(s, \sigma\mathrm{St}_{\mathrm{GSp}(4)}) = L(s, \sigma\nu^{3/2})$. Table A.6 lists the semi-simplifications of V_1/V_2 for all π. Using these two tables, it is easy to verify the claims of the theorem. □

In the remainder of this section, we will examine the relationship between the P_3-filtration and zeta integrals, rather than $L(s,\pi)$. This requires some notation. Let π be a generic, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. Let D be the degree of $L(s,\pi)$. More precisely, if $L(s,\pi) = 1$, let $D = 0$. If $L(s,\pi) \neq 1$, then write

$$L(s,\pi) = \frac{1}{(1-a_1q^{-s})^{n_1} \cdots (1-a_dq^{-s})^{n_d}}$$

where $a_1, \ldots, a_d \in \mathbb{C}^\times$ are distinct, and $n_1, \ldots, n_d$ are positive integers; in this case, let

$$D = n_1 + \cdots + n_d.$$

Using this terminology, Theorem 4.2.1 has the following immediate corollary:

Corollary 4.2.2. *Let the notation be as in Theorem 4.2.1. If* $V_1 = V_2$*, then let* $D_{\rm un} = 0$*; if* $V_1 \neq V_2$*, then let* $D_{\rm un}$ *be the number of quotients* U_{l+1}/U_l *for* $1 \leq l \leq M$ *such that* χ_l *is unramified. Then* $D = D_{\rm un}$.

Continue to let π be a generic, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character, and set $V = \mathcal{W}(\pi, \psi_{-1,1})$. Assume $D > 0$. Let

$$a_1 = r_1 e^{i\theta_1},\ \ldots,\ a_d = r_d e^{i\theta_d}$$

where $r_1, \ldots, r_d$ are positive real numbers, and $0 \leq \theta_1, \ldots, \theta_d < 2\pi$. Since the set of meromorphic functions $Z(s,W)$ for $W \in V$ is exactly the set of functions of the form $P(q^s, q^{-s})L(s,\pi)$ for $P(X,Y) \in \mathbb{C}[X,Y]$, it follows that the poles of the $Z(s,W)$ for $W \in V$ lie among the complex numbers

$$s_1 + \frac{2\pi n i}{\log q},\ \ldots,\ s_d + \frac{2\pi n i}{\log q},$$

where

$$s_1 = \frac{\log r_1 + i\theta_1}{\log q},\ \ldots,\ s_d = \frac{\log r_d + i\theta_d}{\log q}.$$

Note that $q^{-s_1} = a_1^{-1}, \ldots, q^{-s_d} = a_d^{-1}$. Fix $1 \leq j \leq d$. There exists $\varepsilon > 0$ such that $Z(s,W)$ is holomorphic in the punctured disc $0 < |s - s_j| < \varepsilon$. Moreover, if $W \in V$, then the Laurent expansion of $Z(s,W)$ has the form

$$Z(s,W) = \frac{\lambda^j_{n_j}(W)}{(s-s_j)^{n_j}} + \cdots + \frac{\lambda^j_1(W)}{(s-s_j)} + \lambda^j_0(W) + \lambda^j_{-1}(W)(s-s_j) + \cdots .$$

We will study the linear functionals

$$\lambda^j_i : V \longrightarrow \mathbb{C}, \qquad 1 \leq j \leq d,\ 1 \leq i \leq n_j$$

and relate them to the irreducible subquotients U_{l+1}/U_l from Theorem 4.2.1. The following lemma describes some of the basic properties of the λ^j_i.

Lemma 4.2.3. *Let* π *be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character, and set* $V = \mathcal{W}(\pi, \psi_{-1,1})$*. Assume* $D = D_{\rm un} > 0$*. The linear functionals* λ^j_i *for* $1 \leq j \leq d$ *and* $1 \leq i \leq n_j$ *are non-zero and trivial on* $V(Z^J)$*, and thus define non-zero linear functionals on* V_{Z^J}*. We have*

$$\lambda^j_i(\pi(\begin{bmatrix}1&-y&&\\&1&&\\&&1&y\\&&&1\end{bmatrix})W) = \psi(y)\lambda^j_i(W),$$

$$\lambda^j_i(\pi(\begin{bmatrix}1&&&\\&1&&\\&x&1&\\&&&1\end{bmatrix})W) = \lambda^j_i(W),$$

$$\lambda_i^j(\pi(\begin{bmatrix} u & & & \\ & u & & \\ & & 1 & \\ & & & 1 \end{bmatrix})W) = \lambda_i^j(W)$$

for $x, y \in F$, $u \in \mathfrak{o}^\times$, $W \in V$, $1 \le j \le d$ *and* $1 \le i \le n_j$. *Moreover,*

$$\lambda_{n_j}^j(\pi(\begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix})W) = q^{-1/2} a_j \lambda_{n_j}^j(W) \tag{4.6}$$

for $W \in V$ *and* $1 \le j \le d$, *and for* $1 \le j \le d$ *and* $1 \le i < n_j$ *there exist* $c_b^i \in \mathbb{C}^\times$, $i+1 \le b \le n_j$, *such that*

$$\lambda_i^j(\pi(\begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix})W) = q^{-1/2} a_j \lambda_i^j(W) + \sum_{b=i+1}^{n_j} c_b^i \lambda_b^j(W) \tag{4.7}$$

for $W \in V$.

Proof. To see that the λ_i^j are non-zero, fix $1 \le j \le d$ and $1 \le i \le n_j$, and let $W \in V$ be such that $Z(s, W) = L(s, \pi)$. Then $Z(s, W)$ has a pole of order n_j at s_j. There exists $W' \in V$ such that $Z(s, W') = (1 - a_j q^{-s})^{n_j - i} Z(s, W)$. The function $Z(s, W')$ has a pole of order i at s_j. This implies that $\lambda_i^j(W') \ne 0$. Next, a computation using the formula for $Z(s, W)$ proves that each λ_i^j is trivial on $V(Z^J)$. Similar computations prove the remaining claims of the lemma. Note in particular that

$$Z(s, \pi(\begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix})W) = q^{-1/2} q^s Z(s, W)$$

for $W \in V$, and that the Laurent expansion of the holomorphic function $q^{-1/2} q^{-s}$ at s_j is

$$q^{-1/2} q^s = \sum_{t=0}^{\infty} q^{-1/2} a_j \frac{(\log q)^t}{t!} (s - s_j)^t. \quad \square$$

We can prove the main result of this section.

Theorem 4.2.4. *Let* π *be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4, F)$ *with trivial central character, let* $V = \mathcal{W}(\pi, \psi_{-1,1})$, *and let*

$$0 \subset V_2 \subset V_1 \subset V_0 = V_{Z^J}$$

be the chain of P_3-subspaces from Theorem 2.5.3. Assume $D = D_{un} > 0$. Let

$$V_2 = U_1 \subsetneqq \cdots \subsetneqq U_M \subsetneqq U_{M+1} = V_1$$

be a filtration by P_3-subspaces such that U_{l+1}/U_l is irreducible for $1 \le l \le M$, and write

$$U_{l+1}/U_l \cong \tau_{\mathrm{GL}(1)}^{P_3}(\chi_l)$$

where χ_l is a character of $F^\times$, for $1 \le l \le M$. Then for $1 \le j \le d$ and $1 \le i \le n_j$ we have $\lambda_i^j(V_1) \ne 0$ and $\lambda_i^j(V_2) = 0$. If l is the unique integer between 1 and M such that $\lambda_i^j(U_l) = 0$ and $\lambda_i^j(U_{l+1}) \ne 0$, then χ_l is unramified,

$$a_j = q^{3/2}\chi_l(\varpi)$$

and the linear functional $\tau_{\mathrm{GL}(1)}^{P_3}(\chi_l) \cong U_{l+1}/U_l \to \mathbb{C}$ induced by λ_i^j is a non-zero multiple of the functional that sends $f \in \tau_{\mathrm{GL}(1)}^{P_3}(\chi_l)$ to

$$\int\limits_F f(\begin{bmatrix} 1 & & \\ x & 1 & \\ & & 1 \end{bmatrix})\, dx.$$

Moreover, the map

$$\{\lambda_i^j : 1 \le j \le d, 1 \le i \le n_j\} \xrightarrow{\sim} \{U_{l+1}/U_l : \chi_l \text{ is unramified}\}$$

sending λ_i^j to U_{l+1}/U_l is a bijection.

Proof. To prove the first assertion of the theorem, let $1 \le j \le d$. By the proof of Proposition 2.6.4, $Z(s, W)$ is holomorphic for $W \in V_2$. Hence, $\lambda_1^j(V_2) = \cdots = \lambda_{n_j}^j(V_2) = 0$. Let $1 \le i \le n_j$. By Lemma 4.2.3, we have $\lambda_i^j \ne 0$. Assume $\lambda_i^j(V_1) = 0$. Then λ_i^j induces a non-zero functional $V_0/V_1 \to \mathbb{C}$. Recalling that every irreducible P_3-subquotient of V_0/V_1 is of the form $\tau_{\mathrm{GL}(2)}^{P_3}(\rho)$ where ρ is an irreducible, admissible representation of $\mathrm{GL}(2, F)$, it follows that there exists such a ρ and a non-zero functional $\lambda : \tau_{\mathrm{GL}(2)}^{P_3}(\rho) \to \mathbb{C}$ such that

$$\lambda(\begin{bmatrix} 1 & & \\ x & 1 & \\ & & 1 \end{bmatrix} f) = \lambda(f) \quad \text{and} \quad \lambda(\begin{bmatrix} 1 & & \\ & 1 & y \\ & & 1 \end{bmatrix} f) = \psi(y)\lambda(f)$$

for $x, y \in F$ and $f \in \tau_{\mathrm{GL}(2)}^{P_3}(\rho)$. However, by definition

$$\begin{bmatrix} 1 & & \\ & 1 & y \\ & & 1 \end{bmatrix} f = f$$

for $y \in F$ and $f \in \tau_{\mathrm{GL}(2)}^{P_3}(\rho)$, so that $\lambda(f) = \psi(y)\lambda(f)$ for $f \in \tau_{\mathrm{GL}(2)}^{P_3}(\rho)$ and $y \in F$. This contradiction implies that $\lambda_i^j(V_1) \ne 0$, as desired.

Now fix $1 \le j \le d$. For $1 \le i \le n_j$, let $1 \le l_i \le M$ be such that $\lambda_i^j(U_{l_i}) = 0$, but $\lambda_i^j(U_{l_i+1}) \neq 0$. We claim that

$$l_1 < \cdots < l_{n_j},$$

that $\chi_{l_1} = \cdots = \chi_{l_{n_j}}$, and this character is unramified, and

$$a_j = q^{3/2}\chi_{l_1}(\varpi) = \cdots = q^{3/2}\chi_{l_{n_j}}(\varpi).$$

To prove this claim we begin with some initial observations. Let $1 \le i \le n_j$. Then λ_i^j induces a non-zero linear functional $\lambda : \tau_{\mathrm{GL}(1)}^{P_3}(\chi_{l_i}) \cong U_{l_i+1}/U_{l_i} \to \mathbb{C}$ via the definition $\lambda(W + U_{l_i}) = \lambda_i^j(W)$. This functional is of the type considered in Lemma 2.5.5. Hence, by Lemma 2.5.5, we have

$$\lambda(\begin{bmatrix} a & & \\ & 1 & \\ & & 1 \end{bmatrix} f) = |a|^{-1}\chi_{l_i}(a)\lambda(f)$$

for $f \in \tau_{\mathrm{GL}(1)}^{P_3}(\chi_{l_i})$ and $a \in F^\times$. This means that

$$\lambda_i^j(\pi(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix})W) = |a|^{-1}\chi_{l_i}(a)\lambda_i^j(W) \tag{4.8}$$

for $W \in U_{l_i+1}$ and $a \in F^\times$. On the other hand, by Lemma 4.2.3, we have

$$i < n_j \implies \lambda_i^j(\pi(\begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix})W) = q^{-1/2}a_j\lambda_i^j(W) + \sum_{b=i+1}^{n_j} c_b^i \lambda_b^j(W), \tag{4.9}$$

$$i = n_j \implies \lambda_{n_j}^j(\pi(\begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix})W) = q^{-1/2}a_j\lambda_{n_j}^j(W), \tag{4.10}$$

for $W \in V_{Z^J}$. By (4.8), (4.9) and (4.10),

$$i < n_j \implies (q^{-1/2}a_j - q\chi_{l_i}(\varpi))\lambda_i^j(W) = -\sum_{b=i+1}^{n_j} c_b^i \lambda_b^j(W), \tag{4.11}$$

$$i = n_j \implies (q^{-1/2}a_j - q\chi_{l_{n_j}}(\varpi))\lambda_{n_j}^j(W) = 0, \tag{4.12}$$

for $W \in U_{l_i+1}$. By Lemma 4.2.3, we also have

$$\lambda_i^j(\pi(\begin{bmatrix} u & & & \\ & u & & \\ & & 1 & \\ & & & 1 \end{bmatrix})W) = \lambda_i^j(W)$$

for $u \in \mathfrak{o}^\times$ and $W \in V_{Z^J}$. By (4.8) we thus obtain

$$(1-\chi_{l_i}(u))\lambda_i^j(W) = 0$$

for $W \in U_{l_i+1}$. Since $\lambda_i^j(U_{l_i+1}) \neq 0$, we get that χ_{l_i} is unramified for $1 \leq i \leq n_j$. Having made these observations, we will now prove our claim by induction, beginning at $k = n_j$, and going down to $k = 1$. For $1 \leq k \leq n_j$, let (P_k) be the following statement: $a_j = q^{3/2}\chi_{l_k}(\varpi) = \cdots = q^{3/2}\chi_{l_{n_j}}(\varpi)$, and if $k \leq i < n_j$, then $l_i < l_{i+1}$. We need to verify (P_{n_j}) and prove the implication $(P_k) \implies (P_{k-1})$ for $1 < k \leq n_j$. To verify (P_{n_j}) it suffices to check that $a_j = q^{3/2}\chi_{l_{n_j}}(\varpi)$. This follows from (4.12) since $\lambda_{n_j}^j(U_{l_{n_j}+1}) \neq 0$. Assume that (P_k) holds. To prove (P_{k-1}) it suffices to prove that $l_{k-1} < l_k$ and $a_j = q^{3/2}\chi_{l_{k-1}}(\varpi)$. Suppose $l_k < l_{k-1}$. By (4.9) with $i = k-1$ we have

$$\lambda_{k-1}^j(\pi(\begin{bmatrix} \varpi &&& \\ & \varpi && \\ && 1 & \\ &&& 1 \end{bmatrix})W) = q^{-1/2}a_j\lambda_{k-1}^j(W) + \sum_{b=k}^{n_j} c_b^{k-1}\lambda_b^j(W) \tag{4.13}$$

for $W \in V_{Z^J}$. Let $W \in U_{l_k+1}$. Since $l_k + 1 \leq l_{k-1}$ we have $W \in U_{l_{k-1}}$. Therefore, by definition,

$$\lambda_{k-1}^j(\pi(\begin{bmatrix} \varpi &&& \\ & \varpi && \\ && 1 & \\ &&& 1 \end{bmatrix})W) = \lambda_{k-1}^j(W) = 0.$$

Also, since $l_k + 1 \leq l_b$ for $k+1 \leq b \leq n_j$, we have $W \in U_{l_b}$ for $k+1 \leq b \leq n_j$. Therefore, $\lambda_b^j(W) = 0$ for $k+1 \leq b \leq n_j$. By (4.13), we get $c_k^{k-1}\lambda_k^j(W) = 0$, so that $\lambda_k^j(W) = 0$ since $c_k^{k-1} \neq 0$. This contradicts $\lambda_k^j(U_{l_k+1}) \neq 0$. Assume $l_k = l_{k-1}$. By Lemma 2.5.5, there exists $c \in \mathbb{C}^\times$ such that

$$\lambda_{k-1}^j(W) = c\lambda_k^j(W)$$

for $W \in U_{l_{k-1}+1} = U_{l_k+1}$. By (4.13), noting again that $l_k + 1 \leq l_b$ for $k+1 \leq b \leq n_j$, we have

$$\lambda_{k-1}^j(\pi(\begin{bmatrix} \varpi &&& \\ & \varpi && \\ && 1 & \\ &&& 1 \end{bmatrix})W) = q^{-1/2}a_j\lambda_{k-1}^j(W) + c_k^{k-1}\lambda_k^j(W)$$

for $W \in U_{l_{k-1}+1} = U_{l_k+1}$. A substitution produces

$$c\lambda_k^j(\pi(\begin{bmatrix} \varpi &&& \\ & \varpi && \\ && 1 & \\ &&& 1 \end{bmatrix})W) = cq^{-1/2}a_j\lambda_k^j(W) + c_k^{k-1}\lambda_k^j(W)$$

for $W \in U_{l_{k-1}+1} = U_{l_k+1}$. By (4.8), this gives

$$cq\chi_{l_k}(\varpi)\lambda_k^j(W) = cq^{-1/2}a_j\lambda_k^j(W) + c_k^{k-1}\lambda_k^j(W)$$

for $W \in U_{l_{k-1}+1} = U_{l_k+1}$. Since (P_k) holds, we have $q\chi_{l_k}(\varpi) = q^{-1/2}a_j$, so that $c_k^{k-1}\lambda_k^j(W) = 0$ for $W \in U_{l_k+1}$. Again, this is a contradiction. We conclude that $l_{k-1} < l_k$. By (4.11) with $i = k-1$, we have

$$(q^{-1/2}a_j - q\chi_{l_{k-1}}(\varpi))\lambda_{k-1}^j(W) = -\sum_{b=k}^{n_j} c_b^{k-1}\lambda_b^j(W)$$

for $W \in U_{l_{k-1}+1}$. Since $l_{k-1}+1 \leq l_k$ and since $l_k < \cdots < l_{n_j}$, we have $\lambda_b^j(W) = 0$ for $k \leq b \leq n_j$. Therefore,

$$(q^{-1/2}a_j - q\chi_{l_{k-1}}(\varpi))\lambda_{k-1}^j(W) = 0$$

for $W \in U_{l_{k-1}+1}$. Since λ_{k-1}^j is non-zero on $U_{l_{k-1}+1}$ by definition, we obtain $q^{-1/2}a_j = q\chi_{l_{k-1}}(\varpi)$, as desired. The proof of our claim is complete.

The claim implies all the remaining assertions of the theorem except that the map from the statement of the theorem is surjective. This follows from the fact that the map is injective and that the two sets have the same cardinality $D = D_{\mathrm{un}}$ by Corollary 4.2.2. □

We note that Theorem 4.2.4 and Theorem 4.2.1 are consistent. That is, by Theorem 4.2.4, if $D = D_{\mathrm{un}} \neq 0$, then

$$\begin{aligned} L(s,\pi) &= \prod_{j=1}^{d} \frac{1}{(1-a_jq^{-s})^{n_j}} \\ &= \prod_{\chi_l \text{ unramified}} \frac{1}{(1-q^{3/2}\chi_l(\varpi)q^{-s})} \\ &= \prod_{\chi_l \text{ unramified}} \frac{1}{(1-\chi_l(\varpi)q^{-(s-3/2)})} \\ &= \prod_{\chi_l \text{ unramified}} L(s-3/2,\chi_l) \\ &= \prod_{l=1}^{M} L(s-3/2,\chi_l). \end{aligned}$$

This is also asserted by Theorem 4.2.1.

4.3 The η Principle

Let π be a generic, irreducible, and admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character, and let $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. Let $n \geq 0$ be an integer, and let $W \in V(n)$. We saw in Proposition 4.1.3 that if $n \geq 2$ and $W = \eta W_1$ for some $W_1 \in V(n-2)$, then W is degenerate, i.e., $Z(s,W) = 0$. In this section we will prove the opposite implication, which we call the η *Principle*: If $W \in V(n)$ is non-zero and degenerate, then $n \geq 2$, and there exists $W_1 \in V(n-2)$ such that $W = \eta W_1$.

The proof of the η Principle consists of a number of results which reduce the problem, and the final statement and argument appear in Theorem 4.3.7. Suppose that π is as in the last paragraph, and $W \in V(n)$ is non-zero and degenerate. Then we need to prove two things: $n \geq 2$, and $W = \eta W_1$ for some $W_1 \in V(n-2)$. As will be noted again in the proof of Theorem 4.3.7, Corollary 7.1.5 immediately tells us that $n = 0$ is impossible, so that $n \geq 1$. That is, a non-zero spherical vector in a spherical representation is always non-degenerate. Thus, we are reduced to proving that $n = 1$ is impossible, and that $W = \eta W_1$ for some $W_1 \in V(n-2)$. We will use P_3-theory to prove these assertions. Throughout this section we will write

$$W' = SW - qW,$$

where

$$SW = \sum_{y \in \mathfrak{p}^{-1}/\mathfrak{o}} \pi\left(\begin{bmatrix} 1 & y & & \\ & 1 & & \\ & & 1 & -y \\ & & & 1 \end{bmatrix}\right)W$$

(see (3.10)). Our first lemma uses some of the algebra from Sect. 3.4 to relate what we want to prove to a P_3 condition about W'. We will say more about the strategy of the proof of the η Principle after the proof of this lemma.

Lemma 4.3.1. *Let (π, V) be a smooth representation of* $\mathrm{GSp}(4,F)$ *for which the center of* $\mathrm{GSp}(4,F)$ *acts trivially. Let $n \geq 0$ be a non-negative integer, and let $W \in V(n)$.*

i) Assume that $n \geq 2$. Then the following are equivalent:
  a) There exists $W_1 \in V(n-2)$ such that $\eta W_1 = W$;
  b) $W' = 0$;
  c) $W' \in V(Z^J)$.

ii) Assume that $n = 1$ and that the subspace of vectors of V fixed by $\mathrm{Sp}(4,F)$ *is trivial. If $W' \in V(Z^J)$, then $W = 0$, or $\theta' W \neq 0$ and there exists $W_2 \in V(0)$, $W_2 \neq 0$, such that $\theta' W = \eta W_2$.*

Proof. i) Assume that $n \geq 2$.
a) $\iff$ b) This is Lemma 3.2.4.
b) $\implies$ c) This is trivial.
c) $\implies$ a) Since $W' \in V(Z^J)$, there exists an integer $k \geq 0$ such that

$$\int\limits_{\mathfrak{p}^{-(n+k)}/\mathfrak{p}^{-n}} \pi\left(\begin{bmatrix} 1 & & & y \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\right)(SW - qW)\,dy = 0.$$

Set

$$W_0 = \int\limits_{\mathfrak{p}^{-(n+k)}/\mathfrak{p}^{-n}} \pi(\begin{bmatrix} 1 & & & y \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})W\,dy;$$

then $SW_0 = qW_0$. By Lemma 3.4.1,

$$W_0 = \theta'^k W + W_1 + W_2$$

where $W_1 \in V(n+k)$, $W_2 \in V(n+k+1)$ and there exist $W_1' \in V(n+k-2)$ and $W_2' \in V(n+k-1)$ such that $W_1 = \eta W_1'$ and $W_2 = \eta W_2'$. By Lemma 3.2.4 each $W_i = \eta W_i'$, $1 \le i \le 2$, is invariant under

$$\begin{bmatrix} 1 & \mathfrak{p}^{-1} & & \\ & 1 & & \\ & & 1 & \mathfrak{p}^{-1} \\ & & & 1 \end{bmatrix}.$$

Hence,

$$\begin{aligned} SW_0 &= S\theta'^k W + SW_1 + SW_2 \\ &= S\theta'^k W + qW_1 + qW_2. \end{aligned}$$

On the other hand, $SW_0 = qW_0 = q\theta'^k W + qW_1 + qW_2$. We obtain

$$S\theta'^k W = q\theta'^k W.$$

The claim a) now follows from Corollary 3.2.8.

ii) Assume that $n = 1$, the subspace of vectors of V fixed by $\mathrm{Sp}(4, F)$ is trivial, and $W' \in V(Z^J)$. The same argument as in the proof of the implication c) $\implies$ a) from above proves that $S\theta'^k W = q\theta'^k W$ for some non-negative integer $k \ge 0$. The conclusion follows from Corollary 3.2.8. □

As we mentioned, Lemma 4.3.1 connects the assertions of the η Principle to P_3-theory. To explain the next steps in the proof of the η Principle, assume π is a generic, irreducible, admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character, and let $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. Let $p : V \to V_{Z^J} = V/V(Z^J)$ be the projection. Then the P_3 condition from Lemma 4.3.1 is $p(W') = 0$. Assuming that $W \in V(n)$, $n \ge 1$, is degenerate, we will prove below that indeed $p(W') = 0$. The proof will have two stages. Let

$$0 \subset V_2 \subset V_1 \subset V_{Z^J} = V/V(Z^J)$$

be the P_3-filtration from Theorem 2.5.3. In the first stage we will prove $p(W') \in V_2$; in the second stage, starting from $p(W') \in V_2$, we will prove $p(W') = 0$. Note that, by Theorem 2.5.3, π is supercuspidal if and only if $V_2 = V_{Z^J}$. Thus, if π is supercuspidal, then the first stage is not required. The proof that $p(W') \in V_2$ proceeds by showing that $p(W')$ cannot have non-zero image in any of irreducible subquotients of V_{Z^J}/V_2. For this we will use the connection between the P_3-filtration of V_{Z^J} and poles of zeta integrals which is explained in Section 4.2.

Lemma 4.3.2. *Let π be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character, and let $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. Let $n \geq 1$ be an integer, and let $W \in V(n)$ be degenerate. Set $W' = SW - qW$. Then $p(W') \in V_2$.*

Proof. Let $f' = p(W')$. We need to prove $f' \in V_2$. The identities

$$\begin{bmatrix} ad-bc&&&\\ &a&b&\\ &c&d&\\ &&&1 \end{bmatrix} \begin{bmatrix} 1&z&&\\ &1&&\\ &&1&-z\\ &&&1 \end{bmatrix} = \begin{bmatrix} 1&zd&&\\ &1&&\\ &&1&-zd\\ &&&1 \end{bmatrix} \begin{bmatrix} ad-bc&-bcz&-bdz&bdz^2\\ &a&b&-bz\\ &c&d&\\ &&&1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1&x&y&\\ &1&&y\\ &&1&-x\\ &&&1 \end{bmatrix} \begin{bmatrix} 1&z&&\\ &1&&\\ &&1&-z\\ &&&1 \end{bmatrix} = \begin{bmatrix} 1&z&&\\ &1&&\\ &&1&-z\\ &&&1 \end{bmatrix} \begin{bmatrix} 1&x&y&-2yz\\ &1&&y\\ &&1&-x\\ &&&1 \end{bmatrix},$$

along with the invariance of W under $\mathrm{K}(\mathfrak{p}^n)$, imply that W' is invariant under the elements of $\mathrm{GSp}(4,F)$ of the form

$$\begin{bmatrix} u&x&y&\\ &a&b&x\\ &c&d&-y\\ &&&1 \end{bmatrix}, \quad \text{where} \quad \begin{bmatrix} a&b\\ c&d \end{bmatrix} \in \mathrm{GL}(2,\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o}&\mathfrak{p}\\ \mathfrak{o}&\mathfrak{o} \end{bmatrix}, \quad x,y \in \mathfrak{o}, \quad u \in \mathfrak{o}^\times.$$

Also,

$$\sum_{y \in \mathfrak{p}^{-1}/\mathfrak{o}} \pi(\begin{bmatrix} 1&y&&\\ &1&&\\ &&1&-y\\ &&&1 \end{bmatrix})W' = 0.$$

It follows that f' is invariant under the subgroup of P_3 consisting of the elements

$$\begin{bmatrix} a&b&x\\ c&d&y\\ &&1 \end{bmatrix}, \quad \text{where} \quad \begin{bmatrix} a&b\\ c&d \end{bmatrix} \in \mathrm{GL}(2,\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o}&\mathfrak{p}\\ \mathfrak{o}&\mathfrak{o} \end{bmatrix}, \quad x,y \in \mathfrak{o},$$

and

$$\sum_{y \in \mathfrak{p}^{-1}/\mathfrak{o}} \begin{bmatrix} 1&&\\ &1&y\\ &&1 \end{bmatrix} f' = 0.$$

Next, consider the chain of P_3-subspaces

$$0 \subset V_2 \subset V_1 \subset V_{Z^J}$$

from Theorem 2.5.3. By this theorem, this chain has the property that V_1/V_2 is of finite length with each irreducible subquotient of the form $\tau_{\mathrm{GL}(1)}^{P_3}(\chi)$ for χ a character of $F^\times$, and V_{Z^J}/V_1 is of finite length, with each irreducible subquotient of the form $\tau_{\mathrm{GL}(2)}^{P_3}(\rho)$ for ρ an irreducible, admissible representation of $\mathrm{GL}(2,F)$. By Lemma 4.3.3 below, and the second property of f' noted above, we see that $f' \in V_1$. Suppose that $f' \notin V_2$. Let

$$V_2 = U_1 \subsetneqq \cdots \subsetneqq U_M \subsetneqq U_{M+1} = V_1$$

be a filtration of P_3-subspaces such that U_l/U_{l+1} is irreducible for $1 \le l \le M$. Write

$$U_{l+1}/U_l \cong \tau_{\mathrm{GL}(1)}^{P_3}(\chi_l),$$

where χ_l is a character of $F^\times$. Let $1 \le l \le M$ be such that $f' \in U_{l+1}$ but $f' \notin U_l$, so that $f' + U_l$ is a non-zero element of $U_{l+1}/U_l \cong \tau_{\mathrm{GL}(1)}^{P_3}(\chi_l)$. Then by Lemma 4.3.4 below and two properties of f' noted above, χ_l is unramified. A computation shows that since W is degenerate, so is W'. Let λ_i^j correspond to U_{l+1}/U_l via the bijection from Theorem 4.2.4. Using the definition of λ_i^j, it follows immediately that $\lambda_i^j(W') = 0$. By Theorem 4.2.4, it follows that $f' + U_l$ is in the kernel of the linear functional on $\tau_{\mathrm{GL}(1)}^{P_3}(\chi_l)$ that sends an element $f \in \tau_{\mathrm{GL}(1)}^{P_3}(\chi_l)$ to

$$\int_F f(\begin{bmatrix} 1 & & \\ x & 1 & \\ & & 1 \end{bmatrix})\, dx.$$

By Lemma 4.3.4 below, this implies that $f' \in U_{l+1}$, a contradiction. □

The following two lemmas are used in the proof of the above Lemma 4.3.2.

Lemma 4.3.3. *Let ρ be an irreducible admissible representation of* $\mathrm{GL}(2,F)$. *Let $f' \in \tau_{\mathrm{GL}(2)}^{P_3}(\rho)$ be such that*

$$\sum_{y \in \mathfrak{p}^{-1}/\mathfrak{o}} \begin{bmatrix} 1 & & \\ & 1 & y \\ & & 1 \end{bmatrix} f' = 0.$$

Then $f' = 0$.

Proof. This follows from the fact that the elements

$$\begin{bmatrix} 1 & & \\ & 1 & y \\ & & 1 \end{bmatrix}$$

of P_3 for $y \in F$ act trivially. □

Lemma 4.3.4. *Let χ be a character of $F^\times$. Suppose that $f' \in \tau_{\mathrm{GL}(1)}^{P_3}(\chi)$ is invariant under the elements*

$$\begin{bmatrix} a & b & x \\ c & d & y \\ & & 1 \end{bmatrix}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2,\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} \end{bmatrix}, \quad x, y \in \mathfrak{o},$$

and that

$$\sum_{y \in \mathfrak{p}^{-1}/\mathfrak{o}} \begin{bmatrix} 1 & & \\ & 1 & y \\ & & 1 \end{bmatrix} f' = 0.$$

We have $f' = 0$ if and only if $f(1) = 0$. If χ is ramified or f' is in the kernel of the linear functional on $\tau_{\mathrm{GL}(1)}^{P_3}(\chi)$ that sends f to

$$\int\limits_F f(\begin{bmatrix} 1 & & \\ x & 1 & \\ & & 1 \end{bmatrix})\, dx,$$

then $f' = 0$.

Proof. We have

$$\begin{bmatrix} g_1 & g_2 & z \\ g_3 & g_4 & x \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y_1 \\ & 1 & y_2 \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & g_1y_1 + g_2y_2 \\ & 1 & g_3y_1 + g_4y_2 \\ & & 1 \end{bmatrix} \begin{bmatrix} g_1 & g_2 & z \\ g_3 & g_4 & x \\ & & 1 \end{bmatrix}$$

for $\left[\begin{smallmatrix} g_1 & g_2 \\ g_3 & g_4 \end{smallmatrix}\right] \in \mathrm{GL}(2,F)$ and $z, x, y_1, y_2 \in F$. Hence,

$$(\begin{bmatrix} 1 & & y_1 \\ & 1 & y_2 \\ & & 1 \end{bmatrix} f')(\begin{bmatrix} g_1 & g_2 & z \\ g_3 & g_4 & x \\ & & 1 \end{bmatrix}) = \psi(g_3y_1 + g_4y_2) f'(\begin{bmatrix} g_1 & g_2 & z \\ g_3 & g_4 & x \\ & & 1 \end{bmatrix}).$$

Since f' is invariant under

$$\begin{bmatrix} 1 & & \mathfrak{o} \\ & 1 & \mathfrak{o} \\ & & 1 \end{bmatrix}$$

we have

$$f'(\begin{bmatrix} g_1 & g_2 & z \\ g_3 & g_4 & x \\ & & 1 \end{bmatrix}) = \psi(g_3y_1 + g_4y_2) f'(\begin{bmatrix} g_1 & g_2 & z \\ g_3 & g_4 & x \\ & & 1 \end{bmatrix})$$

for $y_1, y_2 \in \mathfrak{o}$, $\left[\begin{smallmatrix} g_1 & g_2 \\ g_3 & g_4 \end{smallmatrix}\right] \in \mathrm{GL}(2,F)$ and $z, x \in F$. Therefore,

$$g_3 \notin \mathfrak{o} \text{ or } g_4 \notin \mathfrak{o} \implies f'(\begin{bmatrix} g_1 & g_2 & z \\ g_3 & g_4 & x \\ & & 1 \end{bmatrix}) = 0.$$

Evaluating

$$\sum_{y \in \mathfrak{p}^{-1}/\mathfrak{o}} \begin{bmatrix} 1 & & \\ & 1 & y \\ & & 1 \end{bmatrix} f' = 0$$

at an arbitrary point $\left[\begin{smallmatrix} g_1 & g_2 & z \\ g_3 & g_4 & x \\ & & 1 \end{smallmatrix}\right]$ of P_3 we also find that

$$\Big(\sum_{y \in \mathfrak{p}^{-1}/\mathfrak{o}} \psi(g_4 y)\Big) f'\Big(\begin{bmatrix} g_1 & g_2 & z \\ g_3 & g_4 & x \\ & & 1 \end{bmatrix}\Big) = 0.$$

This implies that

$$g_4 \in \mathfrak{p} \implies f'\Big(\begin{bmatrix} g_1 & g_2 & z \\ g_3 & g_4 & x \\ & & 1 \end{bmatrix}\Big) = 0.$$

Combining together what we know, we have

$$f'\Big(\begin{bmatrix} g_1 & g_2 & z \\ g_3 & g_4 & x \\ & & 1 \end{bmatrix}\Big) \neq 0 \implies g_3 \in \mathfrak{o} \text{ and } g_4 \in \mathfrak{o}^\times.$$

Now suppose $p \in P_3$ is such that $f'(p) \neq 0$. We can write

$$p = \begin{bmatrix} 1 & & z \\ & 1 & x \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & y & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} s & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & t & \\ & & 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & \\ k_3 & k_4 & \\ & & 1 \end{bmatrix}$$

with $x, y, z \in F$, $s, t \in F^\times$ and $\left[\begin{smallmatrix} k_1 & k_2 \\ k_3 & k_4 \end{smallmatrix}\right] \in \mathrm{GL}(2, \mathfrak{o})$. We get

$$f'(p) = \psi(x)\chi(s) f'\Big(\begin{bmatrix} k_1 & k_2 & \\ tk_3 & tk_4 & \\ & & 1 \end{bmatrix}\Big).$$

As $f'(p) \neq 0$ we must have $tk_3 \in \mathfrak{o}$ and $tk_4 \in \mathfrak{o}^\times$. Moreover,

$$\begin{aligned} f'(p) &= \psi(x)\chi(s) f'\Big(\begin{bmatrix} k_1 & k_2 & \\ tk_3 & tk_4 & \\ & & 1 \end{bmatrix}\Big) \\ &= \psi(x)\chi(s) f'\Big(\begin{bmatrix} 1 & -(tk_4)^{-1}k_2 & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & \\ tk_3 & tk_4 & \\ & & 1 \end{bmatrix}\Big) \\ &= \psi(x)\chi(s) f'\Big(\begin{bmatrix} k_4^{-1}(k_1k_4 - k_2k_3) & & \\ tk_3 & tk_4 & \\ & & 1 \end{bmatrix}\Big) \\ &= \psi(x)\chi(sk_4^{-1}) f'\Big(\begin{bmatrix} k_1k_4 - k_2k_3 & & \\ tk_3 & tk_4 & \\ & & 1 \end{bmatrix}\Big). \end{aligned}$$

The element $\left[\begin{smallmatrix} k_1k_4-k_2k_3 & \\ tk_3 & tk_4 \end{smallmatrix}\right]$ is contained in $\mathrm{GL}(2,\mathfrak{o}) \cap \left[\begin{smallmatrix} \mathfrak{o} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} \end{smallmatrix}\right]$. Hence,

$$f'(p) = \psi(x)\chi(sk_4^{-1})f'(\begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix}).$$

This implies that $f' = 0$ if and only if $f'(1) = 0$.

Assume χ is ramified. Let $u \in \mathfrak{o}^\times$ with $\chi(u) \neq 1$. Then

$$f'(1) = f'(\begin{bmatrix}u&&\\&1&\\&&1\end{bmatrix}) = \chi(u)f'(1)$$

implies $f'(1) = 0$, so that $f' = 0$.

Finally, assume that

$$\int\limits_F f'(\begin{bmatrix}1&&\\x&1&\\&&1\end{bmatrix})\,dx = 0.$$

Using the useful identity (2.8) and the invariance properties of f', we have

$$\begin{aligned} 0 &= \int\limits_F f'(\begin{bmatrix}1&&\\x&1&\\&&1\end{bmatrix})\,dx \\ &= \int\limits_{\mathfrak{o}} f'(\begin{bmatrix}1&&\\x&1&\\&&1\end{bmatrix})\,dx + \int\limits_{F\setminus\mathfrak{o}} f'(\begin{bmatrix}1&&\\x&1&\\&&1\end{bmatrix})\,dx \\ &= f'(1) + \int\limits_{F\setminus\mathfrak{o}} f'(\begin{bmatrix}-x^{-1}&&\\&-x&\\&&1\end{bmatrix}\begin{bmatrix}&1&\\-1&&\\&&1\end{bmatrix})\,dx. \end{aligned} \tag{4.14}$$

Let $x \in F \setminus \mathfrak{o}$ and $y \in \mathfrak{o}$. Then

$$\begin{aligned} f'(\begin{bmatrix}-x^{-1}&&\\&-x&\\&&1\end{bmatrix}\begin{bmatrix}&1&\\-1&&\\&&1\end{bmatrix}) &= f'(\begin{bmatrix}-x^{-1}&&\\&-x&\\&&1\end{bmatrix}\begin{bmatrix}&1&\\-1&&\\&&1\end{bmatrix}\begin{bmatrix}1&&y\\&1&\\&&1\end{bmatrix}) \\ &= f'(\begin{bmatrix}-x^{-1}&&\\&-x&\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&y\\&&1\end{bmatrix}\begin{bmatrix}&1&\\-1&&\\&&1\end{bmatrix}) \\ &= f'(\begin{bmatrix}1&&\\&1&-xy\\&&1\end{bmatrix}\begin{bmatrix}-x^{-1}&&\\&-x&\\&&1\end{bmatrix}\begin{bmatrix}&1&\\-1&&\\&&1\end{bmatrix}) \\ &= \psi(-xy)f'(\begin{bmatrix}-x^{-1}&&\\&-x&\\&&1\end{bmatrix}\begin{bmatrix}&1&\\-1&&\\&&1\end{bmatrix}). \end{aligned}$$

Since $x \notin \mathfrak{o}$ and this identity holds for all $y \in \mathfrak{o}$, we obtain

$$f'(\begin{bmatrix} -x^{-1} & & \\ & -x & \\ & & 1 \end{bmatrix} \begin{bmatrix} & 1 & \\ -1 & & \\ & & 1 \end{bmatrix}) = 0.$$

By (4.14), we get $f'(1) = 0$. Hence, $f' = 0$. □

Now we will carry out the second stage of the proof that $p(W') = 0$. The basis for this stage is the following P_3-theory theorem. This theorem is about the obstruction to the existence of a naive "Kirillov model" for generic, irreducible, admissible representations of $\mathrm{GSp}(4, F)$ with trivial central character. In analogy to the case of $\mathrm{GL}(n)$ as presented in 5.18–5.20 of [BZ], one might wonder if the map that sends an element of $\mathcal{W}(\pi, \psi_{c_1,c_2})$ to its restriction to the Klingen parabolic subgroup Q is injective. A bit of thought reveals, however, that this is naive: elements of $V(Z^J)$ have trivial restrictions to Q. The following theorem asserts that, at least when π is supercuspidal, then the elements of $V(Z^J)$ are exactly the elements of V which have trivial restrictions, so that the following sequence is exact:

$$0 \to V(Z^J) \to V \to \{W|_Q : W \in V = \mathcal{W}(\pi, \psi_{c_1,c_2})\} \to 0.$$

We do not know if this sequence is exact if π is not supercuspidal. Instead, for an arbitrary representation, the theorem below involves just $p^{-1}(V_2)$. If the sequence were exact for all π, then the first stage of our argument that $p(W') = 0$ could be eliminated.

Theorem 4.3.5. *Let π be a generic, irreducible, admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character, and let $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. As in Section 2.5, let $p : V \to V_{Z^J} = V/V(Z^J)$ be the projection, and let $V_2 \subset V_{Z^J}$ be the irreducible P_3-subspace from Theorem 2.5.3. Let $W' \in V$, and assume $p(W') \in V_2$. Then $p(W') = 0$ if and only if $W'(Q) = 0$.*

Proof. A computation using that Q normalizes Z^J proves that if $p(W') = 0$, so that $W' \in V(Z^J)$, then $W'(Q) = 0$. To prove the converse, let V_0 be the space of elements $W \in p^{-1}(V_2)$ such that $W(Q) = 0$. We need to prove that $p(V_0) = 0$. To prove this, we note that $p(V_0)$ is a P_3-subspace of V_2. Since V_2 is an irreducible P_3-subspace of V_{Z^J}, we have $p(V_0) = 0$ or $P(V_0) = V_2$. Assume $p(V_0) = V_2$; we will obtain a contradiction. Consider the non-zero linear Whittaker functional $V_{Z^J} \to \mathbb{C}$ that sends $W + V(Z^J)$ to $W(1)$. Since $p(V_0) = V_2$, V_2 lies in the kernel of this functional. Therefore, V_{Z^J}/V_2 is non-degenerate in the sense of the definition in 5.7 of [BZ]. However, V_{Z^J}/V_2 is degenerate. See 5.15 of [BZ]. □

The previous theorem reduces the problem of showing that $p(W') = 0$ to proving $W'(Q) = 0$. We prove this in the next lemma. For the proof we will need the following fact. If π is a generic, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character, $V = W(\pi,\psi_{c_1,c_2})$, and $W \in V(n)$, $n \geq 0$, is degenerate, then

$$v(a) \leq v(b) \implies W(\begin{bmatrix} a &&& \\ & b && \\ && cb^{-1} & \\ &&& ca^{-1} \end{bmatrix}) = 0 \tag{4.15}$$

for all $a,b,c \in F^\times$. If $v(a) < v(b)$, then this is Lemma 4.1.2; if $v(a) = v(b)$, then this follows immediately from Lemma 4.1.1.

Lemma 4.3.6. *Let π be a generic, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character, and let $V = \mathcal{W}(\pi,\psi_{c_1,c_2})$. Let $n \geq 0$ be an integer, and let $W \in V(n)$ be degenerate. Set $W' = SW - qW$. Then $W'(Q) = 0$.*

Proof. We will prove the sufficient statement that for every $z \in \mathfrak{o}$

$$\pi(\begin{bmatrix} 1 & z && \\ & 1 && \\ && 1 & -z \\ &&& 1 \end{bmatrix})\tilde{W} - \tilde{W}$$

vanishes on Q, where $\tilde{W} = \eta^{-1}W$. By the transformation properties of Whittaker functions, it will suffice to show that

$$\tilde{W}(\begin{bmatrix} a &&& \\ & b && \\ & x & cb^{-1} & \\ &&& ca^{-1} \end{bmatrix}\begin{bmatrix} 1 & z && \\ & 1 && \\ && 1 & -z \\ &&& 1 \end{bmatrix}) = \tilde{W}(\begin{bmatrix} a &&& \\ & b && \\ & x & cb^{-1} & \\ &&& ca^{-1} \end{bmatrix})$$

for $a,b,c \in F^\times$, $x \in F$ and $z \in \mathfrak{o}$. Now

$$\begin{bmatrix} a &&& \\ & b && \\ & x & cb^{-1} & \\ &&& ca^{-1} \end{bmatrix}\begin{bmatrix} 1 & z && \\ & 1 && \\ && 1 & -z \\ &&& 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{az}{b} && \\ & 1 && \\ && 1 & -\frac{az}{b} \\ &&& 1 \end{bmatrix}\begin{bmatrix} a &&& \\ & b && \\ & x & cb^{-1} & \\ &&& ca^{-1} \end{bmatrix}.$$

Hence,

$$\tilde{W}(\begin{bmatrix} a &&& \\ & b && \\ & x & cb^{-1} & \\ &&& ca^{-1} \end{bmatrix}\begin{bmatrix} 1 & z && \\ & 1 && \\ && 1 & -z \\ &&& 1 \end{bmatrix}) = \psi(c_1\frac{az}{b})\tilde{W}(\begin{bmatrix} a &&& \\ & b && \\ & x & cb^{-1} & \\ &&& ca^{-1} \end{bmatrix}).$$

We thus need to prove

$$\psi(c_1\frac{az}{b})\tilde{W}(\begin{bmatrix} a &&& \\ & b && \\ & x & cb^{-1} & \\ &&& ca^{-1} \end{bmatrix}) = \tilde{W}(\begin{bmatrix} a &&& \\ & b && \\ & x & cb^{-1} & \\ &&& ca^{-1} \end{bmatrix});$$

to prove this, it will suffice to show that

$$v(a/b) < 0 \implies \tilde{W}(\begin{bmatrix} a & & & \\ & b & & \\ & x & cb^{-1} & \\ & & & ca^{-1} \end{bmatrix}) = 0.$$

Assume $v(a/b) < 0$, i.e., $v(a) < v(b)$. We have

$$\begin{aligned} \tilde{W}(\begin{bmatrix} a & & & \\ & b & & \\ & x & cb^{-1} & \\ & & & ca^{-1} \end{bmatrix}) &= W(\begin{bmatrix} a\varpi & & & \\ & b & & \\ & x & cb^{-1} & \\ & & & ca^{-1}\varpi^{-1} \end{bmatrix}) \\ &= W(\begin{bmatrix} a\varpi & & & \\ & b & & \\ & & cb^{-1} & \\ & & & ca^{-1}\varpi^{-1} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & y & 1 & \\ & & & 1 \end{bmatrix}). \end{aligned}$$

where $y = c^{-1}bx$. If $v(y) \geq 0$, we get

$$W(\begin{bmatrix} a\varpi & & & \\ & b & & \\ & x & cb^{-1} & \\ & & & ca^{-1}\varpi^{-1} \end{bmatrix}) = W(\begin{bmatrix} a\varpi & & & \\ & b & & \\ & & cb^{-1} & \\ & & & ca^{-1}\varpi^{-1} \end{bmatrix}),$$

which is 0, since W is degenerate and $v(a\varpi) \leq v(b)$ (see (4.15)). Assume $v(y) < 0$. Using the useful identity (2.8), we have

$$\begin{aligned} &\tilde{W}(\begin{bmatrix} a & & & \\ & b & & \\ & x & cb^{-1} & \\ & & & ca^{-1} \end{bmatrix}) \\ &= W(\begin{bmatrix} a\varpi & & & \\ & b & & \\ & & cb^{-1} & \\ & & & ca^{-1}\varpi^{-1} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & y & 1 & \\ & & & 1 \end{bmatrix}) \\ &= W(\begin{bmatrix} a\varpi & & & \\ & b & & \\ & & cb^{-1} & \\ & & & ca^{-1}\varpi^{-1} \end{bmatrix} \\ &\qquad \times \begin{bmatrix} 1 & & & \\ & 1 & y^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & -y^{-1} & & \\ & & -y & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & y^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix}) \end{aligned}$$

$$= W(\begin{bmatrix} a\varpi & & & \\ & b & & \\ & & cb^{-1} & \\ & & & ca^{-1}\varpi^{-1} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & y^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & -y^{-1} & & \\ & & -y & \\ & & & 1 \end{bmatrix})$$

$$= W(\begin{bmatrix} 1 & & & \\ & 1 & y^{-1}c^{-1}b^2 & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} a\varpi & & & \\ & b & & \\ & & cb^{-1} & \\ & & & ca^{-1}\varpi^{-1} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & -y^{-1} & & \\ & & -y & \\ & & & 1 \end{bmatrix})$$

$$= \psi(y^{-1}c^{-1}b^2)W(\begin{bmatrix} a\varpi & & & \\ & by^{-1} & & \\ & & ycb^{-1} & \\ & & & ca^{-1}\varpi^{-1} \end{bmatrix}).$$

This is zero since W is degenerate,

$$\begin{aligned} v(y) &\le -1 \\ 1 &\le -v(y) \\ v(a)+1 &\le v(b)-v(y) \\ v(a\varpi) &\le v(by^{-1}), \end{aligned}$$

and (4.15) holds. □

Theorem 4.3.7 (η Principle). *Let π be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character, and let $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. Let $n \ge 0$ be an integer. If W is non-zero and degenerate, then $n \ge 2$, and there exists $W_1 \in V(n-2)$ such that $W = \eta W_1$.*

Proof. Assume that W is degenerate.

Suppose first $n \ge 2$; we will prove that there exists $W_1 \in V(n-2)$ such that $W = \eta W_1$. As above, let $W' = SW - qW$, and let $V_2 \subset V_{Z^J}$ be as in Theorem 2.5.3. By Lemma 4.3.2, $p(W') \in V_2$. By Lemma 4.3.6, $W'(Q) = 0$. By Theorem 4.3.5, $p(W') = 0$. By Lemma 4.3.1, there exists $W_1 \in V(n-2)$ such that $W = \eta W_1$.

Finally, let us prove $n \ge 2$. Since $W \ne 0$, by Corollary 7.1.5 below we cannot have $n = 0$, i.e., a non-zero spherical vector has a non-zero zeta integral. Suppose $n = 1$; we will obtain a contradiction. Arguing exactly as in the last paragraph, we have $p(W') = 0$. Since $W \ne 0$, by Lemma 4.3.1, we have $\theta' W \ne 0$ and $\theta' W = \eta W_2$ for some $W_2 \in V(0)$, $W_2 \ne 0$. It follows that π is spherical. By, for example, Theorem 5.2.2, $\dim V(1) = 2$. Consider $\theta W_2, \theta' W_2 \in V(1)$. By Proposition 4.1.3, we have $Z(s, \theta W_2) = q^{-s+3/2} Z(s, W_2)$ and $Z(s, \theta' W_2) = qZ(s, W_2)$. Since by Corollary 7.1.5 the zeta integral $Z(s, W_2)$ is non-zero, it follows that θW_2 and $\theta' W_2$ are linearly independent and thus form a basis for $V(1)$. Write $W = a\theta W_2 + b\theta' W_2$, with $a, b \in \mathbb{C}$ and either $a \ne 0$ or $b \ne 0$. We have $0 = Z(s, W) = (aq^{-s+3/2} + bq)Z(s, W_2)$. Since $Z(s, W_2) \ne 0$, this is a contradiction. □

Corollary 4.3.8. *Let $n \geq 0$. If $W \in V(n)$ and W vanishes on the diagonal subgroup of* $\mathrm{GSp}(4, F)$ *then $W = 0$.*

Proof. By Theorem 4.3.7, since W is degenerate, $W = \eta W_1$ for some $W_1 \in V(n-2)$. By the assumption on W, W_1 also vanishes on the diagonal subgroup. Therefore, $W_1 = \eta W_2$ for some $W_2 \in V(n-4)$. Continuing, there exists $k \geq 0$ such that $V(n-2k) = 0$ and $W = \eta^k W_k$ for some $W_k \in V(n-2k) = 0$; that is, $W = 0$. $\square$

Proposition 4.3.9. *Let π be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4, F)$ *with trivial central character and let $V = W(\pi, \psi_{c_1,c_2})$. Assume that π admits non-zero paramodular vectors, and let n be the smallest non-negative integer such that $V(n) \neq 0$. Let $W \in V(n)$ be non-zero. Then the vectors*

$$\theta'^i \theta^j \eta^k W, \qquad i, j, k \geq 0,$$

are linearly independent. Consequently,

$$\dim V(m) \geq \left[\frac{(m-n+2)^2}{4}\right] = (m-n) + 1 + \left[\frac{(m-n)^2}{4}\right] \quad \textit{for} \quad m \geq n.$$

Here, for $x \in \mathbb{R}$, $[x]$ is the unique integer such that $[x] \leq x < [x] + 1$.

Proof. By Theorem 4.3.7, i.e., by the η Principle, we have $Z(s, W) \neq 0$. For $t \geq 0$ a non-negative integer let $d(t)$ be the dimension of the subspace of $V(n+t)$ spanned by the vectors of the form $\theta'^i \theta^j \eta^k W$ where i, j and k are non-negative integers such that $i + j + 2k = t$. We claim that the vectors $\theta'^i \theta^j \eta^k W \in V(n+t)$ are linearly independent. We will prove this by induction on t. If $t = 0$ this is clear. Assume the claim holds for all t' with $t' < t$; we will prove it holds for t. Suppose there is a linear relation

$$0 = \sum_{i+j+2k=t} c(i,j,k) \theta'^i \theta^j \eta^k W$$

with $c(i,j,k) \in \mathbb{C}$. Then

$$-\sum_{i+j=t} c(i,j,0) \theta'^i \theta^j W = \eta\Big(\sum_{\substack{i+j+2k=t,\\ k>0}} c(i,j,k) \theta'^i \theta^j \eta^{k-1} W \Big).$$

By Proposition 4.1.3, this vector is degenerate. Hence, by Proposition 4.1.3 again,

$$0 = \Big(\sum_{i+j=t} c(i,j,0) q^i q^{3j/2} (q^{-s})^j \Big) Z(s, W).$$

Since $Z(s, W)$ is non-zero, this implies that $c(i,j,0) = 0$ for $i + j = t$. Therefore, since η is invertible,

$$0 = \sum_{\substack{i+j+2k=t,\\ t>0}} c(i,j,k) \theta'^i \theta^j \eta^{k-1} W$$

$$0 = \sum_{\substack{i+j+2(k-1)=t-2,\\ k>0}} c(i,j,k)\theta'^i\theta^j\eta^{k-1}W.$$

By the induction hypothesis, $c(i,j,k) = 0$ for $k > 0$, proving the claim. It follows from the claim that $d(t)$ is the number of solutions to $i + j + 2k = t$ where i, j and k are non-negative integers. Evidently, $d(t)$ is the number of solutions with $k = 0$, i.e., $t + 1$, plus the number of solutions with $k \geq 1$, i.e., $d(t-2)$. That is, $d(t) = (t+1) + d(t-2)$. Using this recursive relation, it is easy to prove by induction that $d(t) = (t+1) + [t^2/4]$. □

4.4 The Existence Theorem for Generic Representations

In the first chapter, while proving basic properties about zeta integrals, we used P_3-theory to show that in a generic representation there exists a vector W such that $Z(s,W)$ is non-zero and constant. As the proof of the following theorem shows, this W can be chosen to be paramodular. In particular, this proves the existence of non-zero paramodular vectors in generic representations.

Theorem 4.4.1 (Existence for Generic Representations). *Let (π, V) be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character. Then V contains non-zero paramodular vectors. Moreover, there exists a paramodular vector W in the Whittaker model of π such that $Z(s,W)$ is constant and non-zero.*

Proof. We will use the objects and notation of the proof of Proposition 2.6.4. Let $W_0 \in X$ be such that $j(p(W_0)) = f_0$. A computation shows that f_0 is invariant under $P_3(\mathfrak{o})$. Using (2.51), we compute

$$\begin{aligned}(j \circ p)\Big(\int\limits_{Q(\mathfrak{o})} \pi(k)W_0\,dk\Big) &= \int\limits_{Q(\mathfrak{o})} (j \circ p)(\pi(k)W_0)\,dk\\ &= \int\limits_{Q(\mathfrak{o})} i(k)(j \circ p)(W_0)\,dk\\ &= \int\limits_{Q(\mathfrak{o})} i(k)f_0\,dk\\ &= \Big(\int\limits_{Q(\mathfrak{o})} dk\Big)f_0.\end{aligned}$$

Hence we may assume that W_0 is invariant under $Q(\mathfrak{o})$. Since W_0 is a smooth vector, W_0 is invariant under $\mathrm{Kl}(\mathfrak{p}^n)$ for some n. In the proof of Proposition

2.6.4 we saw that $Z(s, W_0) \neq 0$ (in fact, it was constant). By Lemma 4.1.5, we have

$$Z(s, \theta' W_0) = q Z(s, W_0) \neq 0.$$

Since $\theta' W_0 \in V(n+1)$, the proof is complete. □

5

Non-supercuspidal Representations

In this chapter we investigate the structure of paramodular vectors in non-supercuspidal, irreducible, admissible representations (π, V) with trivial central character. In all cases we determine the minimal paramodular level N_π and prove that $\dim V(N_\pi) = 1$. In fact, we determine $\dim V(n)$ for all $n \geq N_\pi$ and prove the Oldforms Principle.

Our method is to realize these representations via the Sally-Tadić classification described in Sect. 2.2 as irreducible subquotients of appropriate full induced representations. The basic structural results underlying our investigations in induced representations are certain explicit double coset decompositions of

$$B\backslash \mathrm{GSp}(4,F)/\mathrm{K}(\mathfrak{p}^n), \quad P\backslash \mathrm{GSp}(4,F)/\mathrm{K}(\mathfrak{p}^n), \quad Q\backslash \mathrm{GSp}(4,F)/\mathrm{K}(\mathfrak{p}^n),$$

where B, P, Q are the standard parabolic subgroups of $\mathrm{GSp}(4)$. Using these explicit decompositions we first compute the dimensions of the spaces of paramodular vectors in full induced representations. Then, by calculating the action of level raising operators in convenient full induced representations, we are able to compute paramodular vectors for all non-supercuspidal representations.

In addition, knowledge of the explicit form of paramodular vectors in non-supercuspidal representations will be used to compute Atkin–Lehner eigenvalues and, in subsequent chapters, Hecke eigenvalues.

5.1 Double Coset Decompositions

In this section we will obtain several double coset decompositions involving the paramodular group. Let s_1 and s_2 be the Weyl group elements as defined in Sect. 2.1. In the following lemma $\mathrm{Si}(\mathfrak{p}^n)$ denotes the Siegel congruence subgroup of level $\mathfrak{p}^n$, as defined in (2.4).

Lemma 5.1.1. *For any $n \geq 1$, the following is a complete system of representatives for* $\mathrm{Si}(\mathfrak{p}^n)\backslash\mathrm{GSp}(4,\mathfrak{o})$.

$$\begin{bmatrix}1&&&\\&1&&\\y&z&1&\\x&y&&1\end{bmatrix}, \qquad x,y,z \in \mathfrak{o}/\mathfrak{p}^n,\ x,y,z \equiv 0 \mod \mathfrak{p}, \tag{5.1}$$

$$\begin{bmatrix}1&&&\\&1&&\\y&z&1&\\x&y&&1\end{bmatrix} s_2, \qquad x,y,z \in \mathfrak{o}/\mathfrak{p}^n,\ x,y \equiv 0 \mod \mathfrak{p}, \tag{5.2}$$

$$\begin{bmatrix}1&&&\\&1&&\\y&z&1&\\x&y&&1\end{bmatrix} s_2s_1, \qquad x,y,z \in \mathfrak{o}/\mathfrak{p}^n,\ x \equiv 0 \mod \mathfrak{p}, \tag{5.3}$$

$$\begin{bmatrix}1&&&\\&1&&\\y&z&1&\\x&y&&1\end{bmatrix} s_2s_1s_2, \qquad x,y,z \in \mathfrak{o}/\mathfrak{p}^n. \tag{5.4}$$

In particular $\#\mathrm{Si}(\mathfrak{p}^n)\backslash\mathrm{GSp}(4,\mathfrak{o}) = q^{3n-3}(q+1)(q^2+1)$.

Proof. By the Bruhat decomposition,

$$\begin{aligned}\mathrm{GSp}(4) &= P \sqcup Ps_2\begin{bmatrix}1&&&\\&1&*&\\&&1&\\&&&1\end{bmatrix} \sqcup Ps_2s_1\begin{bmatrix}1&*&&*\\&1&&\\&&1&*\\&&&1\end{bmatrix} \sqcup Ps_2s_1s_2\begin{bmatrix}1&&*&*\\&1&*&*\\&&1&\\&&&1\end{bmatrix}\\ &= P \sqcup P\begin{bmatrix}1&&&\\&1&&\\&*&1&\\&&&1\end{bmatrix} s_2 \sqcup P\begin{bmatrix}1&&&\\&1&&\\*&*&1&\\&*&&1\end{bmatrix} s_2s_1 \sqcup P\begin{bmatrix}1&&&\\&1&&\\*&*&1&\\*&*&&1\end{bmatrix} s_2s_1s_2\end{aligned}$$

over any field. In particular, over the field $k = \mathfrak{o}/\mathfrak{p}$ with q elements we obtain the assertion of the lemma for $n = 1$. Now assume that $n > 1$. It is easy to see that

$$\mathrm{Si}(\mathfrak{p}) = \bigsqcup_{x,y,z\in\mathfrak{o}/\mathfrak{p}^{n-1}} \mathrm{Si}(\mathfrak{p}^n)\begin{bmatrix}1&&&\\&1&&\\y\varpi&z\varpi&1&\\x\varpi&y\varpi&&1\end{bmatrix}.$$

The lemma now follows from the following simple fact: Let G be a group, H a subgroup, and $G = \sqcup_g Hg$ with some representatives $g \in G$. Let further L be a subgroup of H and $H = \sqcup_h Lh$. Then $G = \sqcup_g \sqcup_h Lhg$. $\square$

For the statement of the following proposition define for $i \geq 1$

$$L_i = \begin{bmatrix} 1 & & & \\ \varpi^i & 1 & & \\ & & 1 & \\ & & -\varpi^i & 1 \end{bmatrix}, \qquad M_i = \begin{bmatrix} 1 & & & \\ & 1 & & \\ \varpi^i & & 1 & \\ & \varpi^i & & 1 \end{bmatrix}. \tag{5.5}$$

In addition, we write $G = \mathrm{GSp}(4)$.

Proposition 5.1.2. *With notations as in (5.5) we have, for any $n \geq 1$, the following complete systems of representatives.*

double coset space	#	representatives
$B(F)\backslash G(F)/\mathrm{K}(\mathfrak{p}^n)$	$\left[\frac{(n+2)^2}{4}\right]$	$1,\ s_1,\ L_i,\ 1 \leq i < n,\ M_j,\ 1 \leq j < n,$ $L_iM_j,\ 1 \leq j < i < n-j$
$Q(F)\backslash G(F)/\mathrm{K}(\mathfrak{p}^n)$	$n+1$	$1,\ s_1,\ L_i,\ 1 \leq i < n$
$P(F)\backslash G(F)/\mathrm{K}(\mathfrak{p}^n)$	$\left[\frac{n+2}{2}\right]$	$1,\ M_i,\ 1 \leq i \leq \frac{n}{2}$

Proof. We will first consider representatives for $P(F)\backslash G(F)/\mathrm{K}(\mathfrak{p}^n)$. It is clear that such representatives can be found amongst representatives for

$$\begin{aligned} P(F)\backslash G(F)/\mathrm{Kl}(\mathfrak{p}^n) &\cong P(\mathfrak{o})\backslash G(\mathfrak{o})/\mathrm{Kl}(\mathfrak{p}^n) \\ &\cong P(\mathfrak{o}/\mathfrak{p}^n)\backslash G(\mathfrak{o}/\mathfrak{p}^n)/Q(\mathfrak{o}/\mathfrak{p}^n) \\ &\cong \mathrm{Si}(\mathfrak{p}^n)\backslash G(\mathfrak{o})/Q(\mathfrak{o}). \end{aligned} \tag{5.6}$$

Representatives for $\mathrm{Si}(\mathfrak{p}^n)\backslash G(\mathfrak{o})$ are given in Lemma 5.1.1, and we have to consider the action of $Q(\mathfrak{o})$ from the right on this space. Since $s_2 \in Q(\mathfrak{o})$, elements of type (5.2) are equivalent in $P(F)\backslash G(F)/\mathrm{Kl}(\mathfrak{p}^n)$ to elements of type (5.1). Similarly, elements of type (5.3) are equivalent to elements of type (5.4). The representatives in (5.4) are obviously all equivalent to $s_2s_1s_2$. But

$$s_2s_1s_2 = \begin{bmatrix} & 1 & & \\ \varpi^n & & & \\ & & & \varpi^{-n} \\ & & 1 & \end{bmatrix} \begin{bmatrix} & & & \varpi^{-n} \\ & & 1 & \\ & -1 & & \\ -\varpi^n & & & \end{bmatrix}.$$

Hence the representatives in (5.4) are all equivalent to the identity matrix. It follows that $P(F)\backslash G(F)/\mathrm{K}(\mathfrak{p}^n)$ is represented by 1 and

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ r & & 1 & \\ s & r & & 1 \end{bmatrix}, \qquad r, s \in \mathfrak{o}/\mathfrak{p}^n,\ r, s \equiv 0 \mod \mathfrak{p}.$$

Let us abbreviate such a matrix by (r, s). We have the following matrix identities.

$$(r,s) = \begin{bmatrix} 1 & r^{-1} & & -s^{-1} \\ -sr^{-1} & 0 & 1 & \\ & & s^{-1}r^2 & -s^{-1}r \\ & & r & 0 \end{bmatrix} (sr^{-1},0) \begin{bmatrix} 1 & & & s^{-1} \\ & s^{-1}r^2 & -1 & \\ & 1 & 0 & \\ & & & 1 \end{bmatrix}, \tag{5.7}$$

$$(r,s) = \begin{bmatrix} 1 & & & \\ -sr^{-1} & 1 & & \\ & & 1 & \\ & & sr^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & sr^{-2} & \\ & & 1 & \\ & & & 1 \end{bmatrix} (r,0) \begin{bmatrix} 1 & & & \\ & 1 & -sr^{-2} & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \tag{5.8}$$

$$(r,s) = \begin{bmatrix} -s^{-1}\varpi^n & -s^{-1}r & & -\varpi^{-n} \\ & 1 & & \\ & & 1 & -r\varpi^{-n} \\ & & & -s\varpi^{-n} \end{bmatrix} \begin{bmatrix} & & & \varpi^{-n} \\ & 1 & & \\ & -s^{-1}r^2 & 1 & -s^{-1}r \\ -\varpi^n & -\varpi^n s^{-1}r & 0 & -\varpi^n s^{-1} \end{bmatrix}. \tag{5.9}$$

Equation (5.7) shows that if $v(r) \le v(s) \le 2v(r)$ and $v(s) \le n$, then (r,s) is equivalent to $(sr^{-1},0)$ in the double coset space $P(F)\backslash G(F)/\mathrm{K}(\mathfrak{p}^n)$. If $v(s) \ge 2v(r)$ then (5.8) shows that (r,s) is equivalent to $(r,0)$. And if $v(s) \le v(r)$ then (5.9) shows that (r,s) is equivalent to 1. This proves that $P(F)\backslash G(F)/\mathrm{K}(\mathfrak{p}^n)$ is represented by 1 and

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ r & & 1 & \\ & r & & 1 \end{bmatrix}, \qquad r \in \mathfrak{o}/\mathfrak{p}^n,\ r \equiv 0 \mod \mathfrak{p}.$$

We can multiply by units, thus assuming that $r = \varpi^m$, $1 \le m \le n$. The relation

$$\begin{aligned} &(\varpi^{n-m},0) \begin{bmatrix} \varpi^m & 0 & 1 & -\varpi^{-n} \\ 0 & 0 & 1 & 0 \\ -\varpi^n & -1 & -\varpi^{n-m} & 0 \\ \varpi^n & 0 & 0 & 0 \end{bmatrix} \\ &\qquad = \begin{bmatrix} 0 & \varpi^{m-n} & 1 & -\varpi^{-n} \\ -\varpi^m & 0 & 1 & 0 \\ & & 0 & -\varpi^{-m} \\ & & \varpi^{n-m} & 0 \end{bmatrix} (\varpi^m,0) \end{aligned} \tag{5.10}$$

shows that $(\varpi^m,0)$ and $(\varpi^{n-m},0)$ are equivalent. One can check that these are the only equivalences between such representatives, and that for $m < n$ none of these is equivalent to 1. This proves the statement about $P(F)\backslash G(F)/\mathrm{K}(\mathfrak{p}^n)$.

To find representatives for $B(F)\backslash G(F)/\mathrm{K}(\mathfrak{p}^n)$ we take the set of representatives we just found for $P(F)\backslash G(F)/\mathrm{K}(\mathfrak{p}^n)$ and multiply each element from the left with a set of representatives for $B(F)\backslash P(F)$. As for the latter we choose

$$1 \qquad \text{and} \qquad s_1 \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix}, \qquad x \in F.$$

It is easy to see that each representative thus obtained is equivalent in $B(F)\backslash G(F)/\mathrm{K}(\mathfrak{p}^n)$ to one of the following:

$$1, \quad s_1, \tag{5.11}$$
$$L_i, \qquad 1 \le i < n, \tag{5.12}$$
$$M_i, \qquad 1 \le i < n, \tag{5.13}$$
$$L_iM_j, \qquad 1 \le i, j < n. \tag{5.14}$$

For example, the identity

$$s_1M_j = \begin{bmatrix} -\varpi^{n-j} & & 1 & \\ & -\varpi^{-j} & & \varpi^{-n} \\ & & \varpi^{j} & \\ & & & \varpi^{j-n} \end{bmatrix} M_{n-j} \begin{bmatrix} & & & \varpi^{-n} \\ & & 1 & \\ & 1 & & \\ \varpi^{n} & & & \end{bmatrix} \tag{5.15}$$

shows that the occurring elements s_1M_j fall under the class (5.13). It now remains to find all equivalences between the elements listed in (5.11) to (5.14). We have identities

$$M_j \begin{bmatrix} 1 & & & \\ & 1 & -\varpi^{i-j} & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & -\varpi^{i-j} & \\ & & 1 & \\ & & & 1 \end{bmatrix} L_iM_j \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ \varpi^{i+j} & & & 1 \end{bmatrix}, \tag{5.16}$$

$$M_{n-j} \begin{bmatrix} & & & \varpi^{-n} \\ & \varpi^{j-i} & -1 & \\ & 1 & 0 & \\ -\varpi^{n} & & & \varpi^{n-i-j} \end{bmatrix} = \begin{bmatrix} \varpi^{i+j-n} & -\varpi^{j-n} & \varpi^{i-n} & \varpi^{-n} \\ & \varpi^{j-i} & -1 & 0 \\ & & \varpi^{i-j} & \varpi^{-j} \\ & & & \varpi^{n-i-j} \end{bmatrix} L_iM_j. \tag{5.17}$$

The equation (5.16) shows that if $n-i \le j \le i$, then L_iM_j is equivalent to M_j, and (5.17) shows that if $n-i \ge j \ge i$, then L_iM_j is equivalent to M_{n-j}. Thus we are left with L_iM_j for $1 \le j < i < n-j$ and $n-j < i < j < n$. But the second type is equivalent to the first type because of the relation

$$L_{n-i}M_{n-j} = \begin{bmatrix} \varpi^{i+j-n} & -\varpi^{j-n} & \varpi^{i-n} & \varpi^{-n} \\ & -\varpi^{j-i} & 2 & \varpi^{-i} \\ & & \varpi^{i-j} & \varpi^{-j} \\ & & & -\varpi^{n-i-j} \end{bmatrix} L_iM_j \begin{bmatrix} & & & \varpi^{-n} \\ & & 1 & \\ & 1 & & \\ \varpi^{n} & & & \end{bmatrix}.$$

Thus, we get the representatives as stated in the proposition. We postpone checking that there are no equivalences between these elements until after we do the case of $Q(F)\backslash G(F)/\mathrm{K}(\mathfrak{p}^n)$.

Turning to $Q(F)\backslash G(F)/\mathrm{K}(\mathfrak{p}^n)$, representatives for $Q(F)\backslash G(F)/\mathrm{K}(\mathfrak{p}^n)$ can be found amongst the representatives for $B(F)\backslash G(F)/\mathrm{K}(\mathfrak{p}^n)$. Since we are now able to conjugate with s_2, an element M_j is equivalent to L_j. For $j < i < n-j$ the identity

$$L_iM_j = \begin{bmatrix} \varpi^{n-i-j} & & \varpi^{-i} & \varpi^{-n} \\ & \varpi^{i-j} & 2 & \varpi^{i-n} \\ & 1 & \varpi^{j-i} & \varpi^{j-n} \\ & & & -\varpi^{i+j-n} \end{bmatrix} L_{n-i} \begin{bmatrix} & & & \varpi^{-n} \\ & & 1 & \\ & 1 & -\varpi^{i-j} & \\ \varpi^n & & & -\varpi^{n-i-j} \end{bmatrix}$$

shows that L_iM_j is equivalent to L_{n-i}. Thus we are left with the elements as listed in the proposition, and it is easy to check that there are no equivalences amongst them.

Finally, we need to check that there are no equivalences between the stated representatives for $B(F)\backslash G(F)/\mathrm{K}(\mathfrak{p}^n)$. This is now easily accomplished by multiplying an equality $B(F)g\mathrm{K}(\mathfrak{p}^n) = B(F)g'\mathrm{K}(\mathfrak{p}^n)$ on the left by $P(F)$ or $Q(F)$, and using that there are no equivalences between the stated representatives for $P(F)\backslash G(F)/\mathrm{K}(\mathfrak{p}^n)$ or $Q(F)\backslash G(F)/\mathrm{K}(\mathfrak{p}^n)$. □

We remark that one alternatively could work with L_0 and M_0, and get the following systems of representatives.

double coset space	#	representatives
$B(F)\backslash G(F)/\mathrm{K}(\mathfrak{p}^n)$	$\left[\frac{(n+2)^2}{4}\right]$	$L_i,\ 0 \le i \le n,\ \ M_j,\ 1 \le j < n,$ $L_iM_j,\ 1 \le j < i < n-j$
$Q(F)\backslash G(F)/\mathrm{K}(\mathfrak{p}^n)$	$n+1$	$L_i,\ 0 \le i \le n$
$P(F)\backslash G(F)/\mathrm{K}(\mathfrak{p}^n)$	$\left[\frac{n+2}{2}\right]$	$M_i,\ 0 \le i \le \frac{n}{2}$

Note that $B(F)s_1\mathrm{K}(\mathfrak{p}^n) = B(F)L_0\mathrm{K}(\mathfrak{p}^n)$, and

$$M_0 = \begin{bmatrix} & 1 & & \varpi^{-n} \\ \varpi^n & & 1 & \\ & & & \varpi^{-n} \\ & & 1 & \end{bmatrix} \begin{bmatrix} & & & -\varpi^{-n} \\ & & -1 & \\ & 1 & & 1 \\ \varpi^n & & \varpi^n & \end{bmatrix} \tag{5.18}$$

shows that $P(F)1\mathrm{K}(\mathfrak{p}^n) = P(F)M_0\mathrm{K}(\mathfrak{p}^n)$.

5.2 Induction from the Siegel Parabolic Subgroup

Let π be an admissible representation of $\mathrm{GL}(2,F)$ admitting a central character ω_π. Let σ be a character of $F^\times$ such that $\omega_\pi\sigma^2 = 1$, so that the representation $\pi \rtimes \sigma$ of $\mathrm{GSp}(4,F)$ has trivial central character. In this section we consider paramodular vectors in $\pi \rtimes \sigma$. We will prove that $\pi \rtimes \sigma$ is paramodular if and only if there exists a non-negative integer n such that $\sigma\pi$ contains a non-zero vector invariant under

$$\Gamma_1(\mathfrak{p}^n) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2,\mathfrak{o}) :\ c \in \mathfrak{p}^n,\ d \in 1 + \mathfrak{p}^n \right\}. \tag{5.19}$$

In general, if τ is an admissible representation of $\mathrm{GL}(2,F)$, and $\tau^{\Gamma_1(\mathfrak{p}^n)} \neq 0$ for some n, then we let N_τ be the smallest such n; for convenience, if $\tau^{\Gamma_1(\mathfrak{p}^n)} = 0$ for all $n \geq 0$, then we also define $N_\tau = \infty$. We call N_τ the *level* of τ. If τ is irreducible, then we let $a(\tau)$ be the conductor of the L-parameter of τ, as defined in (2.50). It is known that if τ is irreducible and $N_\tau < \infty$, then $N_\tau = a(\tau)$. See [JPSS] and [K]. We will prove that the minimal paramodular level of $\pi \rtimes \sigma$ is $n := N_{\sigma\pi} + 2a(\sigma)$; in fact, we will determine all the paramodular vectors in $\pi \rtimes \sigma$. We note that the statement that n is the minimal paramodular level of $\pi \rtimes \sigma$ is consistent with, and motivated by, one of our main results, Theorem 7.5.9. For suppose, for example, that $\pi \rtimes \sigma$ is irreducible and paramodular. Then π is irreducible. Let $\mu : W_F' \to \mathrm{GL}(2,\mathbb{C})$ be the L-parameter of π. The L-parameter of $\pi \rtimes \sigma$ is defined by

$$w \longmapsto \begin{bmatrix} \sigma(w)\det(\mu(w)) & & \\ & \sigma(w)\mu(w) & \\ & & \sigma(w) \end{bmatrix} \in \mathrm{GSp}(4,\mathbb{C});$$

see Sect. 2.4. The conductor of this L-parameter is $a(\sigma\pi) + 2a(\sigma) = N_{\sigma\pi} + 2a(\sigma)$. Theorem 7.5.9 asserts that this is the minimal paramodular level of $\pi \rtimes \sigma$.

First we require a lemma. We remind the reader of the A' notation introduced in (2.1).

Lemma 5.2.1. *Let n be a positive integer. Let $u \in F^\times$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2,F)$ be given. For $1 \leq i \leq n/2$ the following statements are equivalent:*

i) There exists an $X \in M(2,F)$ of the form $X = \begin{bmatrix} x & y \\ z & x \end{bmatrix}$ such that

$$\begin{bmatrix} A & \\ & uA' \end{bmatrix} \begin{bmatrix} 1 & X \\ & 1 \end{bmatrix} \in P(F) \cap M_i \mathrm{K}(\mathfrak{p}^n) M_i^{-1}. \tag{5.20}$$

Here M_i is as in (5.5).

ii) The following conditions are satisfied:

- $u \in \mathfrak{o}^\times$,
- $\det(A) \in \mathfrak{o}^\times$,
- $u \det(A)^{-1} \in 1 + \mathfrak{p}^i$,
- $A \in \begin{bmatrix} \mathfrak{o} & \mathfrak{p}^{-i} \\ \mathfrak{p}^{n-i} & \mathfrak{o} \end{bmatrix}$.

Proof. A calculation shows that $M_i^{-1} \begin{bmatrix} A & \\ & uA' \end{bmatrix} \begin{bmatrix} 1 & X \\ & 1 \end{bmatrix} M_i \in \mathrm{K}(\mathfrak{p}^n)$ if and only if $u \in \mathfrak{o}^\times$ and the following (5.21) to (5.24) are fulfilled:

$$A(1 + \varpi^i X) \in \begin{bmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o} \end{bmatrix}, \tag{5.21}$$

$$AX \in \begin{bmatrix} \mathfrak{o} & \mathfrak{p}^{-n} \\ \mathfrak{o} & \mathfrak{o} \end{bmatrix}, \tag{5.22}$$

$$uA' - A - \varpi^i AX \in \begin{bmatrix} \mathfrak{p}^{n-i} & \mathfrak{p}^{-i} \\ \mathfrak{p}^{n-i} & \mathfrak{p}^{n-i} \end{bmatrix}, \tag{5.23}$$

$$uA' - \varpi^i AX \in \begin{bmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o} \end{bmatrix}. \tag{5.24}$$

We first show that ii) follows from i). Consider the homomorphism

$$P(F) \cap M_i \mathrm{K}(\mathfrak{p}^n) M_i^{-1} \longrightarrow F^\times \times F^\times,$$
$$\begin{bmatrix} A & * \\ & uA' \end{bmatrix} \longmapsto (u, \det(A)).$$

Its image is a compact subgroup of $F^\times \times F^\times$, therefore contained in $\mathfrak{o}^\times \times \mathfrak{o}^\times$. This shows $u \in \mathfrak{o}^\times$ and $\det(A) \in \mathfrak{o}^\times$. By (5.23) and (5.24) we get

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \begin{bmatrix} \mathfrak{p}^{n-i} & \mathfrak{p}^{-i} \\ \mathfrak{p}^{n-i} & \mathfrak{p}^{n-i} \end{bmatrix} + \begin{bmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o} \end{bmatrix} = \begin{bmatrix} \mathfrak{o} & \mathfrak{p}^{-i} \\ \mathfrak{p}^{n-i} & \mathfrak{o} \end{bmatrix},$$

hence $a, d \in \mathfrak{o}$, $b \in \mathfrak{p}^{-i}$ and $c \in \mathfrak{p}^{n-i}$. Let us first assume that $i < n/2$. Then $bc \in \mathfrak{p}^{n-2i} \subset \mathfrak{p}$, and it follows from $ad - bc \in \mathfrak{o}^\times$ that $a, d \in \mathfrak{o}^\times$. By (5.22) and (5.23) we get

$$uA' - A \in \begin{bmatrix} \mathfrak{p}^{n-i} & \mathfrak{p}^{-i} \\ \mathfrak{p}^{n-i} & \mathfrak{p}^{n-i} \end{bmatrix} + \begin{bmatrix} \mathfrak{p}^i & \mathfrak{p}^{i-n} \\ \mathfrak{p}^i & \mathfrak{p}^i \end{bmatrix} = \begin{bmatrix} \mathfrak{p}^i & \mathfrak{p}^{i-n} \\ \mathfrak{p}^i & \mathfrak{p}^i \end{bmatrix}.$$

Since $A' = \frac{1}{ad-bc} \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$, it follows that $(\frac{u}{ad-bc} - 1)a \in \mathfrak{p}^i$, hence $\frac{u}{ad-bc} \in 1 + \mathfrak{p}^i$. Therefore all the conditions in ii) are fulfilled. — Now assume that $i = n/2$. Then we argue similarly, except that we cannot conclude from $(\frac{u}{ad-bc} - 1)a \in \mathfrak{p}^{n/2}$ that $\frac{u}{ad-bc} \in 1 + \mathfrak{p}^{n/2}$. But we also have $(\frac{u}{ad-bc} - 1)d \in \mathfrak{p}^{n/2}$, so we can make our conclusion if a or d is a unit. Assume therefore that neither a nor d is a unit. Then it follows from $ad - bc \in \mathfrak{o}^\times$ that $v(b) = -n/2$ and $v(c) = n/2$. Adding (5.21) and (5.24) gives

$$uA' + A \in \begin{bmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o} \end{bmatrix}.$$

By the lower left coefficient we see $(\frac{u}{ad-bc} - 1)c \in \mathfrak{p}^n$, and consequently $\frac{u}{ad-bc} \in 1 + \mathfrak{p}^{n/2}$.

Now assume u and A are given such that ii) holds. We define

$$X := \varpi^{-i} \Big(\frac{u}{ad - bc} A^{-1} \begin{bmatrix} a & \varpi^i b \\ \varpi^i c & d \end{bmatrix} - 1 \Big).$$

It is then easy to verify that X has the required form and that (5.21) to (5.24) are fulfilled. □

Theorem 5.2.2. *Let π be an admissible representation of* $\mathrm{GL}(2,F)$*, and let σ be a character of $F^\times$. We assume that $\omega_\pi\sigma^2 = 1$, so that the induced representation $\pi \rtimes \sigma$ has trivial central character.*

i) The minimal paramodular level of $\pi \rtimes \sigma$ is $n := N_{\sigma\pi} + 2a(\sigma)$. Assume that this number is finite, so that $N_{\sigma\pi}$ is finite, $N_{\sigma\pi} = a(\sigma\pi)$, and $n = a(\sigma\pi) + 2a(\sigma)$. Then

$$\dim\big((\pi \rtimes \sigma)^{\mathrm{K}(\mathfrak{p}^n)}\big) = \dim\big((\sigma\pi)^{\Gamma_1(\mathfrak{p}^{a(\sigma\pi)})}\big).$$

In particular, if π is irreducible, then $\dim\big((\pi \rtimes \sigma)^{\mathrm{K}(\mathfrak{p}^n)}\big) = 1$.

ii) If π is irreducible and infinite-dimensional, so that $N_{\sigma\pi}$ is finite and $N_{\sigma\pi} = a(\sigma\pi)$, then

$$\dim\big((\pi \rtimes \sigma)^{\mathrm{K}(\mathfrak{p}^m)}\big) = \begin{cases} \Big[\dfrac{(m-n+2)^2}{4}\Big] & \text{if } m \geq n = a(\sigma\pi) + 2a(\sigma), \\ 0 & \text{if } m < n. \end{cases}$$

iii) If $\pi = \chi\mathbf{1}_{\mathrm{GL}(2)}$ and $\sigma\chi$ is unramified, then

$$\dim\big((\pi \rtimes \sigma)^{\mathrm{K}(\mathfrak{p}^m)}\big) = \begin{cases} \Big[\dfrac{m}{2}\Big] - a(\sigma) + 1 & \text{if } m \geq n = 2a(\sigma), \\ 0 & \text{if } m < n. \end{cases}$$

iv) If $\pi = \chi\mathbf{1}_{\mathrm{GL}(2)}$ and $\sigma\chi$ is ramified, then $\pi \rtimes \sigma$ has no non-zero paramodular invariant vectors.

Proof. Let V be the space of π. The standard space of $\pi \rtimes \sigma$ consists of smooth functions $f:\ \mathrm{GSp}(4,F) \to V$ such that

$$f(\begin{bmatrix} A & * \\ & uA' \end{bmatrix} g) = |u^{-1}\det(A)|^{3/2}\sigma(u)\pi(A)f(g).$$

Assume f is $\mathrm{K}(\mathfrak{p}^m)$ invariant for some m. By Proposition 5.1.2, f is determined by its values on M_i, $0 \leq i \leq \frac{m}{2}$, where we put $M_0 := 1$. For $0 \leq i \leq \frac{m}{2}$ set $v_i := f(M_i)$. Let $0 \leq i \leq \frac{m}{2}$, and assume that $v_i \neq 0$. Using

$$f(\begin{bmatrix} A & * \\ & uA' \end{bmatrix} M_i k) = |u^{-1}\det(A)|^{3/2}\sigma(u)\pi(A)v_i$$

for $\begin{bmatrix} A & * \\ & uA' \end{bmatrix} \in P(F)$, $k \in \mathrm{K}(\mathfrak{p}^m)$ we conclude that we must have

$$\sigma(u)\pi(A)v_i = v_i \qquad \text{for all } \begin{bmatrix} A & * \\ & uA' \end{bmatrix} \in P(F) \cap M_i\mathrm{K}(\mathfrak{p}^m)M_i^{-1}. \tag{5.25}$$

It follows from Lemma 5.2.1 that for given $u \in (1+\mathfrak{p}^i) \cap \mathfrak{o}^\times$ there exists an element $\begin{bmatrix} 1 & * \\ & u1 \end{bmatrix}$ in $P(F) \cap M_i\mathrm{K}(\mathfrak{p}^m)M_i^{-1}$. Since $v_i \neq 0$ we have $\sigma((1+\mathfrak{p}^i) \cap \mathfrak{o}^\times) = 1$. In other words, we have

$$i \geq a(\sigma). \tag{5.26}$$

It further follows from Lemma 5.2.1 and (5.25) that

$$\sigma(\det(A))\pi(A)v_i = v_i \qquad \text{for all } A \in \begin{bmatrix} \mathfrak{o} & \mathfrak{p}^{-i} \\ \mathfrak{p}^{m-i} & \mathfrak{o} \end{bmatrix}, \ \det(A) \in \mathfrak{o}^\times. \tag{5.27}$$

Therefore, the representation $\sigma\pi$, which has trivial central character, has a non-zero vector fixed under $\begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^{m-2i} & \mathfrak{o}^\times \end{bmatrix}$. In case $i < \frac{m}{2}$ this is the congruence subgroup $\mathrm{GL}(2,\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^{m-2i} & \mathfrak{o} \end{bmatrix}$. In case $i = \frac{m}{2}$ it follows that $\sigma\pi$ is unramified. In any case we have $N_{\sigma\pi} < \infty$ and $N_{\sigma\pi} = a(\sigma\pi) \leq m - 2i$. Together with (5.26) it follows that $m \geq a(\sigma\pi) + 2a(\sigma)$, if there exists a non-zero $\mathrm{K}(\mathfrak{p}^m)$ invariant f. In other words, if a non-zero paramodular vector exists, then $N_{\sigma\pi} < \infty$ and the minimal paramodular level is greater than or equal to $n = N_{\sigma\pi} + 2a(\sigma) = a(\sigma\pi) + 2a(\sigma)$.

It also follows from the above considerations that if $n < \infty$, then a non-zero $\mathrm{K}(\mathfrak{p}^n)$-invariant function f must be supported on $P(F)M_{a(\sigma)}\mathrm{K}(\mathfrak{p}^n)$, and on this double coset we necessarily have

$$f(\begin{bmatrix} A & * \\ & uA' \end{bmatrix} M_{a(\sigma)}k) = |u^{-1}\det(A)|^{3/2}\sigma(u)\pi(A)v \qquad \text{for } k \in \mathrm{K}(\mathfrak{p}^n), \tag{5.28}$$

where $v \in V$ is invariant under $(\sigma\pi)(B)$, $B \in \begin{bmatrix} \mathfrak{o} & \mathfrak{p}^{-a(\sigma)} \\ \mathfrak{p}^{a(\sigma\pi)+a(\sigma)} & \mathfrak{o} \end{bmatrix}$, $\det(B) \in \mathfrak{o}^\times$. Conversely, if $n < \infty$, then we can *define* f by formula (5.28). Thus, if $n < \infty$, then the space of $\mathrm{K}(\mathfrak{p}^n)$-invariant f is isomorphic to the space of $v \in V$ invariant under $(\sigma\pi)(B)$ with B as above. But this space in turn is isomorphic to the space of $v \in V$ invariant under $(\sigma\pi)(\Gamma_1(\mathfrak{p}^{a(\sigma\pi)}))$. This proves i).

The more general formula in ii) follows also from the above considerations. We saw that a non-zero $\mathrm{K}(\mathfrak{p}^m)$ invariant function f must be supported on the cosets $P(F)M_i\,\mathrm{K}(\mathfrak{p}^m)$ with

$$a(\sigma) \leq i \leq \frac{1}{2}\big(m - a(\sigma\pi)\big).$$

For each i we shall count how many possibilities we have for the vector $v_i = f(M_i)$. We saw that a necessary and sufficient condition for v_i is that it is invariant under $(\sigma\pi)(B)$ for $B \in \begin{bmatrix} \mathfrak{o} & \mathfrak{p}^{-i} \\ \mathfrak{p}^{m-i} & \mathfrak{o} \end{bmatrix}$, $\det(B) \in \mathfrak{o}^\times$. The space of such v_i is isomorphic to the space of vectors in V invariant under the congruence subgroup $\Gamma_0(\mathfrak{p}^{m-2i})$ (defined by the "c" coefficient being in $\mathfrak{p}^{m-2i}$). By the well-known structure of oldforms in an irreducible, admissible, infinite-dimensional GL(2) representation, the dimension of this space is $m - 2i - a(\sigma\pi) + 1$. The dimension of the space of possible $\mathrm{K}(\mathfrak{p}^m)$ invariant functions f is therefore given by

$$\sum_{i=a(\sigma)}^{[(m-a(\sigma\pi))/2]} (m-2i-a(\sigma\pi)+1).$$

This sum is evaluated to $\left[\frac{(m-n+2)^2}{4}\right]$, proving ii). The formula in iii) is derived similarly; instead of $m-2i-a(\sigma\pi)+1$ all the dimensions are 1. Assertion iv) already follows from (5.27). □

Corollary 5.2.3. *Let χ_1, χ_2, σ be characters of $F^\times$ such that $\chi_1\chi_2\sigma^2 = 1$. Put $n := a(\chi_1\sigma) + a(\chi_2\sigma) + 2a(\sigma)$. Then*

$$\dim\left((\chi_1 \times \chi_2 \rtimes \sigma)^{\mathrm{K}(\mathfrak{p}^m)}\right) = \begin{cases} \left[\dfrac{(m-n+2)^2}{4}\right] & \text{if } m \geq n, \\ 0 & \text{if } m < n. \end{cases}$$

In particular, the minimal paramodular level of $\chi_1 \times \chi_2 \rtimes \sigma$ is n, and the space of $\mathrm{K}(\mathfrak{p}^n)$-invariant vectors is one-dimensional.

Proof. We apply Theorem 5.2.2 with $\pi = \chi_1 \times \chi_2$. The induced representation $\sigma\pi = \sigma\chi_1 \times \sigma\chi_2$ is known to have a unique newform of level $a(\sigma\chi_1) + a(\sigma\chi_2)$. □

5.3 Representations of Type IIIb and IVc

Next we treat the case of induction from the Klingen parabolic subgroup. There are two subcases that require special attention; we consider these in the present section and examine the general case in the next section.

The special cases are representations of type IIIb and IVc. Both representations are constituents of induced representations $\chi \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}$ with characters χ and σ of $F^\times$. The transformation property for the functions in the standard space of this representation is

$$f(\begin{bmatrix} y & * & * \\ & A & * \\ & & y^{-1}\det(A) \end{bmatrix} g) = |y^2 \det(A)^{-1}|\chi(y)\sigma(\det(A))f(g).$$

In particular, we have

$$f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & a & \\ & & & a \end{bmatrix} g) = \sigma(a)f(g) \qquad \text{for all } a \in \mathfrak{o}^\times.$$

If f is paramodular of level m, then, by Proposition 5.1.2, it is determined by its values on 1, s_1, and L_i, $1 \leq i \leq m-1$. Each of these elements commutes with the matrix $\mathrm{diag}(1,1,a,a)$. It follows that $f(g) = \sigma(a)f(g)$ for each $a \in \mathfrak{o}^\times$ and each of these representatives g. Hence, if σ is ramified, any $\mathrm{K}(\mathfrak{p}^m)$ invariant f must be zero. As a consequence, *IIIb and IVc do not have paramodular vectors if σ is ramified*; we proved this earlier using different methods in Theorem 3.4.3. In the following we consider unramified σ, which will be written in the form $\sigma = \nu^{-s/2}$ for some $s \in \mathbb{C}$. We then must have $\chi = \nu^s$ in order for the center to act trivially. Hence we shall consider the induced representation $\nu^s \rtimes \nu^{-s/2}\mathbf{1}_{\mathrm{GSp}(2)}$. Its standard space V_s consists of functions $f:\ \mathrm{GSp}(4,F) \to \mathbb{C}$ with the transformation property

$$f(\begin{bmatrix} y & * & & * \\ & A & & * \\ & & & y^{-1}\det(A) \end{bmatrix} g) = |y^2 \det(A)^{-1}|^{s/2+1} f(g) \qquad (5.29)$$

for all $g \in \mathrm{GSp}(4,F)$. The representation $\nu^s \rtimes \nu^{-s/2}\mathbf{1}_{\mathrm{GSp}(2)}$ is reducible if and only if $q^s \in \{1, q^2, q^{-2}\}$. If $q^s \in \{1, q^2, q^{-2}\}$, then $\nu^s \rtimes \nu^{-s/2}\mathbf{1}_{\mathrm{GSp}(2)}$ has length two, and its constituents are tabulated in (2.11) and (2.9).

Type IIIb

We shall first investigate unramified representations of type IIIb. In the case of trivial central character they are of the form $\nu^s \rtimes \nu^{-s/2}\mathbf{1}_{\mathrm{GSp}(2)}$ with $q^s \notin \{q^{-2}, 1, q^2\}$. Let $f_0 \in V_s$ be the unique $\mathrm{GSp}(4,\mathfrak{o})$ invariant vector. By Proposition 5.1.2, a $\mathrm{K}(\mathfrak{p}^m)$ invariant vector in $\nu^s \rtimes \nu^{-s/2}\mathbf{1}_{\mathrm{GSp}(2)}$ is determined by its values on

$$\mathbf{1}, \quad u_m, \quad L_1, \quad \ldots \quad L_{m-1}, \qquad (5.30)$$

and it is easy to see that these values can be prescribed arbitrarily. Hence $\dim V_s(m) = m+1$.

Lemma 5.3.1. *We identify the space $V_s(m)$ with $\mathbb{C}^{m+1}$ via evaluating functions at the elements listed in (5.30). Then the $(m+1) \times m$–matrices corresponding to the linear operators θ and θ' from $V_s(m-1)$ to $V_s(m)$ are given as follows. For $m = 1$,*

$$\theta: \begin{bmatrix} q^{s/2+1}(q+1) \\ q(q^{s+1}+1) \end{bmatrix}, \qquad \theta': \begin{bmatrix} q(q^{s+1}+1) \\ q^{s/2+1}(q+1) \end{bmatrix}.$$

For $m = 2$,

$$\theta: \begin{bmatrix} q^{s/2+1}(q+1) & 0 \\ 0 & q(q^{s+1}+1) \\ q^{s/2+1} & q \end{bmatrix}, \qquad \theta': \begin{bmatrix} q(q^{s+1}+1) & 0 \\ 0 & q^{s/2+1}(q+1) \\ q & q^{s/2+1} \end{bmatrix}.$$

For $m \geq 3$,

$$\theta : \; q^{s/2+1}\begin{bmatrix} q+1 & & & & & \\ 0 & q^{s/2+1}+q^{-s/2} & & & & \\ \vdots & q^{1+(1-m)(s/2+1)} & 1 & & & \\ & & q & \ddots & & \\ & & & \ddots & 1 & \\ 1 & 0 & \dots & 0 & q & \end{bmatrix},$$

$$\theta' : \; q\begin{bmatrix} q^{s+1}+1 & & & & & \\ 0 & q^{s/2+1}+q^{s/2} & & & & \\ \vdots & q^{s+1+(1-m)(s/2+1)} & 1 & & & \\ & & q^{s+1} & \ddots & & \\ & & & \ddots & 1 & \\ 1 & 0 & \dots & 0 & q^{s+1} & \end{bmatrix}.$$

Proof. These are straightforward computations using Lemma 3.2.2; no special matrix identities are needed. □

Lemma 5.3.2. *Let f_0 be the* $\mathrm{GSp}(4,\mathfrak{o})$ *invariant vector in the representation* $\nu^s \rtimes \nu^{-s/2}\mathbf{1}_{\mathrm{GSp}(2)}$*, where* $q^s \notin \{q^{-2}, 1\}$*. Then the vectors*

$$\theta^m f_0, \quad \theta^{m-1}\theta' f_0, \quad \dots \quad {\theta'}^m f_0 \tag{5.31}$$

are a basis for $V_s(m)$ *for any* $m \geq 1$.

Proof. We prove the statement by induction on m. The matrices for θ, θ' : $V_s(m-1) \to V_s(m)$, $m \geq 1$, were calculated in Lemma 5.3.1: write $\theta = [c_1 \cdots c_m]$ and $\theta' = [c'_1 \cdots c'_m]$ where the c_i and the c'_i are column vectors. Suppose $m = 1$. Since $V(0)$ is spanned by f_0 and $\dim V(1) = 2$, to prove the statement for $m = 1$ it suffices to check that θf_0 and $\theta' f_0$ are linearly independent. This follows from the fact that for $m = 1$ we have $\det(A_1) \neq 0$, $A_1 = [c_1 c'_1]$; this uses $q^s \neq q^{-2}$ and $q^s \neq 1$. Now assume that the statement holds for all k with $1 \leq k < m$. To prove the statement for m it will suffice to prove that the dimension of $\theta V_s(m-1) + \theta' V_s(m-1)$ is $\dim V_s(m) = m+1$; this is because $\theta V_s(m-1) + \theta' V_s(m-1)$ is spanned by the $m+1$ vectors in (5.31) thanks to the induction hypothesis. If $m = 2$, then $\det(A_2) \neq 0$, where $A_2 = [c_1 c_2 c'_3]$, implies that $\theta V_s(m-1) + \theta' V_s(m-1)$ has dimension $m+1$; this uses $q^s \neq 1$. If $m \geq 3$, then $\det A_m \neq 0$, where $A_m = [c_1 c'_2 c_3 \cdots c_m c'_m]$, implies that the dimension of $\theta V_s(m-1) + \theta' V_s(m-1)$ is $m+1$; this uses $q^s \neq 1$. □

It follows from $\dim V_s(m) = m+1$ that the vectors ηf_0, $\theta^2 f_0$, $\theta\theta' f_0$ and ${\theta'}^2 f_0$ of level 2 cannot be linearly independent. Indeed we have:

Lemma 5.3.3. *Assume that* $q^s \notin \{q^{-2}, 1\}$*. Then the trivial representation* $\mathbf{1}_{\mathrm{GSp}(4)}$ *is not contained in* $\nu^s \rtimes \nu^{-1/2}\mathbf{1}_{\mathrm{GSp}(2)}$*, so that the discussion at the* *end of Sect. 3.1 and after Lemma 3.2.3 about the space of all paramodular vectors V_{para} applies. In the algebra of operators generated by θ, θ' and η on V_{para} in $\nu^s \rtimes \nu^{-s/2}\mathbf{1}_{\mathrm{GSp}(2)}$ we have the relation*

$$q^{1-s}(1-q^s)^2\eta + \theta^2 - q^{-s/2}(1+q^s)\theta\theta' + \theta'^2 = 0.$$

Proof. The trivial representation is not contained in $\nu^s \rtimes \nu^{-1/2}\mathbf{1}_{\mathrm{GSp}(2)}$ if this representation is irreducible. If it is reducible, then necessarily $q^s = q^2$; in this case the trivial representation is a quotient and not a subrepresentation of $\nu^s \rtimes \nu^{-1/2}\mathbf{1}_{\mathrm{GSp}(2)}$ by (2.9). By Lemma 5.3.2 (and the fact that all our operators commute), it is enough to verify this relation on the spherical vector f_0. This is a calculation using the matrices from Lemma 5.3.1; the matrix for $\eta:\, V_s(0) \to V_s(2)$ is easily computed to be the transpose of $[q^{s+2}, q^{s+2}, q^{s+2}]$. □

We summarize:

Proposition 5.3.4. *In the representation $\nu^s \rtimes \nu^{-s/2}\mathbf{1}_{\mathrm{GSp}(2)}$ of type IIIb, where $q^s \notin \{q^{-2}, 1, q^2\}$, the space of $\mathrm{K}(\mathfrak{p}^m)$ invariant vectors has dimension $m+1$. If f_0 is a non-zero $\mathrm{GSp}(4,\mathfrak{o})$ invariant vector, then this space is spanned by*

$$\theta^m f_0, \quad \theta^{m-1}\theta' f_0, \quad \ldots \quad \theta'^m f_0.$$

In particular, the Oldforms Principle holds for $\nu^s \rtimes \nu^{-s/2}\mathbf{1}_{\mathrm{GSp}(2)}$.

Type IVc

As above we consider the induced representation $\nu^s \rtimes \nu^{-s/2}\mathbf{1}_{\mathrm{GSp}(2)}$. In the case that $q^s \in \{q^{-2}, q^2\}$ it is not irreducible but has length 2. For $q^s = q^2$ it contains the representation $L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2})$ of type IVc as a subrepresentation, and $\mathbf{1}_{\mathrm{GSp}(4)}$ as the quotient; see (2.9).

Proposition 5.3.5. *The representation $L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$ of type IVc, where σ is unramified and quadratic, has minimal paramodular level $\mathfrak{p}$. The dimension of the space of $\mathrm{K}(\mathfrak{p}^m)$ invariant vectors is m, for any $m \geq 1$. If f_1 is a non-zero $\mathrm{K}(\mathfrak{p})$ invariant vector, then this space is spanned by*

$$\theta^{m-1} f_1, \quad \theta^{m-2}\theta' f_1, \quad \ldots \quad \theta'^{m-1} f_1.$$

In particular, the Oldforms Principle holds for representations of type IVc.

Proof. We may assume that $\sigma = 1_{F^\times}$. As we saw before, the space of $\mathrm{K}(\mathfrak{p}^m)$ invariant vectors in the standard space V_s of the full induced representation $\nu^s \rtimes \nu^{-s/2}\mathbf{1}_{\mathrm{GSp}(2)}$ has dimension $m+1$. For $q^s = q^2$ we have IVc and $\mathbf{1}_{\mathrm{GSp}(4)}$ as irreducible constituents. It follows that the dimension of the space of $\mathrm{K}(\mathfrak{p}^m)$ invariant vectors in IVc is m, for any $m \geq 0$. In particular, the minimal paramodular level of this representation is $\mathfrak{p}$. Now consider $\nu^s \rtimes \nu^{-s/2}\mathbf{1}_{\mathrm{GSp}(2)}$

with $q^s = q^{-2}$. This induced representation contains $\sigma\mathbf{1}_{\mathrm{GSp}(4)}$ as a subrepresentation (as is evident from (5.29)) and IVc as the resulting quotient. The formulas in Lemma 5.3.1 are still valid. As in the proof of Lemma 5.3.2 we define the matrix A_m, $m \geq 1$. We see that A_m is invertible, except in the case $m = 1$. It is clear that $m = 1$ is exceptional, since θf_0 and $\theta' f_0$ must both lie in the one-dimensional space V_{triv} of the subrepresentation $\mathbf{1}_{\mathrm{GSp}(4)}$. However, for $m \geq 2$ it follows that the images of θ and θ', as operators on $V_s(m-1)$, together span all of $V_s(m)$. Consequently, for $m \geq 2$, the images of θ and θ' as operators

$$V_s(m-1)/V_{\mathrm{triv}} \longrightarrow V_s(m)/V_{\mathrm{triv}},$$

span all of $V_s(m)/V_{\mathrm{triv}}$. But these quotients are the spaces of paramodular vectors of level $m-1$ resp. m in IVc. It follows that the space of $\mathrm{K}(\mathfrak{p}^m)$ invariant vectors in IVc is indeed spanned by the vectors of level m obtained by repeatedly applying θ and θ' to f_1 and taking linear combinations. □

5.4 Induction from the Klingen Parabolic Subgroup

Now let π be an admissible representation of $\mathrm{GL}(2,F)$ admitting a central character ω_π, and let χ be a character of $F^\times$ with $\chi\omega_\pi = 1$, so that the induced representation $\chi \rtimes \pi$ of $\mathrm{GSp}(4,F)$ has trivial central character. In this section we consider paramodular vectors in $\chi \rtimes \pi$. We will prove that the minimal paramodular level of $\chi \rtimes \pi$ is $n := 2N_\pi$, and generally determine the paramodular vectors in $\chi \rtimes \pi$. Here, N_π is defined as at the beginning of Sect. 5.2, so that N_π is the smallest non-negative integer n such that $\pi^{\Gamma_1(\mathfrak{p}^n)} \neq 0$ if such an n exists, and $N_\pi = \infty$ otherwise. That n is the minimal paramodular level is again consistent with, and motivated by, our main results. For suppose, for example, that $\chi \rtimes \pi$ is irreducible and paramodular. Then π is irreducible; let $\mu : W_F' \to \mathrm{GL}(2,\mathbb{C})$ be its L-parameter. By Sect. 2.4, this L-parameter is defined by

$$w \longmapsto \begin{bmatrix} \chi(w)\det(\mu(w))\mu(w)' & \\ & \mu(w) \end{bmatrix} \in \mathrm{GSp}(4,\mathbb{C}). \tag{5.32}$$

The conductor of this parameter is $a(\chi\pi) + a(\pi) = 2a(\pi) = 2N_\pi$. Theorem 7.5.9 asserts that this is the minimal paramodular level of $\pi \rtimes \sigma$.

Lemma 5.4.1. *Let n be a positive integer. Let $y \in F^\times$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2,F)$ be given. For $1 \leq i < n$ the following statements are equivalent:*

i) There exist $\lambda, \mu, \kappa \in F$ such that

$$\begin{bmatrix} y & & \\ & A & \\ & & y^{-1}\det(A) \end{bmatrix} \begin{bmatrix} 1 & \lambda & \mu & \kappa \\ & 1 & & \mu \\ & & 1 & -\lambda \\ & & & 1 \end{bmatrix} \in Q(F) \cap L_i \mathrm{K}(\mathfrak{p}^n) L_i^{-1}. \tag{5.33}$$

Here L_i is as in (5.5).

ii) The following conditions are satisfied:

- $y, a, d \in \mathfrak{o}^\times$.
- $a - y \in \begin{cases} \mathfrak{p}^{n-i} & \text{if } 2i \geq n, \\ \mathfrak{p}^i & \text{if } 2i < n. \end{cases}$
- $b \in \begin{cases} \mathfrak{o} & \text{if } 2i \geq n, \\ \mathfrak{p}^{2i-n} & \text{if } 2i < n. \end{cases}$
- $c \in \mathfrak{p}^{n-i}$.

Proof. Multiplying out matrices, it is easy to see that the conditions in ii) are necessary for i) to hold. To see that they are sufficient, assume first that $2i \geq n$. Then we can put $\lambda = \mu = \kappa = 0$, and a computation shows that (5.33) holds. In case that $2i < n$, we put

$$\lambda = \varpi^{-i}(ay^{-1} - 1), \qquad \mu = \varpi^{-i}by^{-1}, \qquad \kappa = \varpi^{-2i}by^{-1},$$

and again it is easy to check that (5.33) holds. □

Theorem 5.4.2. *Let χ be a character of $F^\times$, and let π be an admissible representation of $\mathrm{GL}(2, F)$. We assume that $\chi\omega_\pi = 1$, so that the induced representation $\chi \rtimes \pi$ has trivial central character.*

i) The minimal paramodular level of $\chi \rtimes \pi$ is $n := 2N_\pi$. Assume that this number is finite, so that N_π is finite and $N_\pi = a(\pi)$. Then

$$\dim\left((\chi \rtimes \pi)^{\mathrm{K}(\mathfrak{p}^n)}\right) = \dim\left(\pi^{\Gamma_1(\mathfrak{p}^{a(\pi)})}\right)$$

(the congruence subgroup Γ_1 was defined in (5.19)). In particular, if π is irreducible, then $\dim\left((\chi \rtimes \pi)^{\mathrm{K}(\mathfrak{p}^n)}\right) = 1$.

ii) If π is irreducible and infinite-dimensional, then N_π is finite, so that $N_\pi = a(\pi)$, and

$$\dim\left((\chi \rtimes \pi)^{\mathrm{K}(\mathfrak{p}^m)}\right) = \begin{cases} \left[\dfrac{(m-n+2)^2}{4}\right] & \text{if } m \geq n = 2a(\pi), \\ 0 & \text{if } m < n. \end{cases}$$

iii) If $\pi = \sigma\mathbf{1}_{\mathrm{GL}(2)}$ and σ is unramified, then

$$\dim\left((\chi \rtimes \pi)^{\mathrm{K}(\mathfrak{p}^m)}\right) = m + 1 \qquad \text{for any } m \geq n = 0.$$

iv) If $\pi = \sigma\mathbf{1}_{\mathrm{GL}(2)}$ and σ is ramified, then $\chi \rtimes \pi$ has no non-zero paramodular invariant vectors.

Proof. Statements iii) and iv) have already been proven in Sect. 5.3. To prove i) and ii), assume first that $N_\pi \geq 1$. Let V be the space of π. The standard space of $\chi \rtimes \pi$ consists of smooth functions $f : \mathrm{GSp}(4, F) \to V$ such that

$$f(\begin{bmatrix} y & * & & * \\ & A & & * \\ & & & y^{-1}\det(A) \end{bmatrix} g) = |y^2 \det(A)^{-1}|\, \chi(y)\, \pi(A) f(g).$$

Let m be a non-negative integer, and assume that f is $\mathrm{K}(\mathfrak{p}^m)$ invariant for some m. By Proposition 5.1.2, f is determined by its values on 1, s_1, and L_i, $1 \leq i < m$. If r is one of these elements, and if $v_r := f(r)$, then we must have

$$\chi(y)\pi(A)v_r = v_r \quad \text{for all } \begin{bmatrix} y & * & & * \\ & A & & * \\ & & & y^{-1}\det(A) \end{bmatrix} \in Q(F) \cap r\mathrm{K}(\mathfrak{p}^m)r^{-1}. \quad (5.34)$$

In particular, it follows that v_1 is $\mathrm{GL}(2,\mathfrak{o})$ invariant. Since we have assumed $N_\pi \geq 1$, we conclude $v_1 = 0$. Moreover, we have

$$\begin{bmatrix} 1 & & & \\ & a & b\varpi^{-m} & \\ & c\varpi^m & d & \\ & & & ad-bc \end{bmatrix} \in Q(F) \cap s_1\mathrm{K}(\mathfrak{p}^m)s_1 \quad \text{for } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2,\mathfrak{o}),$$

from which it follows that $v_{s_1} = 0$. Thus f is non-zero at most on the cosets represented by the L_i, $1 \leq i < m$. We write $v_i := f(L_i)$ for $1 \leq i < m$. Let $1 \leq i < m$ and assume that $v_i \neq 0$ and $2i \geq m$. It follows from (5.34) and Lemma 5.4.1 that

$$\pi(\begin{bmatrix} a & b \\ c & d \end{bmatrix})v_i = \omega_\pi(a)v_i \qquad \text{for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^{m-i} & \mathfrak{o}^\times \end{bmatrix}. \quad (5.35)$$

Therefore $N_\pi < \infty$ and $m - i \geq N_\pi = a(\pi)$. It also follows from (5.34) and Lemma 5.4.1 that

$$\pi(\begin{bmatrix} a & b \\ c & d \end{bmatrix})v_i = v_i \qquad \text{for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \begin{bmatrix} 1+\mathfrak{p}^i & \mathfrak{o} \\ \mathfrak{p}^i & \mathfrak{o}^\times \end{bmatrix}, \quad (5.36)$$

and therefore $i \geq a(\pi)$. We conclude that $m \geq 2a(\pi)$. Assume that $v_i \neq 0$ for some $1 \leq i < m$ with $2i < m$. Then it follows from (5.34) and Lemma 5.4.1 that

$$\pi(\begin{bmatrix} a & b \\ c & d \end{bmatrix})v_i = \omega_\pi(a)v_i \qquad \text{for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{p}^{2i-m} \\ \mathfrak{p}^{m-i} & \mathfrak{o}^\times \end{bmatrix}. \quad (5.37)$$

Since

$$\begin{bmatrix} \mathfrak{o}^\times & \mathfrak{p}^{2i-m} \\ \mathfrak{p}^{m-i} & \mathfrak{o}^\times \end{bmatrix} = \begin{bmatrix} \varpi^{2i-m} & \\ & 1 \end{bmatrix} \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^i & \mathfrak{o}^\times \end{bmatrix} \begin{bmatrix} \varpi^{-(2i-m)} & \\ & 1 \end{bmatrix},$$

we conclude that $N_\pi < \infty$ and $i \geq N_\pi = a(\pi)$. Since $m > 2i$ and $2i \geq 2a(\pi) = n$, we have $m > n = 2a(\pi)$. Thus, for a non-zero $\mathrm{K}(\mathfrak{p}^m)$ invariant f to exist, we must have $m \geq n = 2a(\pi)$. Moreover, if $n < \infty$, then a $\mathrm{K}(\mathfrak{p}^n)$-invariant f

is supported on $Q(F)L_{a(\pi)}\mathrm{K}(\mathfrak{p}^n)$. Using Lemma 5.4.1, it is easy to see that f is well-defined by the formula

$$f(\begin{bmatrix} y & * & & * \\ & A & & * \\ & & & y^{-1}\det(A) \end{bmatrix} L_{a(\pi)}k) = |y^2\det(A)^{-1}|\,\chi(y)\,\pi(A)v \qquad (k\in\mathrm{K}(\mathfrak{p}^n))$$

if and only if $v\in V$ has the property

$$\pi\begin{bmatrix} a & b \\ c & d \end{bmatrix}v=\omega_\pi(a)v \qquad \text{for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix}\in\begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^{a(\pi)} & \mathfrak{o}^\times \end{bmatrix}.$$

The space of such v is isomorphic to $V^{\Gamma_1(\mathfrak{p}^{a(\pi)})}$, proving i).

For part ii) we shall count dimensions similarly as in Theorem 5.2.2. We saw above that a non-zero $\mathrm{K}(\mathfrak{p}^m)$-invariant function f must be supported on the cosets $Q(F)L_i\,\mathrm{K}(\mathfrak{p}^m)$ with

$$a(\pi)\le i\le m-a(\pi).$$

If $a(\pi)\le i<m/2$, then we have the condition (5.37) on $v_i=f(M_i)$. The dimension of the space of such vectors is $i-a(\pi)+1$. If $m/2\le i\le m-a(\pi)$, then we have the condition (5.35) on $v_i=f(M_i)$, and the dimension of the space of such vectors is $m-i-a(\pi)+1$. Our dimension count therefore reads

$$\sum_{i=a(\pi)}^{[(m-1)/2]}(i-a(\pi)+1)+\sum_{i=[(m+1)/2]}^{m-a(\pi)}(m-i-a(\pi)+1).$$

Evaluating these expressions gives $\big[\frac{(m-n+2)^2}{4}\big]$, proving ii).

If π is unramified, i.e., if $a(\pi)=0$, then $\chi=\omega_\pi^{-1}$ is also unramified. In this case the arguments are similar, except that there are now paramodular vectors supported on the double cosets $Q(F)1\mathrm{K}(\mathfrak{p}^m)$ and $Q(F)s_1\mathrm{K}(\mathfrak{p}^m)$. □

5.5 Saito–Kurokawa Representations

In this section we determine the paramodular vectors in a certain family of irreducible, admissible, non-generic representations of $\mathrm{GSp}(4,F)$ with trivial central character. Studying the paramodular vectors in these representations will allow us to determine the paramodular vectors in representations of type IIb, Vb, Vc, VIc and XIb. The definition of this family requires the following proposition.

Proposition 5.5.1. *Let π be an irreducible, admissible, infinite-dimensional representation of $\mathrm{GL}(2,F)$ with trivial central character, and let σ be a character of $F^\times$. Assume that $\pi\not\cong\nu^{3/2}\times\nu^{-3/2}$. Then the representation $\nu^{1/2}\pi\rtimes\nu^{-1/2}\sigma$ of $\mathrm{GSp}(4,F)$ has a unique irreducible quotient $Q(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$ and* *a unique irreducible subrepresentation* $G(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$. *The representation* $Q(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ *is non-generic,* $G(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ *is generic, and these two representations are the only constituents of* $\nu^{1/2}\pi \rtimes \nu^{-1/2}\sigma$. *Moreover:*

i) If $\pi \cong \chi \times \chi^{-1}$ *for a character* χ *of* $F^\times$*, then*

$$Q(\nu^{1/2}\pi, \nu^{-1/2}\sigma) \cong \chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \chi^{-1}\sigma \qquad (IIb),$$
$$G(\nu^{1/2}\pi, \nu^{-1/2}\sigma) \cong \chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \chi^{-1}\sigma \qquad (IIa).$$

ii) If $\pi \cong \xi\mathrm{St}_{\mathrm{GL}(2)}$ *with a non-trivial quadratic character* ξ *of* $F^\times$*, then*

$$Q(\nu^{1/2}\pi, \nu^{-1/2}\sigma) \cong L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma) \qquad (Vb),$$
$$G(\nu^{1/2}\pi, \nu^{-1/2}\sigma) \cong \delta([\xi, \nu\xi], \nu^{-1/2}\sigma) \qquad (Va).$$

iii) If $\pi \cong \mathrm{St}_{\mathrm{GL}(2)}$*, then*

$$Q(\nu^{1/2}\pi, \nu^{-1/2}\sigma) \cong L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma) \qquad (VIc),$$
$$G(\nu^{1/2}\pi, \nu^{-1/2}\sigma) \cong \tau(S, \nu^{-1/2}\sigma) \qquad (VIa).$$

iv) If π *is supercuspidal, then*

$$Q(\nu^{1/2}\pi, \nu^{-1/2}\sigma) \cong L(\nu^{1/2}\pi, \nu^{-1/2}\sigma) \qquad (XIb),$$
$$G(\nu^{1/2}\pi, \nu^{-1/2}\sigma) \cong \delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma) \qquad (XIa).$$

Proof. Assume $\pi \cong \chi \times \chi^{-1}$ for a character of $F^\times$. We may assume that $e(\chi) \geq 0$; by assumption, $\chi \neq \nu^{\pm 3/2}$. There are isomorphisms

$$\begin{aligned}
\nu^{1/2}\pi \rtimes \nu^{-1/2}\sigma &\cong \nu^{1/2}\chi \times \nu^{1/2}\chi^{-1} \rtimes \nu^{-1/2}\sigma \\
&\cong \nu^{1/2}\chi \rtimes (\nu^{1/2}\chi^{-1} \rtimes \nu^{-1/2}\sigma) \\
&\cong \nu^{1/2}\chi \rtimes (\chi^{-1}\sigma \times \nu^{-1/2}\sigma) \\
&\cong \nu^{1/2}\chi \rtimes (\nu^{-1/2}\sigma \times \chi^{-1}\sigma) \\
&\cong \nu^{1/2}\chi \rtimes (\nu^{-1/2}\chi \rtimes \chi^{-1}\sigma) \\
&\cong \nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \chi^{-1}\sigma.
\end{aligned}$$

Here, the fourth step is justified because $\chi^{-1}\sigma \times \nu^{-1/2}\sigma$ is irreducible; this uses $e(\chi) \geq 0$ and $\chi \neq \nu^{3/2}$. There is an exact sequence of GSp(4) representations

$$0 \to \chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \chi^{-1}\sigma \to \nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \chi^{-1}\sigma \to \chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \chi^{-1}\sigma \to 0.$$

Since $\chi^2 \neq \nu^{\pm 1}$ and $\chi \neq \nu^{\pm 3/2}$ by assumption, $\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \chi^{-1}\sigma$ and $\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \chi^{-1}\sigma$ are irreducible and of type IIa and IIb, respectively. Moreover, as can be seen from the expression of these two representations as Langlands quotients from Sect. 2.2, $\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \chi^{-1}\sigma \ncong \chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \chi^{-1}\sigma$. Considering the possibilities for $e(\chi)$, and using the above isomorphisms one sees that $\nu^{1/2}\pi \rtimes \nu^{-1/2}\sigma$ is a standard module; see Sect. 8 of [Tad] for the definition. Since $\nu^{1/2}\pi \rtimes \nu^{-1/2}\sigma$ is a standard module, it admits a unique irreducible quotient, which is $\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \chi^{-1}\sigma$. Therefore, $\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \chi^{-1}\sigma$ is the unique irreducible subrepresentation of $\nu^{1/2}\pi \rtimes \nu^{-1/2}\sigma$. Finally, by Theorem 2 of [Rod], $\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \chi^{-1}\sigma$ and $\nu^{1/2}\pi \rtimes \nu^{-1/2}\sigma$ each have nonzero, unique up to scalars, Whittaker functionals. It follows that $\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \chi^{-1}\sigma$ is generic and $\mathbf{1}_{\mathrm{GL}(2)} \rtimes \chi^{-1}\sigma$ is non-generic. This proves all the assertions about $\nu^{1/2}\pi \rtimes \nu^{-1/2}\sigma$. The arguments for the other cases are similar. $\square$

We now define the *Saito–Kurokawa representations* of $\mathrm{GSp}(4,F)$ to be the representations $Q(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ for π an irreducible, admissible, infinite-dimensional representation of $\mathrm{GL}(2,F)$ with trivial central character such that $\pi \ncong \nu^{3/2} \times \nu^{-3/2}$, and σ a character of $F^\times$. By Proposition 5.5.1, the Saito–Kurokawa representations of $\mathrm{GSp}(4,F)$ include all the representations of type IIb, Vb, VIc, XIb; all the representations of type Vc are also included because $Q(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\xi\sigma)$ is the type Vc representation $L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\xi\sigma)$ for all non-trivial quadratic characters of $F^\times$ and characters σ of $F^\times$. Evidently, the representation $Q(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ has central character σ^2, so that $Q(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ has trivial central character if and only if $\sigma^2 = 1$.

Turning to the problem of determining paramodular vectors in Saito–Kurokawa representations, the following lemma gives a necessary condition for a Saito–Kurokawa representation to admit non-zero paramodular vectors.

Lemma 5.5.2. *Let π be an irreducible, admissible, infinite-dimensional representation of $\mathrm{GL}(2,F)$ with trivial central character such that $\pi \ncong \nu^{3/2} \times \nu^{-3/2}$, and let σ be a character of $F^\times$ with $\sigma^2 = 1$. If $Q(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ is paramodular, then σ is unramified.*

Proof. This follows from Theorem 3.4.3. One can also use the results already proven in this chapter to prove some cases. In the case of IIb it also follows from Theorem 5.2.2 iv) since this representation is equal to $\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \chi^{-1}\sigma$. In the case of Vb it also follows from Theorem 5.2.2 iv) since this representation is a constituent of $\nu^{1/2}\xi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\xi\sigma$. In the case of VIc it follows from Theorem 5.4.2 iv) since this representation is a constituent of $1_{F^\times} \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}$. $\square$

The Minimal Level

Let (π, V_π) be an irreducible, admissible, infinite-dimensional representation of $\mathrm{GL}(2,F)$ with trivial central character such that $\pi \ncong \nu^{3/2} \times \nu^{-3/2}$. For a complex parameter s let V_s be the standard space of the representation $\nu^s\pi \rtimes \nu^{-s}$. Explicitly, V_s is the space of smooth functions $f : G(F) \to V_\pi$ that satisfy the transformation property

$$f(\begin{bmatrix} A & * \\ & uA' \end{bmatrix} g) = |u^{-1}\det(A)|^{s+3/2}\,\pi(A)f(g), \qquad \begin{bmatrix} A & * \\ & uA' \end{bmatrix} \in P(F).$$

Let $n = a(\pi)$ be the level of π. By Theorem 5.2.2 there is an essentially unique non-zero $\mathrm{K}(\mathfrak{p}^n)$-invariant vector f_0 in V_s, and no paramodular vectors at better levels. By the proof of Theorem 5.2.2, this newform f_0 is supported on $P(F)\mathrm{K}(\mathfrak{p}^n)$, and is on this double coset given by

$$f_0(\begin{bmatrix} A & * \\ & uA' \end{bmatrix} k) = |u^{-1}\det(A)|^{s+3/2}\,\pi(A)v_0, \qquad k \in \mathrm{K}(\mathfrak{p}^n), \tag{5.38}$$

where $v_0 \in V_\pi$ is $\Gamma_1(\mathfrak{p}^n)$-invariant (hence v_0 is the unique newform for π). We shall now apply the level raising operators $\theta : V_s(n) \to V_s(n+1)$ and $\theta' : V_s(n) \to V_s(n+1)$ to f_0.

Lemma 5.5.3. *Let the notations be as above. We have:*

i) $(\theta f_0)(1) = q^{s+3/2}v_0 + q\,\pi(\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix})v_0$.
ii) $(\theta f_0)(M_i) = 0$ *for* $1 \le i \le \frac{n+1}{2}$.
iii) $(\theta' f_0)(1) = q^{s+3/2}\,\pi(\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix})v_0 + qv_0$.
iv) $(\theta' f_0)(M_i) = 0$ *for* $1 \le i \le \frac{n+1}{2}$.

Proof. By Lemma 3.2.2 i) we have for any $g \in \mathrm{GSp}(4,F)$

$$(\theta f_0)(g) = f_0(g \begin{bmatrix} 1 &&& \\ & 1 && \\ && \varpi & \\ &&& \varpi \end{bmatrix}) + \sum_{c \in \mathfrak{o}/\mathfrak{p}} f_0(g \begin{bmatrix} 1 &&& \\ & \varpi && \\ && 1 & \\ &&& \varpi \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 & c\varpi^{-1} & \\ && 1 & \\ &&& 1 \end{bmatrix}).$$

For $g = 1$ we obtain the formula in i) by the transformation properties of f_0. To prove ii), assume that $n \ge 1$. For $g = M_i$, where $1 \le i \le (n+1)/2$, we get

$$(\theta f_0)(M_i) = q^{s+3/2} f_0(M_{i-1}) + \pi(\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix}) \sum_{c \in \mathfrak{o}/\mathfrak{p}} f_0(M_i \begin{bmatrix} 1 &&& \\ & 1 & c\varpi^{-1} & \\ && 1 & \\ &&& 1 \end{bmatrix}).$$

Using

$$M_i \begin{bmatrix} 1 &&& \\ & 1 & c\varpi^{-1} & \\ && 1 & \\ &&& 1 \end{bmatrix} = \begin{bmatrix} 1 & \varpi^{-i} && c^{-1}\varpi^{1-2i} \\ & -c\varpi^{-1} && \\ && -c^{-1}\varpi & c^{-1}\varpi^{1-i} \\ &&& 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 &&& \\ & 1 && \\ -c\varpi^{i-1} && 1 & \\ & -c\varpi^{i-1} && 1 \end{bmatrix} \begin{bmatrix} 1 &&& -c^{-1}\varpi^{1-2i} \\ & -c^{-1}\varpi & -1 & \\ & 1 && \\ &&& 1 \end{bmatrix}, \tag{5.39}$$

we get

$$\begin{aligned}
(\theta f_0)(M_i) &= q^{s+3/2} f_0(M_{i-1}) + \pi(\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix}) f_0(M_i) \\
&\quad + \sum_{c \in (\mathfrak{o}/\mathfrak{p})^\times} |\varpi^{-1}|^{s+3/2} \pi(\begin{bmatrix} 1 & \varpi^{-i} \\ & -c \end{bmatrix}) f_0(\begin{bmatrix} 1 & & & \\ & 1 & & \\ -c\varpi^{i-1} & & 1 & \\ & -c\varpi^{i-1} & & 1 \end{bmatrix}) \\
&= q^{s+3/2} f_0(M_{i-1}) + \pi(\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix}) f_0(M_i) \\
&\quad + q^{s+3/2} \sum_{c \in (\mathfrak{o}/\mathfrak{p})^\times} \pi(\begin{bmatrix} 1 & \varpi^{-i} \\ & -c \end{bmatrix}) f_0(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \varpi^{i-1} & & 1 & \\ & \varpi^{i-1} & & 1 \end{bmatrix}) \\
&= q^{s+3/2} f_0(M_{i-1}) + \pi(\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix}) f_0(M_i) \\
&\quad + q^{s+3/2} \sum_{c \in (\mathfrak{o}/\mathfrak{p})^\times} \pi(\begin{bmatrix} 1 & c\varpi^{-i} \\ & 1 \end{bmatrix}) f_0(M_{i-1}) \\
&= q^{s+3/2} \sum_{c \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & c\varpi^{-i} \\ & 1 \end{bmatrix}) f_0(M_{i-1}) + \pi(\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix}) f_0(M_i). \qquad (5.40)
\end{aligned}$$

Assume first that $n \geq 2$. Then the last term in (5.40) vanishes; if $1 \leq i \leq n/2$ it vanishes because f_0 is supported on $P(F)\mathrm{K}(\mathfrak{p}^n)$, and if $i = (n+1)/2$ it vanishes for the same reason because the identity (5.10) shows that $M_{(n+1)/2}$ and $M_{(n-1)/2}$ determine the same double coset in $P(F)\backslash G(F)/\mathrm{K}(\mathfrak{p}^n)$. Hence, if $n \geq 2$,

$$(\theta f_0)(M_i) = q^{s+3/2} \sum_{c \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & c\varpi^{-i} \\ & 1 \end{bmatrix}) f_0(M_{i-1}).$$

If $2 \leq i \leq (n+1)/2$, this is zero because f_0 is supported on $P(F)\mathrm{K}(\mathfrak{p}^n)$. For $i = 1$ we use (5.18) to compute $f_0(M_0) = q^{-n(s+3/2)}\varepsilon(\frac{1}{2}, \pi)v_0$, and consequently

$$(\theta f_0)(M_1) = q^{(1-n)(s+3/2)} \varepsilon\big(\frac{1}{2}, \pi\big) \sum_{c \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & c\varpi^{-1} \\ & 1 \end{bmatrix}) v_0. \qquad (5.41)$$

One checks easily that this vector in the GL(2) representation π is invariant under $\begin{bmatrix} \mathfrak{o}^\times & \mathfrak{p}^{-1} \\ & \mathfrak{o}^\times \end{bmatrix}$ and $\begin{bmatrix} 1 & \\ \mathfrak{p}^n & 1 \end{bmatrix}$. Therefore, $\pi(\begin{bmatrix} \varpi & \\ & 1 \end{bmatrix})(\theta f_0)(M_1)$ is invariant under the subgroup generated by $\begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ & \mathfrak{o}^\times \end{bmatrix}$ and $\begin{bmatrix} 1 & \\ \mathfrak{p}^{n-1} & 1 \end{bmatrix}$. This subgroup contains $\Gamma_1(\mathfrak{p}^{n-1})$; since n is the minimal level of π, it follows that the expression in (5.41) must be zero.

Now assume that $n=1$. Then $\pi=\chi\mathrm{St}_{\mathrm{GL}(2)}$ with an unramified quadratic character χ. We have to check ii) only for $i=1$, in which case (5.40) says

$$(\theta f_0)(M_1)=q^{s+3/2}\sum_{c\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&c\varpi^{-1}\\&1\end{bmatrix})f_0(M_0)+\pi(\begin{bmatrix}1&\\&\varpi\end{bmatrix})f_0(M_1). \qquad (5.42)$$

Since f_0 is $\mathrm{K}(\mathfrak{p})$-invariant, we have $f_0(M_1)=f_0(1)=v_0$. By (5.18), $f_0(M_0)=q^{-(s+3/2)}\varepsilon(\frac{1}{2},\pi)v_0$. The paper [Sch1] contains the explicit form of the newform v_0 in a standard model for $\chi\mathrm{St}_{\mathrm{GL}(2)}$. Using this explicit form, it is easy to compute $\varepsilon(1/2,\pi)=-\chi(\varpi)$, and to evaluate the sum. The result is $(\theta f_0)(M_1)=0$ (the summation amounts to applying a Hecke operator; see our Table 6.1 on p. 215).

The computations for iii) and iv) are similar, using Lemma 3.2.2 ii) and the matrix identity

$$M_i\begin{bmatrix}1&&&c\varpi^{-n-1}\\&1&&\\&&1&\\&&&1\end{bmatrix}=\begin{bmatrix}-c\varpi^{-1}&&&\\\varpi^{n-i}&1&c^{-1}\varpi^{n+1-2i}&\\&&1&\\&&c^{-1}\varpi^{n+1-i}&-c^{-1}\varpi\end{bmatrix}$$
$$\begin{bmatrix}1&&&\\&1&&\\-c\varpi^{i-1}&&1&\\&-c\varpi^{i-1}&&1\end{bmatrix}\begin{bmatrix}-c^{-1}\varpi&&&-\varpi^{-n}\\&1&-c^{-1}\varpi^{n+1-2i}&\\&&1&\\\varpi^{n}&&&\end{bmatrix}. \qquad (5.43)$$

□

Lemma 5.5.4. *Using the same notations as above, the following two statements are equivalent.*

i) θf_0 *and* $\theta' f_0$ *are linearly dependent.*
ii) The complex parameter s *has the property that* $q^{2s+1}=1$.

In this case we have $\theta f_0=\pm\theta' f_0$.

Proof. This follows easily from Lemma 5.5.3. □

Proposition 5.5.5. *Let* π *be an irreducible, admissible, infinite-dimensional representation of* $\mathrm{GL}(2,F)$ *with trivial central character of level* n *such that* $\pi\not\cong\nu^{3/2}\times\nu^{-3/2}$, *and let* σ *be an unramified character of* $F^\times$ *such that* $\sigma^2=1$.

i) The Saito–Kurokawa representation $Q(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$ *(type IIb, Vb, VIc or XIb) has a paramodular vector of level* n*, unique up to scalars, and no paramodular vectors at lower levels.*
ii) The generic constituent $G(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$ *of* $\nu^{1/2}\pi\rtimes\nu^{-1/2}$ *(type IIa, Va, VIa or XIa) has a paramodular vector of level* $n+1$*, unique up to scalars, and no paramodular vectors at lower levels.*

Proof. i) We may assume that $\sigma = 1$. We consider the following induced representations:

$$V_{1/2} = \nu^{1/2}\pi \rtimes \nu^{-1/2} = \underbrace{G(\nu^{1/2}\pi, \nu^{-1/2})}_{\text{sub}} + \underbrace{Q(\nu^{1/2}\pi, \nu^{-1/2})}_{\text{quot}},$$

$$V_{-1/2} = \nu^{-1/2}\pi \rtimes \nu^{1/2} = \underbrace{Q(\nu^{1/2}\pi, \nu^{-1/2})}_{\text{sub}} + \underbrace{G(\nu^{1/2}\pi, \nu^{-1/2})}_{\text{quot}}.$$

We know by Theorem 5.2.2 that the minimal level of both induced representations above is $n = a(\pi)$. More precisely, we have $\dim V_{\pm 1/2}(n) = 1$ and $\dim V_{\pm 1/2}(n+1) = 2$. The essentially unique newform in V_s is given by (5.38). Suppose that our assertion is wrong. Then $G(\nu^{1/2}\pi, \nu^{-1/2})$ has a unique newform at level n. In other words, the function f_0 lies in the subspace $G(\nu^{1/2}\pi, \nu^{-1/2})$ of $V_{1/2}$. By Lemma 5.5.4, the vectors θf_0 and $\theta' f_0$ are linearly independent. Hence, under our assumption, $Q(\nu^{1/2}\pi, \nu^{-1/2})$ has no paramodular vectors at level n or $n+1$.

Now let $\tilde{V}$ be the submodule of $V_{-1/2}$ carrying the non-generic representation $Q(\nu^{1/2}\pi, \nu^{-1/2})$, and let $p: V_{-1/2} \to V_{-1/2}/\tilde{V}$ be the projection. Let f_1 be the newform of level n in $V_{-1/2}$. Since, under our assumption, $\tilde{V}$ has no paramodular vectors at level n, we have $p(f_1) \neq 0$. By Lemma 5.5.4, the vectors θf_1 and $\theta' f_1$ are linearly dependent. The projection p commutes with θ and θ', hence $\theta(p(f_1))$ and $\theta'(p(f_1))$ are also linearly dependent. But $f_0 := p(f_1) \neq 0$ is the newform in $V_{-1/2}/\tilde{V} \cong G(\nu^{1/2}\pi, \nu^{-1/2})$, hence this contradicts the linear independence of θf_0 and $\theta' f_0$ shown above.

ii) Changing notation, let $\tilde{V}$ be the submodule of $V_{1/2}$ carrying the representation $G(\nu^{1/2}\pi, \nu^{-1/2})$, and let $p: V_{1/2} \to V_{1/2}/\tilde{V}$ be the projection. Let $f_0 \in V_{1/2}$ be the newform of level n. By Lemma 5.5.4 the vectors θf_0 and $\theta' f_0$ are linearly independent. On the other hand, again by Lemma 5.5.4, $\theta(p(f_0))$ and $\theta'(p(f_0))$ are linearly dependent, since $p(f_0)$ is the newform in $V_{1/2}/\tilde{V} \cong Q(\nu^{1/2}\pi, \nu^{-1/2})$. It follows that some non-zero linear combination of θf_0 and $\theta' f_0$ lies in $\tilde{V}$. □

For later use we note that the arguments in this proof show the following. Let f_1 be the essentially unique $\mathrm{K}(\mathfrak{p}^n)$-invariant vector in $V_{-1/2} = \nu^{-1/2}\pi \rtimes \nu^{1/2}$. Then f_1 lies in the subspace realizing $Q(\nu^{1/2}\pi, \nu^{-1/2})$, and we have $\theta f_1 = \theta' f_1$. Let f_0 be the essentially unique $\mathrm{K}(\mathfrak{p}^n)$-invariant vector in $V_{1/2} = \nu^{1/2}\pi \rtimes \nu^{-1/2}$. Then $\theta f_0 - \theta' f_0$ is a non-zero $\mathrm{K}(\mathfrak{p}^{n+1})$ invariant vector in the subspace realizing $G(\nu^{1/2}\pi, \nu^{-1/2})$.

Paramodular Dimensions

We shall next determine the dimensions of the spaces of paramodular vectors in $Q(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ and $G(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$.

Lemma 5.5.6. *Let π be an irreducible, admissible, infinite-dimensional representation of* $\mathrm{GL}(2,F)$ *with trivial central character of level n such that $\pi \not\cong \nu^{3/2} \times \nu^{-3/2}$, and let σ be an unramified character of $F^\times$ such that $\sigma^2 = 1$. Let f_0 be the newform of level n in $Q(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$. Then, for any $m \geq n$, the vectors $\theta^d\eta^e f_0$, $d, e \geq 0$, $d + 2e = m - n$, are linearly independent.*

Proof. Again we may assume that $\sigma = 1$. We shall realize $Q(\nu^{1/2}\pi, \nu^{-1/2})$ as a subspace of $V_{-1/2}$. Define a linear map

$$\varphi : \sum_{d,e \geq 0} \mathbb{C}\,\theta^d\eta^e f_0 \longrightarrow V_\pi, \qquad f \longmapsto f(1). \tag{5.44}$$

By the formula for f_0 given in (5.38), the vector $\varphi(f_0)$ is the newform in V_π. A straightforward computation (cf. Lemma 5.5.3) shows that

$$\varphi(\eta f) = q\pi(\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix})\varphi(f), \qquad \varphi(\theta f) = q\varphi(f) + q\pi(\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix})\varphi(f).$$

Hence, on the GL(2) side, the η and θ operators correspond to operators producing oldforms from newforms. It is known that these GL(2) operators produce linearly independent vectors, proving that the sum in (5.44) is direct. □

We can now compute all the dimensions of the spaces of paramodular vectors for Saito–Kurokawa representations with σ unramified. Again let π be an irreducible, admissible, infinite-dimensional representation of $\mathrm{GL}(2,F)$ with trivial central character of level n such that $\pi \not\cong \nu^{3/2} \times \nu^{-3/2}$, and let σ be an unramified character of $F^\times$ such that $\sigma^2 = 1$. Let V_Q be the space of $Q(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ and let V_G be the space of $G(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$. By Proposition 5.5.5 i), n is the minimal level of $Q(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$. It follows from Lemma 5.5.6 that for the dimensions of the spaces of paramodular vectors in this representation we have the estimate

$$\dim V_Q(m) \geq \left[\frac{m-n+2}{2}\right] \qquad \text{for any } m \geq n. \tag{5.45}$$

By Proposition 5.5.5 ii), the minimal level of $G(\nu^{1/2}\pi, \nu^{-1/2})$ is $n+1$. By Lemma 4.3.9 we get the estimate

$$\dim V_G(m) \geq \left[\frac{(m-n+1)^2}{4}\right] \qquad \text{for any } m \geq n+1 \tag{5.46}$$

for the dimensions of the spaces of paramodular vectors in $G(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$. Since

$$\left[\frac{m-n+2}{2}\right] + \left[\frac{(m-n+1)^2}{4}\right] = \left[\frac{(m-n+2)^2}{4}\right]$$

is the full dimension of the space of $\mathrm{K}(\mathfrak{p}^m)$ invariant vectors in the induced representation $\nu^{1/2}\pi \rtimes \nu^{-1/2}\sigma$ by Theorem 5.2.2, the estimates in (5.45) and

(5.46) must actually be equalities. By Lemma 5.5.6 we find that the vectors $\theta^d\eta^e f_0$, $d,e\geq 0$, $d+2e=m-n$, span the space of $\mathrm{K}(\mathfrak{p}^m)$ invariant vectors. In particular, the Oldforms Principle holds for Saito–Kurokawa representations.

Lemma 5.5.7. *Let ξ and σ be characters of $F^\times$ such that such that $\xi^2=\sigma^2=1$ and $\xi\neq 1$. Let V be the space of the Vd type representation $L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$. If ξ and σ are unramified, then*

$$\dim V(n)=\frac{1+(-1)^n}{2}\qquad \text{for all } n\geq 0.$$

If ξ or σ are ramified, then the Vd type representation $L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$ has no paramodular vectors.

Proof. We have by (2.10)

$$\begin{aligned}\nu^{1/2}\xi\mathbf{1}_{\mathrm{GL}(2)}\rtimes\nu^{-1/2}\xi\sigma&=\mathrm{Vb}+\mathrm{Vd}\\&=L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)+L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma).\end{aligned}$$

We have by Proposition 5.5.1,

$$Q(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)=L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma).$$

Moreover, Vd is also a constituent of $\nu^{1/2}\xi\mathbf{1}_{\mathrm{GL}(2)}\rtimes\nu^{-1/2}\sigma$; see (2.10). The last statement of the lemma therefore follows from Theorem 5.2.2 iv). Assume that ξ and σ are unramified. Then, by Theorem 5.2.2 iii), the dimension of the space of $\mathrm{K}(\mathfrak{p}^m)$ invariant vectors in $\nu^{1/2}\xi\mathbf{1}_{\mathrm{GL}(2)}\rtimes\xi\nu^{-1/2}\sigma$ is $\left[\frac{m}{2}\right]+1$, for any $m\geq 0$. By what we proved above, the dimension of the space of $\mathrm{K}(\mathfrak{p}^m)$ invariant vectors in the Saito–Kurokawa representation $Q(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$ is $\left[\frac{m+1}{2}\right]$. Since

$$\left[\frac{m}{2}\right]+1-\left[\frac{m+1}{2}\right]=\frac{1+(-1)^m}{2},$$

the assertion follows. □

Atkin–Lehner Eigenvalues

Saito–Kurokawa representations exhibit a special behavior with respect to Atkin–Lehner involutions.

Proposition 5.5.8. *Let π be an irreducible, admissible, infinite-dimensional representation of $\mathrm{GL}(2,F)$ with trivial central character of level n such that $\pi\not\cong\nu^{3/2}\times\nu^{-3/2}$.*

i) The Atkin–Lehner eigenvalue of the newform of level n of the Saito–Kurokawa representation $Q(\nu^{1/2}\pi,\nu^{-1/2})$ (type IIb, Vb, VIc or XIb) is $\varepsilon(1/2,\pi)$.

ii) The Atkin–Lehner eigenvalue of the newform of level $n+1$ of the generic constituent $G(\nu^{1/2}\pi, \nu^{-1/2})$ of $\nu^{1/2}\pi \rtimes \nu^{-1/2}$ (type IIa, Va, VIa or XIa) is $-\varepsilon(1/2, \pi)$.

Proof. i) As above, we shall realize $Q(\nu^{1/2}\pi, \nu^{-1/2})$ as a subrepresentation of $V_{-1/2}$. By (5.38), the newform is supported on $P(F)\mathrm{K}(\mathfrak{p}^n)$ and is on this double coset given by

$$f_0(\begin{bmatrix} A & * \\ & uA' \end{bmatrix}k) = |u^{-1}\det(A)|\, v_0 \qquad (k \in \mathrm{K}(\mathfrak{p}^n)),$$

where v_0 is the newform in the space of π. It is therefore enough to compute $(u_n f_0)(1)$, where u_n is the Atkin–Lehner element. It can be written as

$$u_n = \begin{bmatrix} & & 1 & \\ & & & -1 \\ \varpi^n & & & \\ & -\varpi^n & & \end{bmatrix} = \begin{bmatrix} & 1 & & \\ -\varpi^n & & & \\ & & & 1 \\ & & -\varpi^n & \end{bmatrix} \begin{bmatrix} & & & \varpi^{-n} \\ & & 1 & \\ & 1 & & \\ \varpi^n & & & \end{bmatrix}.$$

Consequently

$$(u_n f_0)(1) = f_0(u_n) = f_0(\begin{bmatrix} & 1 & & \\ -\varpi^n & & & \\ & & & 1 \\ & & -\varpi^n & \end{bmatrix}) = \pi(\begin{bmatrix} & 1 \\ -\varpi^n & \end{bmatrix})v_0.$$

But $\begin{bmatrix} & 1 \\ -\varpi^n & \end{bmatrix}$ is an Atkin–Lehner element of level n for the GL(2) representation π. The newform v_0 has eigenvalue $\varepsilon(1/2, \pi)$ under the action of this element.

ii) We realize the generic constituent $G(\nu^{1/2}\pi, \nu^{-1/2})$ as a subrepresentation of $V_{1/2}$. Let $f_0 \in V_{1/2}$ be the newform of level n in the full induced representation (which is not an element of $G(\nu^{1/2}\pi, \nu^{-1/2})$). As in part i) we compute $u_n f_0 = \varepsilon(1/2, \pi) f_0$. By the remarks following the proof of Proposition 5.5.5 ii), the newform of level $n+1$ in $\delta(\nu^{1/2}\pi, \nu^{-1/2})$ is given by $\theta f_0 - \theta' f_0$. We compute

$$\begin{aligned} u_{n+1}(\theta f_0 - \theta' f_0) &= u_{n+1}(\theta f_0 - u_{n+1}\theta u_n f_0) \\ &= u_{n+1}\theta f_0 - \theta u_n f_0 \\ &= \theta' u_n f_0 - \theta u_n f_0 \\ &= \varepsilon(1/2, \pi)(\theta' f_0 - \theta f_0), \end{aligned}$$

proving our assertion. □

Note: These results on Atkin–Lehner eigenvalues are expected, because the L-parameter of $Q(\nu^{1/2}\pi, \nu^{-1/2})$ is $\varphi_\pi \oplus \varphi_{\mathbf{1}}$ and the L-parameter of $\delta(\nu^{1/2}\pi, \nu^{-1/2})$ is $\varphi_\pi \oplus \varphi_{\mathrm{St}}$.

Summary

We summarize the properties of Saito–Kurokawa representations.

Theorem 5.5.9. *Let π be an irreducible, admissible, infinite-dimensional representation of* $\mathrm{GL}(2,F)$ *with trivial central character of level n such that $\pi \not\cong \nu^{3/2} \times \nu^{-3/2}$, and let σ be a character of $F^\times$ such that $\sigma^2 = 1$. Then $Q(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ has no paramodular vectors if σ is ramified. If σ is unramified, we have the following properties.*

i) The minimal level of $Q(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ is n.
ii) The dimension of the space of paramodular vectors of level m is $[\frac{m-n+2}{2}]$, for any $m \geq n$.
iii) A basis for the space of paramodular vectors of level m is provided by the vectors $\theta^d\eta^e f_0$, $d, e \geq 0$, $d + 2e = m - n$, where f_0 is the unique newform of level n.

If $\sigma = 1$, we have furthermore:

iv) The operator $\theta - \theta'$ is zero on the full space of paramodular vectors. In other words, θ commutes with Atkin–Lehner involutions.
v) Every paramodular vector is an Atkin–Lehner eigenvector with eigenvalue $\varepsilon(1/2, \pi)$.

If σ is the non-trivial, unramified, quadratic character, then the operator $\theta + \theta'$ is zero on the full space of paramodular vectors.

Proof. All the statements have been proved earlier in this section. As for iv) and v), note that we need only verify these statements on the newform, since we know that the Oldforms Principle holds for Saito–Kurokawa representations. □

Thus we see that the map $\pi \mapsto Q(\nu^{1/2}\pi, \nu^{-1/2})$ provides a level-preserving and Atkin–Lehner preserving local Saito–Kurokawa lifting. We saw in the proof of Lemma 5.5.6 that there is a natural map from the space of paramodular vectors in $Q(\nu^{1/2}\pi, \nu^{-1/2})$ to the space of Γ_1 vectors in π. The image is precisely the local analogue of the "certain space" of Skoruppa and Zagier; see [SZ].

Type IVb

Let σ be a character of $F^\times$ such that $\sigma^2 = 1$. We shall now treat the representation $L(\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$ of type IVb, which is a subrepresentation of $\nu^{3/2}\mathbf{1}_{\mathrm{GL}(2)} \rtimes \nu^{-3/2}\sigma$. The quotient is $\sigma\mathbf{1}_{\mathrm{GSp}(4)}$. It follows from Theorem 5.2.2 iv) that the full induced representation, and therefore also IVb, has no paramodular vectors if σ is ramified. We shall therefore assume that $\sigma = 1$. For a complex parameter s consider the induced representation $\nu^s\mathbf{1}_{\mathrm{GL}(2)} \rtimes \nu^{-s}$.

Its standard space V_s consists of functions $f:\ \mathrm{GSp}(4,F)\to\mathbb{C}$ with the transformation property

$$f(\begin{bmatrix} A & * \\ & uA' \end{bmatrix} g) = |u^{-1}\det(A)|^{3/2+s} f(g)$$

for all $g\in\mathrm{GSp}(4,F)$. A paramodular vector of level m is determined by its values on 1 and M_i, $1\le i\le \frac{m}{2}$. These values can be prescribed arbitrarily, so that $\dim V_s(m)=[\frac{m+2}{2}]$. It follows that the space of $\mathrm{K}(\mathfrak{p}^m)$ invariant vectors in $L(\nu^2,\nu^{-1}\mathrm{St}_{\mathrm{GSp}(2)})$ has dimension $[\frac{m}{2}]$. In particular, the minimal level is $\mathfrak{p}^2$. We see that the growth of the dimensions is "Saito–Kurokawa like", even though IVb is not a Saito–Kurokawa representation.

Lemma 5.5.10. *Let $s\in\mathbb{C}$. We identify the space $V_s(m)$ with $\mathbb{C}^{[(m+2)/2]}$ via evaluating functions at 1 and M_i, $1\le i\le\frac{m}{2}$. The operator $\theta:\ V_s(0)\to V_s(1)$ is given by multiplication with the number $q(q^{1/2+s}+1)$. The operator $\theta:\ V_s(1)\to V_s(2)$ is represented by the matrix $\begin{bmatrix} q(q^{1/2+s}+1) \\ q+1\end{bmatrix}$. If $m\ge 4$ is even, then the operator $\theta:\ V_s(m-1)\to V_s(m)$ is represented by the $\frac{m+2}{2}\times\frac{m}{2}$ matrix*

$$\begin{bmatrix} q(q^{1/2+s}+1) &&&&& \\ q^{1+(2-m)(3/2+s)} & 1 &&&& \\ & q^{5/2+s} & \ddots &&& \\ && \ddots & 1 && \\ &&& q^{5/2+s} & 1 \\ &&&& q^{3/2+s}(q+1) \end{bmatrix}.$$

If $m\ge 3$ is odd, then $\theta:\ V_s(m-1)\to V_s(m)$ is represented by the $\frac{m+1}{2}\times\frac{m+1}{2}$ matrix

$$\begin{bmatrix} q(q^{1/2+s}+1) &&&& \\ q^{1+(2-m)(3/2+s)} & 1 &&& \\ & q^{5/2+s} & \ddots && \\ && \ddots & 1 & \\ &&& q^{5/2+s} & 1 \end{bmatrix}.$$

The operator $\eta:\ V_s(m-2)\to V_s(m)$ is represented by the $[\frac{m+2}{2}]\times[\frac{m}{2}]$ matrix

$$\begin{bmatrix} q^{3/2+s} &&& \\ q^{(3-m)(3/2+s)} & 0 && \\ & q^{3/2+s} & \ddots & \\ && \ddots & 0 \\ &&& q^{3/2+s} \end{bmatrix}.$$

Proof. These are straightforward computations using the formulas in the proof of Lemma 5.5.3. □

Proposition 5.5.11. *Let f_2 be the newform of level 2 in $L(\nu^2, \nu^{-1}\mathrm{St}_{\mathrm{GSp}(2)})$. Then, for any $m \geq 2$, the vectors $\theta^d\eta^e f_2$, $d, e \geq 0$, $d+2e = m-2$, are linearly independent and span the space of $\mathrm{K}(\mathfrak{p}^m)$-invariant vectors. In particular, the Oldforms Principle holds for representations of type IVb.*

Proof. Since we already know that the dimension of the space of $\mathrm{K}(\mathfrak{p}^m)$-invariant vectors is $[\frac{m}{2}]$, we just have to prove that this space is spanned by the vectors $\theta^d\eta^e f_2$ with $d + 2e = m - 2$. We realize $L(\nu^2, \nu^{-1}\mathrm{St}_{\mathrm{GSp}(2)})$ as a quotient of $V_{-3/2}$ and use the matrix representations of Lemma 5.5.10. For $m \geq 3$ consider the operators η and θ^2 from $V_s(m-2)$ to $V_s(m)$. Adding the last column of the matrix for η to the matrix for θ^2 yields an invertible matrix, i.e., the image of η and of θ^2 spans all of $V_s(m)$. The same is then true for η and θ^2 considered as operators

$$V_s(m-2)/\mathbf{1}_{\mathrm{GSp}(4)} \longrightarrow V_s(m)/\mathbf{1}_{\mathrm{GSp}(4)}.$$

But these are the spaces of paramodular vectors of level $m-2$ resp. m in $L(\nu^2, \nu^{-1}\mathrm{St}_{\mathrm{GSp}(2)})$, proving our assertion. □

Type VId

Now consider the representation $L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$ of type VId. It is a quotient of the degenerate principal series representation $\nu^{1/2}\mathbf{1}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma$, which by Theorem 5.2.2 iv) implies that it has no paramodular vectors if σ is ramified. Hence assume that σ is trivial. Then $\nu^{1/2}\mathbf{1}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}$ has a non-zero $\mathrm{GSp}(4, \mathfrak{o})$ invariant vector. The other constituent of $\nu^{1/2}\mathbf{1}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}$ is the tempered $\tau(T, \nu^{-1/2})$. We know by Theorem 3.4.3 that $\tau(T, \nu^{-1/2})$ has no paramodular vectors. Therefore the structure of paramodular vectors in $\nu^{1/2}\mathbf{1}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}$ and in $L(\nu, 1_{F^\times} \rtimes \nu^{-1/2})$ is the same.

Proposition 5.5.12. *Let f_0 be the newform of level 0 in $L(\nu, 1_{F^\times} \rtimes \nu^{-1/2})$. Then, for any $m \geq 0$, the vectors $\theta^d\eta^e f_0$, $d, e \geq 0$, $d + 2e = m$, are linearly independent and span the space of $\mathrm{K}(\mathfrak{p}^m)$ invariant vectors. In particular, the Oldforms Principle holds for representations of type VId.*

Proof. The argument is the same as in the proof of Proposition 5.5.11. □

We see that VId is another representation with a Saito–Kurokawa like structure of paramodular vectors, without actually being a Saito–Kurokawa representation.

Characterization of Saito–Kurokawa Representations

Proposition 5.5.13. *Let (τ, V) be an irreducible, admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character that has non-zero paramodular vectors. The following statements are equivalent.*

i) There exists a non-zero paramodular vector $f \in V$ such that $\theta f = \theta' f$.
ii) For <u>*all*</u> *paramodular vectors $f \in V$ we have $\theta f = \theta' f$.*
iii) θ commutes with Atkin–Lehner involutions on the space of paramodular vectors in V.
iv) τ is one of the following representations.
- $\mathbf{1}_{\mathrm{GSp}(4)}$ *(type IVd).*
- $L(\nu^2, \nu^{-1}\mathrm{St}_{\mathrm{GSp}(2)})$ *(type IVb).*
- $L(\nu\xi, \xi \rtimes \nu^{-1/2})$ *with unramified ξ of order 2 (type Vd).*
- $L(\nu, 1_{F^\times} \rtimes \nu^{-1/2})$ *(type VId).*
- $Q(\nu^{1/2}\pi, \nu^{-1/2})$ *with π an irreducible, admissible, infinite-dimensional representation of $\mathrm{GL}(2,F)$ with trivial central character such that $\pi \not\cong \nu^{3/2} \times \nu^{-3/2}$ (type IIb, Vb, VIc or XIb).*

If there exists a non-zero paramodular vector $f \in V$ such that θf and $\theta' f$ are linearly dependent, then necessarily $\theta f = \theta' f$ or $\theta f = -\theta' f$. In the latter case π is a twist of one of the representations in iv) with the unique non-trivial, unramified, quadratic character of $F^\times$.

Proof: Assume there exists a non-zero paramodular vector f such that θf and $\theta' f$ are linearly dependent. We claim that τ is not generic; suppose otherwise. By the η Principle, Theorem 4.3.7, we may assume that $Z(s,f) \neq 0$. By Proposition 4.1.1, $Z(s,\theta f) = q^{-s+3/2}Z(s,f)$ and $Z(s,\theta' f) = qZ(s,f)$. Since θf and $\theta' f$ are linearly dependent and $Z(s,f) \neq 0$, the holomorphic functions q and $q^{-s+3/2}$ are linearly dependent, a contraction. Since non-generic supercuspidals have no paramodular vectors by Theorem 3.4.3, τ must be a constituent of an induced representation. It cannot be of type IIIb by Proposition 5.3.4. It cannot be of type IVc by Proposition 5.3.5. It cannot be of type VIb, VIIIb or IXb since these representations do not have paramodular vectors. Hence τ must be of one of the types listed under iv). In each case we have verified before that the representation in question has no paramodular vectors if it is twisted with a ramified character (for type Vd, the quadratic character ξ must also be unramified in order for paramodular vectors to exist). Hence π is one of the representations in iv), or an unramified twist of such a representation. The condition of trivial central character forces the twisting character to be quadratic.

Let f_0 be the unique newform for any of the representations listed in iv). Then we can easily verify that $\theta f_0 = \theta' f_0$. This is trivial for $\mathbf{1}_{\mathrm{GSp}(4)}$ and type Vd (which has paramodular dimensions $1, 0, 1, 0, \ldots$). For VId it is a very easy computation. For the Saito–Kurokawa representations we verified it in Lemma 5.5.3 (with $s = -1/2$). Now for each of these representations we also verified the Oldforms Principle. Hence, if $\theta - \theta'$ annihilates the newform, it annihilates all paramodular vectors.

If τ is a representation for which $\theta v = \theta' v$ holds for paramodular vectors v, and if σ is the unique non-trivial, unramified, quadratic character, then the twist $\sigma\tau$ has the property that $\theta v = -\theta' v$ on paramodular vectors v. This is immediate from the explicit formulas in Lemma 3.2.2.

The arguments given so far prove the last statements of the proposition, as well as the implications i) $\Rightarrow$ iv) $\Rightarrow$ ii). The implications ii) $\Rightarrow$ i) and ii) $\Leftrightarrow$ iii) are trivial. □

5.6 Summary

The following theorem summarizes our results on new- and oldforms for non-supercuspidal representations proved so far.

Theorem 5.6.1. *Let (π, V) be an irreducible, admissible, non-supercuspidal representation of $\mathrm{GSp}(4, F)$ with trivial central character. For any non-negative integer m let $V(m)$ be the space of $\mathrm{K}(\mathfrak{p}^m)$-invariant vectors.*

i) The dimension $\dim V(m)$ is given as in Table A.12 on page 291.
ii) If π is paramodular, and if N_π is the minimal paramodular level, then $\dim V(N_\pi) = 1$.
iii) If π is generic, then π is paramodular. Generic representations are characterized by the formula

$$\dim V(m) = \left[\frac{(m - N_\pi + 2)^2}{4}\right] \qquad \text{for } m \geq N_\pi. \tag{5.47}$$

iv) The Oldforms Principle holds for π: Every oldform can be obtained by repeatedly applying the level raising operators θ, θ' and η to the newform and taking linear combinations.

Proof. We know from Theorem 4.4.1 that generic representations are paramodular. This follows also from the results of the current chapter, as the following arguments will show. For type I representations we obtain formula (5.47) from Corollary 5.2.3; for type IIa we obtain it from Theorem 5.2.2 ii); for type IIIa we obtain it from Theorem 5.4.2 ii). The dimension formulas for types IIb and IIIb follow from Theorem 5.2.2 iii), iv) and Theorem 5.4.2 iii), iv), respectively.

The dimensions for group IV are easily obtained since this group contains the trivial representation. We know from (2.9) how the full induced representation $\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma$ decomposes into irreducible constituents. We further know the dimensions for $\nu^{3/2}\mathbf{1}_{\mathrm{GL}(2)} \rtimes \nu^{-3/2}\sigma$ from Theorem 5.2.2, and the dimensions for $\nu^2 \rtimes \nu^{-1}\sigma\mathbf{1}_{\mathrm{GSp}(2)}$ from Theorem 5.4.2. Hence we obtain the dimensions for IVb and IVc. Subtracting everything from the dimensions of the full induced representation (Corollary 5.2.3), we get the dimensions for IVa. In particular, (5.47) holds for twists of the Steinberg representation.

The dimensions for groups V and VI are obtained similarly. For group V, the starting point is Lemma 5.5.7, which gives the dimensions for Vd. The rest follows from (2.10) and Theorem 5.2.2. For group VI we use (2.11) and the fact that VIb has no paramodular vectors; see Theorem 3.4.3.

The results for the generic group VII representations are immediate from Theorem 5.4.2 ii). The same theorem, together with Theorem 3.4.3, give the dimensions for groups VIII and IX.

Theorem 5.2.2 ii) gives the dimensions for the generic group X representations. The dimensions for XIa and XIb in the unramified case were obtained in Sect. 5.5. In the ramified case they follow from Theorem 3.4.3. The proofs of i), ii), and iii) are now complete.

iv) The Oldforms Principle holds for generic representations by the dimension formula (5.47) together with Proposition 4.3.9. It holds for representations of type IIb, Vb,c, VIc and XIb by iii) of Theorem 5.5.9. It holds for representations of type IIIb by Proposition 5.3.4, for type IVb by Proposition 5.5.11, for type IVc by Proposition 5.3.5, for type Vd by Lemma 5.5.7, and for type VId by Proposition 5.5.12. It trivially holds for type IVd, the one-dimensional representations. Representations of type VIb, VIIIb and XIb are never paramodular by Theorem 3.4.3. This covers all cases. □

5.7 Atkin–Lehner Eigenvalues

In this final section we prove part of one of our main results, Theorem 7.5.9. Let π be an irreducible, admissible, non-supercuspidal, paramodular representation of $\mathrm{GSp}(4,F)$ with trivial central character, and let φ_π be the L-parameter attached to π as in Sect. 2.4. By ii) of Theorem 5.6.1, if N_π is the minimal paramodular level of π, then $\dim V(N_\pi) = 1$. Let $v \in V(N_\pi)$ be non-zero. Since $V(N_\pi)$ is one-dimensional, we have $\pi(u_{N_\pi})v = \varepsilon_\pi v$ where u_{N_π} is the Atkin–Lehner element as in (2.2). We call ε_π the Atkin–Lehner eigenvalue of v. We will prove that

$$\varepsilon(s,\varphi_\pi) = \varepsilon_\pi q^{-N_\pi(s-1/2)}.$$

In other words, the two invariants N_π and ε_π of a newform in π determine the ε-factor of the L-parameter of π.

The first step is to compute all the Atkin–Lehner eigenvalues of the paramodular newforms in non-supercuspidal representations. We recall some facts from the GL(2) theory. Let (π, V) be an irreducible, admissible, infinite-dimensional representation of $\mathrm{GL}(2,F)$ with trivial central character. Let n be the least positive integer such that V contains a non-zero vector v invariant under $\Gamma_0(\mathfrak{p}^n) = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2,\mathfrak{o}) : \ c \in \mathfrak{p}^n\}$. Then the dimension of the space of $\Gamma_0(\mathfrak{p}^n)$ invariant vectors is 1. The Atkin–Lehner element $\begin{bmatrix} & 1 \\ \varpi^n & \end{bmatrix}$ acts on this space by multiplication with a sign, and this sign is given by the value of the ε-factor of π at 1/2,

$$\pi(u_n)v = \varepsilon(1/2,\pi)v. \tag{5.48}$$

A proof of this fact can be found in [Sch1].

Lemma 5.7.1. *i) Let σ be a character of $F^\times$, and let π be an irreducible, admissible, infinite-dimensional representation of $\mathrm{GL}(2,F)$. Assume that $\omega_\pi\sigma^2 = 1$, so that the induced representation $\pi \rtimes \sigma$ has trivial central character. Then the minimal paramodular level of $\pi \rtimes \sigma$ is $n = a(\sigma\pi) + 2a(\sigma)$, and $\dim V(n) = 1$. The Atkin–Lehner element u_n acts on $V(n)$ with eigenvalue $\sigma(-1)\varepsilon(1/2,\sigma\pi)$.*

ii) Let χ be a character of $F^\times$, and let π be an irreducible, admissible, infinite-dimensional representation of $\mathrm{GL}(2,F)$. Assume that $\omega_\pi\chi = 1$, so that the induced representation $\chi \rtimes \pi$ has trivial central character. Then the minimal paramodular level of $\chi \rtimes \pi$ is $n = 2a(\pi)$, and $\dim V(n) = 1$. The Atkin–Lehner element u_n acts on $V(n)$ with eigenvalue $\chi(-1)$.

Proof. The assertions about the minimal level and the fact that $\dim V(n) = 1$ have already been proven in Theorems 5.2.2 and 5.4.2. In the proof of these theorems, an explicit form of a non-zero $\mathrm{K}(\mathfrak{p}^n)$-invariant vector in the standard models of the induced representations was given. In the Siegel case $\pi \rtimes \sigma$, a $\mathrm{K}(\mathfrak{p}^n)$-invariant vector f is supported on $P(F)M_{a(\sigma)}\mathrm{K}(\mathfrak{p}^n)$, and in the Klingen case $\chi \rtimes \pi$ such a vector is supported on $Q(F)L_{a(\pi)}\mathrm{K}(\mathfrak{p}^n)$; see (5.5) for notation. We shall carry out the calculation only in the Siegel case. In this case $f(M_{a(\sigma)})$ is the local newform in the representation space of π. Since

$$u_n = \begin{bmatrix} & & 1 & \\ & & & -1 \\ \varpi^n & & & \\ & -\varpi^n & & \end{bmatrix} = \begin{bmatrix} & 1 & & \\ \varpi^n & & & \\ & & & 1 \\ & & \varpi^n & \end{bmatrix} \begin{bmatrix} & & & -\varpi^{-n} \\ & & 1 & \\ & -1 & & \\ \varpi^n & & & \end{bmatrix},$$

we have

$$\begin{aligned} (\pi(u_n)f)(M_{a(\sigma)}) &= f(M_{a(\sigma)}u_n) \\ &= f(M_{a(\sigma)} \begin{bmatrix} & 1 & & \\ \varpi^n & & & \\ & & & 1 \\ & & \varpi^n & \end{bmatrix}) \\ &= f(\begin{bmatrix} & 1 & & \\ \varpi^n & & & \\ & & & 1 \\ & & \varpi^n & \end{bmatrix} M_{a(\sigma)}) \\ &= \sigma(-1)(\sigma\pi)(\begin{bmatrix} & 1 \\ \varpi^n & \end{bmatrix})f(M_{a(\sigma)}) \\ &= \sigma(-1)\varepsilon(1/2,\sigma\pi)f(M_{a(\sigma)}). \end{aligned}$$

For the last equality see (5.48); note that $\sigma\pi$ has trivial central character. This concludes the proof. □

Theorem 5.7.2. *Let (π, V) be an irreducible, admissible, non-supercuspidal, paramodular representation of* $\mathrm{GSp}(4,F)$ *with trivial central character. Let N_π be the minimal paramodular level; by ii) of Theorem 5.6.1,* $\dim V(N_\pi) = 1$. *Let ε_π be the eigenvalue of the Atkin–Lehner involution $\pi(u_{N_\pi})$ on $V(N_\pi)$. Then ε_π is as given in Table A.12 in the appendix.*

Proof. The entry for type I representations follows from Lemma 5.7.1 i) with $\pi = \chi_1 \times \chi_2$. The same lemma gives the Atkin–Lehner eigenvalue for type IIa representations. See Proposition 5.5.8 i) for type IIb. Eigenvalues for type IIIa representations follow from Lemma 5.7.1 ii). By Theorem 3.4.3, type IIIb representations are not paramodular if σ is ramified. If σ is unramified, then the trivial central character condition $\chi\sigma^2 = 1$ forces χ also to be unramified. Hence the representation has a $\mathrm{GSp}(4,\mathfrak{o})$ invariant vector, whose Atkin–Lehner eigenvalue is one.

Consider representations of type IV, V or VI. If the inducing characters are unramified, so that the representation in question is Iwahori-spherical, its Atkin–Lehner eigenvalue can easily be determined by direct computations in the induced models. See Theorem 3.2.9 and Table A.13.

By Theorem 3.4.3, the representations IVb,c,d are not paramodular if σ is ramified. In view of the middle row of Table (2.9), we can use Lemma 5.7.1 ii) to compute the eigenvalue of $\sigma\mathrm{St}_{\mathrm{GSp}(4)}$. If σ is unramified, all the group IV representations are Iwahori-spherical.

The eigenvalues for the Saito–Kurokawa representations Vb,c were determined in Proposition 5.5.8. Making use of Table (2.10) and Theorem 3.4.3, one can determine the eigenvalues for Va, except in the Iwahori-spherical case, where they follow from direct computations.

Since VIb is never paramodular by Theorem 3.4.3, one can use the middle row of Table (2.11) and Lemma 5.7.1 ii) to compute the Atkin–Lehner eigenvalue of VIa. Representations of type VIc are Saito–Kurokawa and were treated in Proposition 5.5.8. Representations of type VId are either not paramodular or Iwahori-spherical.

For group VII we can use Lemma 5.7.1 ii). For groups VIII and IX we can use the same lemma, together with the fact that VIIIb and IXb are never paramodular (Theorem 3.4.3).

Eigenvalues for group X follow from Lemma 5.7.1 i). The same lemma can be used for group XIa if σ is ramified, since in this case XIb is not paramodular by Theorem 3.4.3. If σ is unramified, the eigenvalues for both XIa and XIb follow from Proposition 5.5.8. □

Theorem 5.7.3. *Let (π, V) be an irreducible, admissible, non-supercuspidal, paramodular representation of* $\mathrm{GSp}(4,F)$ *with trivial central character. Let $\varphi_\pi : W_F' \to \mathrm{GSp}(4,\mathbb{C})$ be the L-parameter assigned to π as in Sect. 2.4. Let N_π be the minimal paramodular level of π, and let ε_π be the eigenvalue of the Atkin–Lehner involution on the one-dimensional space $V(N_\pi)$. Then*

$$\varepsilon(s, \varphi_\pi) = \varepsilon_\pi q^{-N_\pi(s-1/2)}.$$

Proof. This follows by comparing Table A.12 and Table A.9. □

For completeness we also give the Atkin–Lehner eigenvalues on spaces of oldforms. Let N_π be the minimal level of an irreducible, admissible, paramodular representation (π, V) of $\mathrm{GSp}(4,F)$ with trivial central character. For any $n \geq N_\pi$ let $V_\pm(n)$ be the subspace of vectors $v \in V(n)$ such that $\pi(u_n)v = \pm v$. Let $\varepsilon \in \pm 1$ be the Atkin–Lehner eigenvalue of the newform. The following table gives the dimensions of the spaces $V_\varepsilon(n)$ and $V_{-\varepsilon}(n)$ for each $n \geq N_\pi$.

$\dim V(n)$	$\dim V_\varepsilon(n)$	$\dim V_{-\varepsilon}(n)$
$\left[\frac{(n-N_\pi+2)^2}{4}\right]$	$\frac{1}{2}(\left[\frac{n-N_\pi}{2}\right]+1)(\left[\frac{n-N_\pi}{2}\right]+2)$	$\frac{1}{2}\left[\frac{n-N_\pi+1}{2}\right](\left[\frac{n-N_\pi+1}{2}\right]+1)$
$\left[\frac{n-N_\pi+2}{2}\right]$	$\left[\frac{n-N_\pi+2}{2}\right]$	0
$n-N_\pi+1$	$\left[\frac{n-N_\pi+2}{2}\right]$	$\left[\frac{n-N_\pi+1}{2}\right]$

(5.49)

The second row of the table applies to representations for which $\dim V(n)$ is given by the formula $\left[\frac{(n-N_\pi+2)^2}{4}\right]$. These are exactly the generic representations; see Table A.12 (the proof for supercuspidal representations will only be complete after we proved uniqueness at the minimal level in Theorem 7.5.1). To prove these formulas, note that, by Proposition 4.3.9, $V(n)$ is spanned by the linearly independent vectors

$$(\theta+\theta')^i(\theta-\theta')^j\eta^k v, \qquad i+j+2k=n-N_\pi,$$

where $v \in V(N_\pi)$ is non-zero. The operators η and $\theta+\theta'$ preserve Atkin–Lehner eigenvalues, while $\theta-\theta'$ changes them. Hence $V_\varepsilon(n)$ is spanned by the vectors $(\theta+\theta')^i(\theta-\theta')^j\eta^k v$ with even j, and $V_{-\varepsilon}(n)$ is spanned by those vectors with odd j. It is easy to count the possibilities.

The third row in table (5.49) applies to Saito–Kurokawa type representations (IIb, IVb, Vb, Vc, VIc, VId, XIb). In these cases the operators θ and η are enough to generate the spaces of oldforms, and they both preserve Atkin–Lehner eigenvalues. See Proposition 5.5.13 for proofs.

The last row in table (5.49) applies to representations of type IIIb and IVc. In these cases the two operators θ and θ' suffice to generate all oldforms. See Sect. 5.3 for proofs.

The only paramodular representations missing from table (5.49) are the unramified twists of the trivial representation and the paramodular representations of type Vd. In the latter case, the dimensions of the spaces $V(n)$ are 1, 0, 1, 0, . . . , and all Atkin–Lehner eigenvalues are the same.

6

Hecke Operators

Let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character. Assume that π is paramodular. In the previous chapter we proved that, for non-supercuspidal π, the space $V(N_\pi)$ is one-dimensional, where N_π is the minimal paramodular level; we will eventually prove this for all paramodular representations. Thanks to uniqueness, any linear operator on $V(N_\pi)$ will act by a scalar, and thus define an invariant. One example will be the Atkin–Lehner eigenvalue ε_π. In this chapter we introduce the paramodular Hecke algebra and study the action of two of its elements on $V(n)$. When $n = N_\pi$, then the eigenvalues of these two operators will define two more important invariants λ_π and μ_π. As we will show in the next chapter, N_π, ε_π, λ_π and μ_π will determine the relevant L- and ε-factors of the representation. Besides ultimately defining the invariants λ_π and μ_π, our two Hecke operators will in fact be an important tool for proving uniqueness at the minimal level and other results.

In the first three sections of this chapter we introduce the two relevant Hecke operators and study their algebraic properties as operators on $V(n)$. These properties are proved in the context of arbitrary smooth representations for which the center acts trivially. Although the context is general, these calculations are rather long. Most of the remainder of this chapter is devoted to the computation of Hecke eigenvalues in non-supercuspidal representations. All non-supercuspidal representations except those of type VII, VIII and IX are treated; the eigenvalues for these omitted cases and for supercuspidal representations will be computed in Theorem 7.5.2 in the next chapter. Finally, in the last section we prove that, if the representation is unitary, our two Hecke operators are self-adjoint. Along with other results this implies that the Hecke operators are simultaneously diagonalizable at the minimal level. This will be an important ingredient in the proof of uniqueness at the minimal level for supercuspidal representations; see Theorem 7.5.1.

6.1 Two Hecke Operators

Let K be an open-compact subgroup of $\mathrm{GSp}(4,F)$. We denote by $\mathcal{H}(K)$ the Hecke algebra of K, i.e., the vector space of left and right K-invariant, compactly supported functions on $\mathrm{GSp}(4,F)$, with the product given by the convolution

$$(T\cdot T')(g)=\int\limits_{\mathrm{GSp}(4,F)} T(gh^{-1})T'(h)\,dh.$$

Here we choose the Haar measure such that $\mathrm{vol}(K)=1$.

Assume that (π,V) is a smooth representation of $\mathrm{GSp}(4,F)$ with trivial central character. Then $\mathcal{H}(K)$ acts on the space V^K of K-invariant vectors by

$$Tv=\int\limits_{\mathrm{GSp}(4,F)} T(g)\pi(g)v\,dg.$$

Again, we fix the Haar measure for which K has volume 1. If T is the characteristic function of KhK, and if $KhK=\bigsqcup_i h_iK$ with representatives h_i, then

$$Tv=\sum_i \pi(h_i)v. \tag{6.1}$$

An alternative formula is

$$Tv=\mathrm{vol}(K\cap hKh^{-1})^{-1}\int\limits_K \pi(k)\pi(h)v\,dk. \tag{6.2}$$

To see this, note that we may assume that the h_i are chosen so that $h_ih^{-1}\in K$. It is easy to verify that $K=\bigsqcup_i h_ih^{-1}(K\cap hKh^{-1})$. On the one hand, we have (6.1); on the other hand, we have

$$\begin{aligned}\int\limits_K \pi(k)\pi(h)v\,dh&=\mathrm{vol}(K\cap hKh^{-1})\sum_i \pi(h_ih^{-1})(\pi(h)v)\\&=\mathrm{vol}(K\cap hKh^{-1})\sum_i \pi(h_i)v\end{aligned}$$

for $v\in V^K$. This proves (6.2).

Now fix a non-negative integer $n\geq 0$ and consider the Hecke algebra $\mathcal{H}(\mathrm{K}(\mathfrak{p}^n))$ for the paramodular group of level $\mathfrak{p}^n$. The elements of interest for us are

$$T_{0,1}=\text{characteristic function of}\quad \mathrm{K}(\mathfrak{p}^n)\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix}\mathrm{K}(\mathfrak{p}^n) \tag{6.3}$$

and

$$T_{1,0} = \text{characteristic function of} \quad \mathrm{K}(\mathfrak{p}^n)\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&\varpi&\\&&&1\end{bmatrix}\mathrm{K}(\mathfrak{p}^n). \tag{6.4}$$

In general these two elements do not commute in $\mathcal{H}(\mathrm{K}(\mathfrak{p}^n))$, but see Proposition 6.2.1.

The Unramified Case

Let $K = \mathrm{K}(\mathfrak{p}^0) = \mathrm{GSp}(4, \mathfrak{o})$. In this case we have the unramified Hecke algebra $\mathcal{H}(K)$, which is known to be commutative. The relevant coset decompositions for the two Hecke algebra elements defined in (6.3) and (6.4) are as follows. For $T_{0,1}$, we have

$$\begin{aligned}K\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix}K &= \bigsqcup_{x,y,z\in\mathfrak{o}/\mathfrak{p}}\begin{bmatrix}1&&y&z\\&1&x&y\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix}K\\
&\sqcup\bigsqcup_{x,z\in\mathfrak{o}/\mathfrak{p}}\begin{bmatrix}1&x&&\\&1&&\\&&1&-x\\&&&1\end{bmatrix}\begin{bmatrix}1&&&z\\&1&&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&1&&\\&&\varpi&\\&&&1\end{bmatrix}K\\
&\sqcup\bigsqcup_{x\in\mathfrak{o}/\mathfrak{p}}\begin{bmatrix}1&&&\\&1&x&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix}K\\
&\sqcup\begin{bmatrix}1&&&\\&1&&\\&&\varpi&\\&&&\varpi\end{bmatrix}K.\end{aligned}\tag{6.5}$$

For $T_{1,0}$, we have

$$\begin{aligned}K\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&\varpi&\\&&&1\end{bmatrix}K &= \bigsqcup_{\substack{z\in\mathfrak{o}/\mathfrak{p}^2\\x,y\in\mathfrak{o}/\mathfrak{p}}}\begin{bmatrix}1&x&&\\&1&&\\&&1&-x\\&&&1\end{bmatrix}\begin{bmatrix}1&&y&z\\&1&&y\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&\varpi&\\&&&1\end{bmatrix}K\\
&\sqcup\bigsqcup_{\substack{c\in\mathfrak{o}/\mathfrak{p}\\d\in\mathfrak{o}/\mathfrak{p}^2}}\begin{bmatrix}1&&c&\\&1&d&c\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&1&\\&&&\varpi\end{bmatrix}K\\
&\sqcup\bigsqcup_{x\in\mathfrak{o}/\mathfrak{p}}\begin{bmatrix}1&x&&\\&1&&\\&&1&-x\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&1&&\\&&\varpi^2&\\&&&\varpi\end{bmatrix}K\end{aligned}$$

$$\sqcup \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & \varpi^2 \end{bmatrix} K$$

$$\sqcup \bigsqcup_{d \in (\mathfrak{o}/\mathfrak{p})^\times} \begin{bmatrix} 1 & & & \\ & 1 & d\varpi^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} K$$

$$\sqcup \bigsqcup_{\substack{u \in (\mathfrak{o}/\mathfrak{p})^\times \\ \lambda \in \mathfrak{o}/\mathfrak{p}}} \begin{bmatrix} 1 & \lambda u\varpi^{-1} & u\varpi^{-1} & \\ & 1 & \lambda^2 u\varpi^{-1} & \lambda u\varpi^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} K. \qquad (6.6)$$

We sketch a proof of these decompositions. Computations show that the coset representatives are contained in the respective double cosets, and that they are disjoint. Thus it suffices to know that the volume of a double coset is equal to the number of cosets in the asserted decomposition; we assume the volume of K is 1. For any $g \in \mathrm{GSp}(4, F)$, the volume of KgK equals the index of $g^{-1}Kg \cap K$ in K. For $g = \mathrm{diag}(\varpi, \varpi, 1, 1)$, we have $g^{-1}Kg \cap K = \mathrm{Si}(\mathfrak{p})$. By Lemma 5.1.1, the index of $\mathrm{Si}(\mathfrak{p})$ in $\mathrm{GSp}(4, \mathfrak{o})$ is $q^3 + q^2 + q + 1$; this is the number of coset in the decomposition (6.5). For $g = \mathrm{diag}(\varpi^2, \varpi, \varpi, 1)$, we have

$$g^{-1}Kg \cap K = K \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} \end{bmatrix}.$$

This group has index q in $\mathrm{Kl}(\mathfrak{p})$, and, by Lemma 3.3.3, the index of $\mathrm{Kl}(\mathfrak{p})$ in $\mathrm{GSp}(4, \mathfrak{o})$ is $q^3 + q^2 + q + 1$. Hence the volume of KgK equals $q(q^3 + q^2 + q + 1)$, which is the number of cosets in the decomposition (6.6).

Hecke Operators for Level $n \geq 1$

Now we present coset decompositions for $T_{0,1}$ and $T_{1,0}$ in the case $n \geq 1$. We begin with a preliminary lemma about double cosets of the form $\mathrm{Kl}(\mathfrak{p}^n)g\mathrm{K}(\mathfrak{p}^n)$.

Lemma 6.1.1. *We have the following disjoint decompositions.*

i) For any $n \geq 1$,

$$\mathrm{Kl}(\mathfrak{p}^n) \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n) = \bigsqcup_{x, y \in \mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n)$$

$$\cup \bigsqcup_{z \in \mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & z & & \\ & 1 & & \\ & & 1 & -z \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n).$$

ii) For any $n \geq 2$,

$$\mathrm{Kl}(\mathfrak{p}^n) \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n)$$

$$= \bigsqcup_{x,y \in \mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y & \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n).$$

For $n = 1$,

$$\mathrm{Kl}(\mathfrak{p}) \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p})$$

$$= \bigsqcup_{x,y,z \in \mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y & z \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}).$$

Proof. Since the argument is similar in both cases, we shall prove only i). Using the Iwahori factorization (2.7) we compute

$$\mathrm{Kl}(\mathfrak{p}^n) \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n)$$

$$= \begin{bmatrix} 1 & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ & 1 & & \mathfrak{o} \\ & & 1 & \mathfrak{o} \\ & & & 1 \end{bmatrix} \begin{bmatrix} \mathfrak{o}^\times & & & \\ & \mathfrak{o} & \mathfrak{o} & \\ & \mathfrak{o} & \mathfrak{o} & \\ & & & \mathfrak{o}^\times \end{bmatrix} \begin{bmatrix} 1 & & & \\ \mathfrak{p}^n & 1 & & \\ \mathfrak{p}^n & & 1 & \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n)$$

$$= \begin{bmatrix} 1 & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ & 1 & & \mathfrak{o} \\ & & 1 & \mathfrak{o} \\ & & & 1 \end{bmatrix} \begin{bmatrix} \mathfrak{o}^\times & & & \\ & \mathfrak{o} & \mathfrak{o} & \\ & \mathfrak{o} & \mathfrak{o} & \\ & & & \mathfrak{o}^\times \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n)$$

$$= \begin{bmatrix} 1 & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ & 1 & & \mathfrak{o} \\ & & 1 & \mathfrak{o} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & \mathfrak{o} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n) \cup \begin{bmatrix} 1 & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ & 1 & & \mathfrak{o} \\ & & 1 & \mathfrak{o} \\ & & & 1 \end{bmatrix} s_2 \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n)$$

$$= \begin{bmatrix} 1 & & \mathfrak{o} & \\ & 1 & & \mathfrak{o} \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & \mathfrak{o} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n) \cup \begin{bmatrix} 1 & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ & 1 & & \mathfrak{o} \\ & & 1 & \mathfrak{o} \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n)$$

$$= \begin{bmatrix} 1 & & \mathfrak{o} & \\ & 1 & \mathfrak{o} & \mathfrak{o} \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n) \cup \begin{bmatrix} 1 & \mathfrak{o} & & \\ & 1 & & \\ & & 1 & \mathfrak{o} \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n)$$

$$= \bigcup_{x,y \in \mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n) \cup \bigcup_{z \in \mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & z & & \\ & 1 & & \\ & & 1 & -z \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n).$$

It is easy to see that this decomposition is disjoint. □

Lemma 6.1.2. *We have the following coset decompositions.*

i) For any $n \geq 1$,

$$\mathrm{K}(\mathfrak{p}^n) \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n) = \bigsqcup_{x,y,z \in \mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & & y & z\varpi^{-n} \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n)$$

$$\sqcup \bigsqcup_{x,z \in \mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & x & & z\varpi^{-n} \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n)$$

$$\sqcup \bigsqcup_{x,y \in \mathfrak{o}/\mathfrak{p}} t_n \begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n)$$

$$\sqcup \bigsqcup_{x \in \mathfrak{o}/\mathfrak{p}} t_n \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n).$$

ii) For any $n \geq 1$,

$$\mathrm{K}(\mathfrak{p}^n) \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n)$$

$$= \bigsqcup_{x,y \in \mathfrak{o}/\mathfrak{p}} \ \bigsqcup_{z \in \mathfrak{o}/\mathfrak{p}^2} \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y & z\varpi^{-n} \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n)$$

$$\sqcup \bigsqcup_{x,y,z\in\mathfrak{o}/\mathfrak{p}} t_n \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y & z\varpi^{-n+1} \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n).$$

Here, t_n is the element defined in (2.3).

Proof. We shall prove only i), the argument for ii) being similar. It is easy to see that all the listed cosets are disjoint. To prove that they exhaust the double coset, note that

$$\mathrm{K}(\mathfrak{p}^n) = \begin{bmatrix} 1 & & & \mathfrak{p}^{-n} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{Kl}(\mathfrak{p}^n) \cup t_n \begin{bmatrix} 1 & & & \mathfrak{p}^{-n+1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{Kl}(\mathfrak{p}^n),$$

by Lemma 3.3.1. It follows that

$$\mathrm{K}(\mathfrak{p}^n) \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n) = \begin{bmatrix} 1 & & & \mathfrak{p}^{-n} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{Kl}(\mathfrak{p}^n) \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n)$$

$$\cup\ t_n \begin{bmatrix} 1 & & & \mathfrak{p}^{-n+1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{Kl}(\mathfrak{p}^n) \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n).$$

Now we substitute the cosets from Lemma 6.1.1 and obtain a decomposition as asserted. □

Dualizing and Twisting

As before, let (π, V) be a smooth representation of $\mathrm{GSp}(4, F)$ such that the center of $\mathrm{GSp}(4, F)$ acts trivially. On the space $V(n)$ we define the dual Hecke operators

$$T_{0,1}^* = u_n \circ T_{0,1} \circ u_n^{-1} \qquad \text{and} \qquad T_{1,0}^* = u_n \circ T_{1,0} \circ u_n^{-1}.$$

These endomorphisms are also induced by elements of $\mathcal{H}(\mathrm{K}(\mathfrak{p}^n))$. Namely, if T is the endomorphism induced by the characteristic function of $\mathrm{K}(\mathfrak{p}^n)g\mathrm{K}(\mathfrak{p}^n)$, then T^* is the endomorphism induced by the characteristic function of the double coset $u_n^{-1}\mathrm{K}(\mathfrak{p}^n)g\mathrm{K}(\mathfrak{p}^n)u_n = \mathrm{K}(\mathfrak{p}^n)u_n^{-1}gu_n\mathrm{K}(\mathfrak{p}^n)$. Hence

$$T_{0,1}^* = \begin{array}{c}\text{endomorphism induced by}\\ \text{the characteristic function of}\end{array} \ \mathrm{K}(\mathfrak{p}^n) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \mathrm{K}(\mathfrak{p}^n)$$

and

$$T_{1,0}^* = \begin{matrix}\text{endomorphism induced by}\\ \text{the characteristic function of}\end{matrix} \quad \mathrm{K}(\mathfrak{p}^n)\begin{bmatrix}\varpi&&&\\&1&&\\&&\varpi^2&\\&&&\varpi\end{bmatrix}\mathrm{K}(\mathfrak{p}^n).$$

Since $\mathrm{K}(\mathfrak{p}^n)$ contains the element

$$\begin{bmatrix}&&&\varpi^{-n}\\&&1&\\&1&&\\\varpi^n&&&\end{bmatrix},$$

we see that $T_{0,1} = T_{0,1}^*$: The Hecke operator $T_{0,1}$ is self-dual. This is not true, however, for $T_{1,0}$, unless $n = 0$. Conjugating the cosets in Lemma 6.1.2 ii) by u_n we see that $T_{1,0}^*$ is the endomorphism induced by the characteristic function of

$$\begin{aligned}
&\mathrm{K}(\mathfrak{p}^n)\begin{bmatrix}\varpi&&&\\&1&&\\&&\varpi^2&\\&&&\varpi\end{bmatrix}\mathrm{K}(\mathfrak{p}^n)\\
&= \bigsqcup_{x,y\in\mathfrak{o}/\mathfrak{p}}\ \bigsqcup_{z\in\mathfrak{o}/\mathfrak{p}^2}\begin{bmatrix}1&x&&\\&1&&\\&&1&-x\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\y\varpi^n&z&1&\\&y\varpi^n&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&1&&\\&&\varpi^2&\\&&&\varpi\end{bmatrix}\mathrm{K}(\mathfrak{p}^n)\\
&\sqcup \bigsqcup_{x,y,z\in\mathfrak{o}/\mathfrak{p}}\begin{bmatrix}1&&x&\\&1&&x\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\y\varpi^n&1&z\varpi&\\&&1&\\&&-y\varpi&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi^2&&\\&&1&\\&&&\varpi\end{bmatrix}\mathrm{K}(\mathfrak{p}^n).
\end{aligned}$$

Next, let ξ be the unique non-trivial, unramified, quadratic character of $F^\times$. Then the structure of paramodular vectors in the twist $\xi\pi$ is the same as for π. In fact, if π and $\xi\pi$ are both realized on the same space V, then the space $V(n)$ of $\mathrm{K}(\mathfrak{p}^n)$-invariant vectors is the same for both representations. Each element in the double coset defining $T_{0,1}$ has multiplier in $\varpi\mathfrak{o}^\times$. Hence, for the representation $\xi\pi$, the endomorphism $T_{0,1}$ of $V(n)$ differs by a sign from the endomorphism $T_{0,1}$ for π. The situation is different for $T_{1,0}$: Since the elements of the double coset defining this Hecke operator have multiplier in $\varpi^2\mathfrak{o}^\times$, the endomorphism $T_{1,0}$ of $V(n)$ is the same for both π and $\xi\pi$. In short, the Hecke operator $T_{1,0}$ is invariant under twisting with ξ, while $T_{0,1}$ changes its sign.

6.2 The Commutation Relation

Let (π, V) be a smooth representation of $\mathrm{GSp}(4, F)$ such that the center of $\mathrm{GSp}(4, F)$ acts trivially, and let $n \geq 2$ be a positive integer. In this section we shall compute the commutator $T_{0,1}T_{1,0} - T_{1,0}T_{0,1}$, considered as an endomorphism of the space $V(n)$ of $\mathrm{K}(\mathfrak{p}^n)$-invariant vectors. Briefly put, $T_{0,1}$ and $T_{1,0}$ commute up to level lowering operators. In particular, they act as commuting endomorphisms at the minimal level; see Corollary 6.2.2 at the end of this section. This result will have important consequences in the next chapter.

In general, if T is the characteristic function of $\mathrm{K}(\mathfrak{p}^n)h\mathrm{K}(\mathfrak{p}^n) = \bigsqcup_i h_i\mathrm{K}(\mathfrak{p}^n)$, and if T' is the characteristic function of $\mathrm{K}(\mathfrak{p}^n)h'\mathrm{K}(\mathfrak{p}^n) = \bigsqcup_j h'_j\mathrm{K}(\mathfrak{p}^n)$, then $T \cdot T'$ acts on a vector $v \in V(n)$ by

$$(T \cdot T')v = \sum_{i,j} \pi(h_i h'_j)v.$$

For convenience, if $h \in \mathrm{GSp}(4, F)$ and $v \in V$, then we will write hv instead of $\pi(h)v$. Let $v \in V$. According to Lemma 6.1.2, we have $T_{0,1}v = Av + Bv + Cv + Dv$ with

$$Av = \sum_{x,y,z \in \mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & & y & z\varpi^{-n} \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} v,$$

$$Bv = \sum_{x,z \in \mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & x & & z\varpi^{-n} \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} v,$$

$$Cv = \sum_{x,y \in \mathfrak{o}/\mathfrak{p}} t_n \begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} v,$$

$$Dv = \sum_{x \in \mathfrak{o}/\mathfrak{p}} t_n \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} v,$$

and $T_{1,0}v = A'v + B'v$ with

$$A'v = \sum_{x,y \in \mathfrak{o}/\mathfrak{p}} \sum_{z \in \mathfrak{o}/\mathfrak{p}^2} \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y & z\varpi^{-n} \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} v,$$

$$B'v = \sum_{x,y,z \in \mathfrak{o}/\mathfrak{p}} t_n \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y & z\varpi^{-n+1} \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} v.$$

It will be convenient to write these formulas as integrals:

$$Av = q^3 \int\limits_{x,y,z\in\mathfrak{o}} \begin{bmatrix} 1 & & y & z\varpi^{-n} \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} v,$$

$$Bv = q^2 \int\limits_{x,z\in\mathfrak{o}} \begin{bmatrix} 1 & x & & z\varpi^{-n} \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} v,$$

$$Cv = q^2 \int\limits_{x,y\in\mathfrak{o}} t_n \begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} v,$$

$$Dv = q \int\limits_{x\in\mathfrak{o}} t_n \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} v,$$

$$A'v = q^4 \int\limits_{x,y,z\in\mathfrak{o}} \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y & z\varpi^{-n} \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} v,$$

$$B'v = q^3 \int\limits_{x,y,z\in\mathfrak{o}} t_n \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y & z\varpi^{-n+1} \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} v.$$

Here, we use the Haar measure on F that gives $\mathfrak{o}$ measure 1. This change of notation is advantageous because in the formulas for $Av, Bv, Cv, Dv, A'v$ and $B'v$, if v is replaced by any vector in V, then the formulas are still meaningful; that is, the formulas may be regarded as defining endomorphisms of V. Consequently, the product $(T_{0,1}T_{1,0})v$ consists of eight terms

$$AA'v, \quad BA'v, \quad CA'v, \quad DA'v, \quad AB'v, \quad BB'v, \quad CB'v, \quad DB'v,$$

and similarly for $(T_{1,0}T_{0,1})v$. We shall compute all these terms.

Proposition 6.2.1. *Let (π, V) be a smooth representation of* GSp(4, F) *such that the center of* GSp(4, F) *acts trivially. For any $n \geq 2$ we have*

$$T_{0,1}T_{1,0} - T_{1,0}T_{0,1} = q^2(\theta\delta_1 - \theta'\delta_2),$$

where both sides are endomorphisms of $V(n)$. Here δ_1 and δ_2 are the level lowering operators defined in Sect. 3.3.

Proof. Let $v \in V(n)$. We start by computing $CA'v$. Using the above expressions we have

$$
\begin{aligned}
CA'v = q^6 \int\limits_{x,y,x',y',z' \in \mathfrak{o}} & t_n \begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \\
& \begin{bmatrix} 1 & x' & & \\ & 1 & & \\ & & 1 & -x' \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y' & z'\varpi^{-n} \\ & 1 & & y' \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} v \\
= q^6 \int\limits_{x,y,x',y',z' \in \mathfrak{o}} & t_n \begin{bmatrix} 1 & x' & & \\ & 1 & & \\ & & 1 & -x' \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y - xx' & xx'^2 - 2yx' \\ & 1 & x & y - xx' \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \\
& \begin{bmatrix} 1 & & y' & z'\varpi^{-n} \\ & 1 & & y' \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} v \\
= q^6 \int\limits_{x,y,x',y',z' \in \mathfrak{o}} & t_n \begin{bmatrix} 1 & x' & & \\ & 1 & & \\ & & 1 & -x' \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y & \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \\
& \begin{bmatrix} 1 & & y' & z'\varpi^{-n-1} \\ & 1 & x & y' \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} v \\
= q^6 \int\limits_{x,y,x',y',z,z' \in \mathfrak{o}} & t_n \begin{bmatrix} 1 & x' & & \\ & 1 & & \\ & & 1 & -x' \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y & \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \\
& \begin{bmatrix} 1 & & y' & (z + z'\varpi)\varpi^{-n-1} \\ & 1 & x & y' \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} v \\
= q^6 \int\limits_{x,y,x',y',z,z' \in \mathfrak{o}} & t_n \begin{bmatrix} 1 & x' & & \\ & 1 & & \\ & & 1 & -x' \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y & z\varpi^{-n+1} \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix}
\end{aligned}
$$

$$\begin{bmatrix}1 & & y' & z'\varpi^{-n}\\ & 1 & x & y'\\ & & 1 & \\ & & & 1\end{bmatrix}\begin{bmatrix}\varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1\end{bmatrix}v$$

$$= B'Av.$$

With similar calculations one computes

$$DA'v = B'Bv, \qquad AA'v = A'Av, \qquad BA'v = A'Bv.$$

So far this proves

$$\begin{aligned}(T_{0,1}T_{1,0} - T_{1,0}T_{0,1})v &= AB'v + BB'v + CB'v + DD'v\\ &\quad - (A'Cv + A'Dv + B'Cv + B'Dv).\end{aligned} \tag{6.7}$$

We shall next compute that the terms in parentheses sum up to $q^2\theta'\delta_2 v$. To begin,

$$\begin{aligned}A'Cv &= q^6 \int_{x,y,x',y',z\in\mathfrak{o}} \begin{bmatrix}1 & x & & \\ & 1 & & \\ & & 1 & -x\\ & & & 1\end{bmatrix}\begin{bmatrix}1 & & y & z\varpi^{-n}\\ & 1 & & y\\ & & 1 & \\ & & & 1\end{bmatrix}\begin{bmatrix}\varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1\end{bmatrix}\\
&\qquad t_n\begin{bmatrix}1 & & y' & \\ & 1 & x' & y'\\ & & 1 & \\ & & & 1\end{bmatrix}\begin{bmatrix}\varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1\end{bmatrix}v\\
&= q^6 \int_{x,y,x',y',z\in\mathfrak{o}} \begin{bmatrix}1 & & y & z\varpi^{-n}\\ & 1 & x' & y\\ & & 1 & \\ & & & 1\end{bmatrix}\begin{bmatrix}\varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1\end{bmatrix}\\
&\qquad t_n\begin{bmatrix}1 & & & \\ & 1 & & \\ x\varpi^{n-1} & & 1 & \\ & x\varpi^{n-1} & & 1\end{bmatrix}\begin{bmatrix}1 & & y' & \\ & 1 & & y'\\ & & 1 & \\ & & & 1\end{bmatrix}\begin{bmatrix}\varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1\end{bmatrix}v\\
&= q^6 \int_{x,y,x',y',z\in\mathfrak{o}} \begin{bmatrix}1 & & y & z\varpi^{-n}\\ & 1 & x' & y\\ & & 1 & \\ & & & 1\end{bmatrix}\begin{bmatrix}\varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1\end{bmatrix}\\
&\qquad t_n\begin{bmatrix}1 & & y' & \\ & 1 & & y'\\ & & 1 & \\ & & & 1\end{bmatrix}\begin{bmatrix}\varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1\end{bmatrix}v\\
&= q^6 \int_{y,x',y',z,z'\in\mathfrak{o}} \begin{bmatrix}1 & & y & (z+z'\varpi)\varpi^{-n}\\ & 1 & x' & y\\ & & 1 & \\ & & & 1\end{bmatrix}\begin{bmatrix}\varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1\end{bmatrix}\end{aligned}$$

$$
\begin{aligned}
&t_n \begin{bmatrix} 1 & & y' & \\ & 1 & & y' \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} v \\
&= q^6 \int\limits_{y,x',y',z,z' \in \mathfrak{o}} \begin{bmatrix} 1 & & y & z\varpi^{-n} \\ & 1 & x' & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \\
&\qquad t_n \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ z'\varpi^{n-1} & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y' & \\ & 1 & & y' \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} v.
\end{aligned}
$$

At this point we use the matrix identity

$$
\begin{aligned}
&\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ z'\varpi^{n-1} & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y' & \\ & 1 & & y' \\ & & 1 & \\ & & & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & & & \\ -y'z'\varpi^{n-1} & 1 & & \\ & & 1 & \\ & & y'z'\varpi^{n-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & & y' & \\ & 1 & & y' \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & y'^2z'\varpi^{n-1} & \\ & & 1 & \\ z'\varpi^{n-1} & & & 1 \end{bmatrix}.
\end{aligned}
$$

It shows that

$$
\begin{aligned}
A'Cv &= q^6 \int\limits_{y,x',y',z,z' \in \mathfrak{o}} \begin{bmatrix} 1 & & y & z\varpi^{-n} \\ & 1 & x' & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \\
&\qquad \begin{bmatrix} 1 & & y'z'\varpi^{-1} & \\ & 1 & & y'z'\varpi^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} t_n \begin{bmatrix} 1 & & y' & \\ & 1 & & y' \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} v \\
&= q^6 \int\limits_{y,x',y',z,z' \in \mathfrak{o}} \begin{bmatrix} 1 & & y & z\varpi^{-n} \\ & 1 & x' & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \\
&\qquad t_n \begin{bmatrix} 1 & & y' & \\ & 1 & & y' \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} v
\end{aligned}
$$

$$= q^6 \int\limits_{y,x',y',z\in\mathfrak{o}/\mathfrak{p}} \begin{bmatrix}1 & y & & z\varpi^{-n}\\ & 1 & x' & y\\ & & 1 & \\ & & & 1\end{bmatrix} t_n \begin{bmatrix}1 & y'\varpi^{-1} & & \\ & 1 & & y'\varpi^{-1}\\ & & 1 & \\ & & & 1\end{bmatrix} \begin{bmatrix}1 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi\end{bmatrix} v$$

$$= q^6 \int\limits_{z,y,x',y'\in\mathfrak{o}} \begin{bmatrix}1 & & & z\varpi^{-n}\\ & 1 & & \\ & & 1 & \\ & & & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & & & \\ y'\varpi^{n-1} & 1 & & \\ & & 1 & \\ & & -y'\varpi^{n-1} & 1\end{bmatrix} \begin{bmatrix}1 & y & & \\ & 1 & x' & y\\ & & 1 & \\ & & & 1\end{bmatrix} \begin{bmatrix}\varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1\end{bmatrix} v.$$

Similar calculations prove the following identities:

$$A'Dv = q^5 \int\limits_{x,x',z\in\mathfrak{o}} \begin{bmatrix}1 & & & z\varpi^{-n}\\ & 1 & & \\ & & 1 & \\ & & & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & & & \\ & & 1 & \\ x'\varpi^{n-1} & & & 1\\ & x'\varpi^{n-1} & & 1\end{bmatrix} \begin{bmatrix}1 & x & & \\ & 1 & & \\ & & 1 & -x\\ & & & 1\end{bmatrix} \begin{bmatrix}\varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1\end{bmatrix} v,$$

$$B'Cv = q^5\eta \int\limits_{y,x',y'\in\mathfrak{o}} \begin{bmatrix}1 & & & \\ y\varpi^{n-1} & 1 & & \\ & & 1 & \\ & & -y\varpi^{n-1} & 1\end{bmatrix} \begin{bmatrix}1 & y' & & \\ & 1 & x' & y'\\ & & 1 & \\ & & & 1\end{bmatrix} \begin{bmatrix}\varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1\end{bmatrix} v,$$

$$B'Dv = q^4\eta \int\limits_{x,x'\in\mathfrak{o}} \begin{bmatrix}1 & & & \\ & & 1 & \\ x\varpi^{n-1} & & & 1\\ & x\varpi^{n-1} & & 1\end{bmatrix} \begin{bmatrix}1 & x' & & \\ & 1 & & \\ & & 1 & -x'\\ & & & 1\end{bmatrix} \begin{bmatrix}\varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1\end{bmatrix} v.$$

Using formula (3.24), it follows that

$$A'Cv + A'Dv = q^2 \sum_{z\in\mathfrak{o}/\mathfrak{p}} \begin{bmatrix}1 & & & z\varpi^{-n}\\ & 1 & & \\ & & 1 & \\ & & & 1\end{bmatrix} \delta_2 v$$

and

$$B'Cv + B'Dv = q^2\,\eta\,\delta_2 v.$$

By (3.7) we obtain

$$A'Cv + A'Dv + B'Cv + B'Dv = q^2\,\theta'\,\delta_2 v. \tag{6.8}$$

We are now going to compute the four remaining terms in (6.7), starting with

$$AB'v = q^6 \int\limits_{x,y,z,x',y',z'\in\mathfrak{o}} \begin{bmatrix} 1 & & y & z\varpi^{-n} \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$
$$t_n \begin{bmatrix} 1 & & y' & \\ & 1 & & y' \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & x' & & z'\varpi^{-n+1} \\ & 1 & & \\ & & 1 & -x' \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} v.$$

Let $(AB')_1$ be the part of this integral where $z' \in \mathfrak{o}^\times$, and let $(AB')_2$ be the part of this integral where $z' \in \mathfrak{p}$. To compute $(AB')_1$, we use the matrix identity

$$t_n \begin{bmatrix} 1 & & & z'\varpi^{-n+1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & & & -z'^{-1}\varpi^{-n-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & & & 1 \\ z'\varpi^{n+1} & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^{-1}z'^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi z' \end{bmatrix}$$

and get

$$(AB')_1 = q^6 \int\limits_{\substack{x,y,z,x',y'\in\mathfrak{o} \\ z'\in\mathfrak{o}^\times}} \begin{bmatrix} 1 & & y & z\varpi^{-n} \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & & & \\ & & 1 & \\ & & & 1 \\ z'\varpi^{n+1} & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y'\varpi^{-1} & \\ & 1 & & y'\varpi^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & x'\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -x'\varpi^{-1} \\ & & & 1 \end{bmatrix} v.$$

A computation shows that conjugating the third matrix by the product of the last two matrices gives an element in $\mathrm{K}(\mathfrak{p}^n)$. Consequently the third matrix can be omitted, and we get

$$(AB')_1 = q^6(1-q^{-1}) \int\limits_{x,y,z,x',y'\in\mathfrak{o}} \begin{bmatrix} 1 & & y & z\varpi^{-n} \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & & y'\varpi^{-1} & \\ & 1 & & y'\varpi^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & x'\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -x'\varpi^{-1} \\ & & & 1 \end{bmatrix} v$$

$$= q^5(q-1) \int\limits_{x,y,z,x' \in \mathfrak{o}} \begin{bmatrix} 1 & & y & z\varpi^{-n} \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & x'\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -x'\varpi^{-1} \\ & & & 1 \end{bmatrix} v. \tag{6.9}$$

Next we compute

$$(AB')_2 = q^5 \int\limits_{x,y,z,x',y' \in \mathfrak{o}} \begin{bmatrix} 1 & & y & z\varpi^{-n} \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$
$$t_n \begin{bmatrix} 1 & & y' & \\ & 1 & & y' \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & x' & & \\ & 1 & & \\ & & 1 & -x' \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} v$$
$$= q^5 \int\limits_{x,y,z,x',y' \in \mathfrak{o}} \begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} t_n$$
$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ z\varpi^n & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y' & \\ & 1 & & y' \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & x'\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -x'\varpi^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} v.$$

A computation shows that conjugating the first matrix on the second line by the product of the last three matrices gives an element in $\mathrm{K}(\mathfrak{p}^n)$. Consequently

$$(AB')_2 = q^5 \int\limits_{x,y,x',y' \in \mathfrak{o}} \begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} t_n$$
$$\begin{bmatrix} 1 & & y' & \\ & 1 & & y' \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & x'\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -x'\varpi^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} v.$$

A further computation shows that the y variable can be brought all the way to the right and will then disappear. Hence

$$(AB')_2 = q^5 \int\limits_{x,x',y' \in \mathfrak{o}} \begin{bmatrix} 1 & & & \\ & 1 & x & \\ & & 1 & \\ & & & 1 \end{bmatrix} t_n$$

$$\begin{bmatrix} 1 & & y' & \\ & 1 & & y' \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & x'\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -x'\varpi^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} v. \tag{6.10}$$

This concludes the preliminary computation of AB'. With similar arguments one obtains $BB'v = (BB')_1 + (BB')_2$, where

$$(BB')_1 = q^4(q-1) \int\limits_{x,z,y' \in \mathfrak{o}} \begin{bmatrix} 1 & x & y'\varpi^{-1} & z\varpi^{-n} \\ & 1 & & y'\varpi^{-1} \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} v \tag{6.11}$$

and

$$(BB')_2 = q^4 \int\limits_{x',y' \in \mathfrak{o}} t_n \begin{bmatrix} 1 & & y'\varpi^{-1} & \\ & 1 & & y'\varpi^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & x' & & \\ & 1 & & \\ & & 1 & -x' \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} v. \tag{6.12}$$

Furthermore, $CB'v = (CB')_1 + (CB')_2$, where

$$(CB')_1 = q^5 \int\limits_{x,y,x' \in \mathfrak{o}} \int\limits_{z' \in \mathfrak{o}^\times} t_n \begin{bmatrix} 1 & & y & z'\varpi^{-n} \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & x'\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -x'\varpi^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} v \tag{6.13}$$

and

$$(CB')_2 = q^4 \int\limits_{x,x',y' \in \mathfrak{o}} \begin{bmatrix} 1 & & y' & \\ & 1 & x & y' \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & x'\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -x'\varpi^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} v. \tag{6.14}$$

Finally, $DB'v = (DB')_1 + (DB')_2$, where

$$(DB')_1 = q^4 \int\limits_{x,y' \in \mathfrak{o}} \int\limits_{z' \in \mathfrak{o}^\times} t_n \begin{bmatrix} 1 & x & y'\varpi^{-1} & z'\varpi^{-n} \\ & 1 & & y'\varpi^{-1} \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} v \tag{6.15}$$

and

$$(DB')_2 = q^3 \int\limits_{x',y' \in \mathfrak{o}} \begin{bmatrix} 1 & & y'\varpi^{-1} & \\ & 1 & & y'\varpi^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & x' & & \\ & 1 & & \\ & & 1 & -x' \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} v. \quad (6.16)$$

Thus we obtain the eight terms (6.9) to (6.16). We rearrange and write $(BB')_2 + (DB')_1 = E_1 + E_2$ with

$$E_1 = q^4 \int\limits_{x,y',z' \in \mathfrak{o}} t_n \begin{bmatrix} 1 & x & y'\varpi^{-1} & z'\varpi^{-n} \\ & 1 & & y'\varpi^{-1} \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} v$$

and

$$E_2 = q^3(q-1) \int\limits_{x',y' \in \mathfrak{o}} t_n \begin{bmatrix} 1 & & y'\varpi^{-1} & \\ & 1 & & y'\varpi^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & x' & & \\ & 1 & & \\ & & 1 & -x' \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} v.$$

Also write $(AB')_2 + (CB')_1 = F_1 + F_2$ with

$$F_1 = q^5 \int\limits_{x,y,x',z' \in \mathfrak{o}} t_n \begin{bmatrix} 1 & & y & z'\varpi^{-n} \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & x'\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -x'\varpi^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} v$$

and

$$F_2 = q^4(q-1) \int\limits_{x,x',y' \in \mathfrak{o}} \begin{bmatrix} 1 & & & \\ & 1 & x & \\ & & 1 & \\ & & & 1 \end{bmatrix} t_n \begin{bmatrix} 1 & & y' & \\ & 1 & & y' \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & x'\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -x'\varpi^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} v.$$

By Lemma 3.3.7 we see that

$$(AB')_1 + F_2 = q(q-1) \sum_{x \in \mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & x\varpi^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \delta_1 v,$$

$$(BB')_1 + E_2 = q(q-1)\begin{bmatrix}1&&&\\&1&&\\&&\varpi&\\&&&\varpi\end{bmatrix}\delta_1 v,$$

$$(CB')_2 + F_1 = q\sum_{x\in\mathfrak{o}/\mathfrak{p}}\begin{bmatrix}1&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix}\begin{bmatrix}1&&&\\&1&x\varpi^{-1}&\\&&1&\\&&&1\end{bmatrix}\delta_1 v,$$

$$(DB')_2 + E_1 = q\begin{bmatrix}1&&&\\&1&&\\&&\varpi&\\&&&\varpi\end{bmatrix}\delta_1 v.$$

By Lemma 3.2.2 it follows that

$$(AB')_1 + F_2 + (BB')_1 + E_2 = q(q-1)\theta\delta_1 v$$

and

$$(CB')_2 + F_1 + (DB')_2 + E_1 = q\theta\delta_1 v.$$

Hence the eight terms (6.9) to (6.16) add up to $q^2\theta\delta_1 v$. Together with (6.7) and (6.8) this proves the asserted formula. □

Corollary 6.2.2. *Let (π, V) be a smooth representation of $\mathrm{GSp}(4,F)$ such that the center of $\mathrm{GSp}(4,F)$ acts trivially. Assume that $V(n) \neq 0$ for some non-negative integer n, and let N_π be the minimal paramodular level of π. If $N_\pi \geq 2$, then the Hecke operators $T_{0,1}$ and $T_{1,0}$ act on $V(N_\pi)$ as a pair of commuting endomorphisms.*

Corollary 6.2.3. *Let (π, V) be a smooth representation of $\mathrm{GSp}(4,F)$ such that the center of $\mathrm{GSp}(4,F)$ acts trivially. At any level $n \geq 2$, the two Hecke operators $T_{0,1}$ and $T_{1,0} + T_{1,0}^*$ commute as endomorphisms of $V(n)$ (and they commute with Atkin–Lehner involutions).*

Proof. This follows by adding the formula in Proposition 6.2.1 to its dual (Atkin–Lehner conjugate). □

6.3 Hecke Operators and Level Raising

In this section we prove formulas about commuting the Hecke operators $T_{0,1}$ and $T_{1,0}$ and the level raising operators θ and θ'. These formulas will be used to compute Hecke eigenvalues at the minimal level in Sect. 6.4. The following result uses the level lowering operators defined in Sect. 3.3.

Proposition 6.3.1. *Let (π, V) be a smooth representation of* $\mathrm{GSp}(4,F)$ *for which the center acts trivially. For any $n \geq 2$ we have the formulas*

$$T_{0,1} \circ \theta' = \theta' \circ T_{0,1} + q^2\theta - \eta \circ \delta_2, \tag{6.17}$$
$$T_{0,1} \circ \theta = \theta \circ T_{0,1} + q^2\theta' - \eta \circ \delta_1, \tag{6.18}$$
$$T_{1,0} \circ \theta' = \theta' \circ T_{1,0} + q^3\theta' - q\,\eta \circ \delta_3. \tag{6.19}$$

For $n = 1$ we have

$$T_{0,1} \circ \theta' = \theta' \circ T_{0,1} + q^2\theta - \eta \circ \delta_2 + \theta' \circ u_1, \tag{6.20}$$
$$T_{0,1} \circ \theta = \theta \circ T_{0,1} + q^2\theta' - \eta \circ \delta_1 + \theta \circ u_1, \tag{6.21}$$
$$T_{1,0} \circ \theta' = \theta' \circ T_{1,0} + q^3\theta' - q\,\eta \circ \delta_3 + q\,\theta \circ u_1. \tag{6.22}$$

For any $n \geq 1$,

$$T_{1,0} \circ \theta = q\,T_{0,1} \circ \theta' - q^2(q+1)\,\theta. \tag{6.23}$$

In these formulas, each term is a linear map from $V(n)$ to $V(n+1)$.

Proof. Formula (6.18) follows from (6.17) by dualizing, observing that $T_{0,1}$ is self-dual. Similarly, (6.21) follows from (6.20). We shall prove (6.17) and (6.20) simultaneously. Let $v \in V(n)$. In contrast to the convention of the proof of Proposition 6.2.1, we will revert to writing $\pi(g)v$ instead of gv for $g \in \mathrm{GSp}(4,F)$; this will avoid confusion at one point in the proof. Using Lemmas 3.2.2 and 6.1.2, we have

$$T_{0,1}\theta' v = A_1 + A_2 + B_1 + B_2 + C_1 + C_2 + D_1 + D_2$$

with

$$A_1 = q^3 \int\limits_{x,y,z \in \mathfrak{o}} \pi(\begin{bmatrix} 1 & & y & z\varpi^{-(n+1)} \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \eta)v,$$

$$A_2 = q^4 \int\limits_{x,y,z,c \in \mathfrak{o}} \pi(\begin{bmatrix} 1 & & y & z\varpi^{-(n+1)} \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & c\varpi^{-n-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v,$$

$$B_1 = q^2 \int\limits_{x,z \in \mathfrak{o}} \pi(\begin{bmatrix} 1 & x & & z\varpi^{-(n+1)} \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \eta)v,$$

$$B_2 = q^3 \int\limits_{x,z,c \in \mathfrak{o}} \pi(\begin{bmatrix} 1 & x & & z\varpi^{-(n+1)} \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & c\varpi^{-n-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v,$$

$$C_1 = q^2 \int\limits_{x,y\in\mathfrak{o}} \pi(t_{n+1} \begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \eta)v,$$

$$C_2 = q^3 \int\limits_{x,y,c\in\mathfrak{o}} \pi(t_{n+1} \begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & c\varpi^{-n-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v,$$

$$D_1 = q \int\limits_{x\in\mathfrak{o}} \pi(t_{n+1} \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \eta)v,$$

$$D_2 = q^2 \int\limits_{x,c\in\mathfrak{o}} \pi(t_{n+1} \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & c\varpi^{-n-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v.$$

Making use of the formula in Lemma 3.2.2 i), a straightforward computation shows that

$$A_1 + B_1 + C_1 + D_1 = q(q+1)\theta v.$$

Next, let

$$R_1 = q^3 \int\limits_{x,y,z\in\mathfrak{o}} \pi(\begin{bmatrix} 1 & & & z\varpi^{-n-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} t_n \begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v,$$

$$R_2 = q^2\eta \int\limits_{x,y\in\mathfrak{o}} \pi(\begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v,$$

$$R_3 = q^2 \int\limits_{x,z\in\mathfrak{o}} \pi(\begin{bmatrix} 1 & & & z\varpi^{-n-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} t_n \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix})v,$$

$$R_4 = q\eta \int\limits_{x\in\mathfrak{o}} \pi(\begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix})v.$$

With this notation we have

$$\theta'(T_{0,1}v) = \eta T_{0,1}v + q \int\limits_{z\in\mathfrak{o}} \pi(\begin{bmatrix} 1 & & & z\varpi^{-(n+1)} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})T_{0,1}v$$

$$= \eta\pi(t_n)T_{0,1}v + A_2 + B_2 + R_1 + R_3$$
$$= C_2 + D_2 + R_2 + R_4 + A_2 + B_2 + R_1 + R_3.$$

Consequently,

$$A_2 + B_2 + C_2 + D_2 = \theta'(T_{0,1}v) - (R_1 + R_2 + R_3 + R_4).$$

Let R_{11} be the part of R_1 where $z \in \mathfrak{o}^\times$, and let R_{12} be the part of R_1 where $z \in \mathfrak{p}$. Similarly, let R_{31} be the part of R_3 where $z \in \mathfrak{o}^\times$, and let R_{32} be the part of R_3 where $z \in \mathfrak{p}$. It follows from the matrix identity

$$t_{n+1}\begin{bmatrix}1&&&z\varpi^{-n-1}\\&1&&\\&&1&\\&&&1\end{bmatrix}t_n = \begin{bmatrix}1&&&-z^{-1}\varpi^{-n-1}\\&1&&\\&&1&\\&&&1\end{bmatrix}t_n\begin{bmatrix}z&&&-\varpi^{1-n}\\&1&&\\&&1&\\&&&z^{-1}\end{bmatrix}$$

that R_{11} and R_{31} are invariant under t_{n+1}. We compute

$$R_{11} = q^3\int\limits_{x,y\in\mathfrak{o}}\int\limits_{z\in\mathfrak{o}^\times}\pi(t_n\begin{bmatrix}1&&&\\&&1&\\&&&1\\z\varpi^{n-1}&&&1\end{bmatrix}\begin{bmatrix}1&&y&\\&1&x&y\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})v$$
$$= q^3\int\limits_{x,y\in\mathfrak{o}}\int\limits_{z\in\mathfrak{o}^\times}\pi(t_n\begin{bmatrix}1&&&\\&1&x&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\-yz\varpi^{n-1}&1&&\\&&1&\\&&yz\varpi^{n-1}&1\end{bmatrix}$$
$$\begin{bmatrix}1&&y&\\&1&y^2z\varpi^{n-1}&y\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&&1&\\&&&1\\z\varpi^{n-1}&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})v.$$

Since $\pi(t_{n+1})R_{11} = R_{11}$, we get

$$R_{11} = q^3\eta\int\limits_{x,y\in\mathfrak{o}}\int\limits_{z\in\mathfrak{o}^\times}\pi(\begin{bmatrix}1&&&\\-yz\varpi^{n-1}&1&&\\&&1&\\&&yz\varpi^{n-1}&1\end{bmatrix}$$
$$\begin{bmatrix}1&&y&\\&1&x&y\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})v$$
$$= q^3\eta\int\limits_{x\in\mathfrak{o}}\int\limits_{y,z\in\mathfrak{o}^\times}\pi(\begin{bmatrix}1&&&\\-z\varpi^{n-1}&1&&\\&&1&\\&&z\varpi^{n-1}&1\end{bmatrix}$$

$$\begin{bmatrix}1&&y&\\&1&x&y\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})v$$

$$+(q-1)q\int\limits_{x\in\mathfrak{o}}\pi(\begin{bmatrix}1&&&\\&1&x&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})v. \qquad (6.24)$$

A similar computation shows that

$$R_{31}=q^2\eta\int\limits_{x,z\in\mathfrak{o}^\times}\pi(\begin{bmatrix}1&x&&\\&1&&\\z\varpi^{n-1}&&1&-x\\&z\varpi^{n-1}&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&1&&\\&&\varpi&\\&&&1\end{bmatrix})v$$

$$+(q-1)\pi(\begin{bmatrix}1&&&\\&1&&\\&&\varpi&\\&&&\varpi\end{bmatrix})v. \qquad (6.25)$$

The two "small" terms in (6.24) and (6.25) add up to $(q-1)\theta v$. Adding up the other terms one easily obtains

$$R_1+R_2+R_3+R_4=q^2\eta\int\limits_{x,z\in\mathfrak{o}}\begin{bmatrix}1&x&&\\&1&&\\z\varpi^{n-1}&&1&-x\\&z\varpi^{n-1}&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&1&&\\&&\varpi&\\&&&1\end{bmatrix}v$$

$$+q^3\eta\int\limits_{x,y,z\in\mathfrak{o}}\begin{bmatrix}1&&&\\z\varpi^{n-1}&1&&\\&&1&\\&&-z\varpi^{n-1}&1\end{bmatrix}\begin{bmatrix}1&&y&\\&1&x&y\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix}v+q\theta v.$$

The formula (6.17) now follows from (3.24), and (6.20) follows from Lemma 3.3.8.

The computations for (6.19) and (6.22) are similar but slightly easier. By Lemma 3.2.2 and Lemma 6.1.2, we have $T_{1,0}(\theta' v)=A_1+A_2+B_1+B_2$ with

$$A_1=q^4\int\limits_{x,y,z\in\mathfrak{o}}\pi(\begin{bmatrix}1&x&y&z\varpi^{-n-1}\\&1&&y\\&&1&-x\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&\varpi&\\&&&1\end{bmatrix})\eta v,$$

$$A_2=q^5\int\limits_{x,y,z,c\in\mathfrak{o}}\pi(\begin{bmatrix}1&x&y&z\varpi^{-n-1}\\&1&&y\\&&1&-x\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&\varpi&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&c\varpi^{-n-1}\\&1&&\\&&1&\\&&&1\end{bmatrix})v,$$

$$B_1 = q^3 \int\limits_{x,y,z\in\mathfrak{o}} \pi(t_{n+1} \begin{bmatrix} 1 & x & y & z\varpi^{-n} \\ & 1 & & y \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix})\eta v,$$

$$B_2 = q^4 \int\limits_{x,y,z,c\in\mathfrak{o}} \pi(t_{n+1} \begin{bmatrix} 1 & x & y & z\varpi^{-n} \\ & 1 & & y \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & c\varpi^{-n-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v.$$

Obvious simplifications show that

$$A_1 + B_1 = q^3\theta' v.$$

We can rewrite

$$\begin{aligned} A_2 + B_2 &= \big(q \int\limits_{c\in\mathfrak{o}} \pi(\begin{bmatrix} 1 & & & c\varpi^{-n-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}) + \eta\pi(t_n)\big) \\ &\qquad q^4 \int\limits_{x,y,z\in\mathfrak{o}} \pi(\begin{bmatrix} 1 & x & y & z\varpi^{-n} \\ & 1 & & y \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix})v \\ &= \big(q \int\limits_{c\in\mathfrak{o}} \pi(\begin{bmatrix} 1 & & & c\varpi^{-n-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}) + \eta\pi(t_n)\big) \\ &\qquad \big(T_{1,0}v - q^3 \int\limits_{x,y,z\in\mathfrak{o}} \pi(t_n \begin{bmatrix} 1 & x & y & z\varpi^{-n+1} \\ & 1 & & y \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix})v\big) \\ &= \theta'(T_{1,0}v) - R, \end{aligned}$$

where

$$\begin{aligned} R &= q^4 \int\limits_{c,x,y,z\in\mathfrak{o}} \pi(t_n \begin{bmatrix} 1 & & & \\ & & 1 & \\ & & & 1 \\ c\varpi^{n-1} & & & 1 \end{bmatrix} \begin{bmatrix} 1 & x & y & z\varpi^{-n+1} \\ & 1 & & y \\ & & 1 & -x \\ & & & 1 \end{bmatrix})\eta^{-1}v \\ &+ q^3 \int\limits_{x,y,z\in\mathfrak{o}} \pi(\eta\, t_{n-1} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & & & 1 \\ z\varpi^{n-1} & & & 1 \end{bmatrix} t_{n-1} \begin{bmatrix} 1 & x & y & \\ & 1 & & y \\ & & 1 & -x \\ & & & 1 \end{bmatrix})\eta^{-1}v. \end{aligned}$$

By (3.26) we have

$$R = q \int\limits_{c\in\mathfrak{o}} \pi(t_n \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ c\varpi^{n-1} & & & 1 \end{bmatrix})\delta_3(v) = q\eta\,\delta_3(v)$$

if $n \geq 2$, and by (3.25) we have

$$R = q \int\limits_{c\in\mathfrak{o}} \pi(t_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ c & & & 1 \end{bmatrix})(\delta_3(v) - \eta^{-1}\theta u_1 v) = q\eta\,\delta_3(v) - q\theta u_1 v$$

if $n = 1$. This proves (6.19) and (6.22).

Next we prove (6.23). By Lemmas 3.2.2 and 6.1.2 we get $T_{1,0}(\theta v) = A + B + C + D$ with

$$A = q^4 \int\limits_{x,z\in\mathfrak{o}} \pi(\begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & z\varpi^{-n-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix})v,$$

$$B = q^5 \int\limits_{y,c,z\in\mathfrak{o}} \pi(\begin{bmatrix} 1 & & y & z\varpi^{-n-1} \\ & 1 & c & y \\ & & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v,$$

$$C = q^3 \int\limits_{x,z\in\mathfrak{o}} \pi(t_{n+1}\begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & z\varpi^{-n} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix})v,$$

$$D = q^4 \int\limits_{y,c,z\in\mathfrak{o}} \pi(t_{n+1}\begin{bmatrix} 1 & & y & z\varpi^{-n} \\ & 1 & c & y \\ & & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v.$$

Using Lemma 3.2.2 ii) we rewrite

$$A = q^4 \int\limits_{x,z,z'\in\mathfrak{o}} \pi(\begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & z\varpi^{-n-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & z'\varpi^{-n-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v,$$

$$= q^3 \int\limits_{x,z\in\mathfrak{o}} \pi(\begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & z\varpi^{-n-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix})\theta' v$$

$$
\begin{aligned}
&- q^3 \int\limits_{x,z\in\mathfrak{o}} \pi(\begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & z\varpi^{-n-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix})\eta v \\
= {}& q^3 \int\limits_{x,z\in\mathfrak{o}} \pi(\begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & z\varpi^{-n-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix})\theta' v \\
&- q^3 \pi(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix})v.
\end{aligned}
$$

Similar manipulations lead to

$$
\begin{aligned}
B &= q^4 \int\limits_{y,z,c\in\mathfrak{o}} \pi(\begin{bmatrix} 1 & & y & z\varpi^{-n-1} \\ & 1 & c & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix})\theta' v \\
&\quad - q^4 \int\limits_{c\in\mathfrak{o}} \pi(\begin{bmatrix} 1 & & & \\ & 1 & c & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{bmatrix})v, \\
C &= q^2 \int\limits_{x\in\mathfrak{o}} \pi(t_{n+1} \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix})\theta' v \\
&\quad - q^2 \pi(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix})v, \\
D &= q^3 \int\limits_{y,c\in\mathfrak{o}} \pi(t_{n+1} \begin{bmatrix} 1 & & y & \\ & 1 & c & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix})\theta' v \\
&\quad - q^3 \int\limits_{c\in\mathfrak{o}} \pi(\begin{bmatrix} 1 & & & \\ & 1 & c & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{bmatrix})v.
\end{aligned}
$$

Adding up everything and observing Lemma 6.1.2 i) proves formula (6.23).
□

Corollary 6.3.2. *Let (π, V) be a smooth representation of* $\mathrm{GSp}(4, F)$ *such that the center of* $\mathrm{GSp}(4, F)$ *acts trivially. Assume that $V(n) \neq 0$ for some non-negative integer n, and let N_π be the minimal paramodular level.*

i) Assume that $N_\pi \geq 2$. Then, for any $v \in V(N_\pi)$,

$$T_{0,1}(\theta \pm \theta')v = (\theta \pm \theta')(T_{0,1}v \pm q^2 v) \tag{6.26}$$

and

$$T_{1,0}(\theta \pm \theta')v = q\theta' T_{0,1}v \pm \theta' T_{1,0}v - q^2\theta v \pm q^3\theta' v. \tag{6.27}$$

In particular, if $v \in V(N_\pi)$ is an eigenvector for $T_{0,1}$ with eigenvalue λ, then $(\theta \pm \theta')v$ is an eigenvector with eigenvalue $\lambda \pm q^2$.

ii) Assume that $N_\pi = 1$. Then, for $v \in V(1)$,

$$T_{0,1}(\theta \pm \theta')v = (\theta \pm \theta')(T_{0,1}v \pm q^2 v + u_1 v) \tag{6.28}$$

and

$$T_{1,0}(\theta \pm \theta')v = q\theta' T_{0,1}v \pm \theta' T_{1,0}v - q^2\theta v \pm q^3\theta' v \pm q(\theta \pm \theta')u_1 v. \tag{6.29}$$

In particular, if $u_1 v = \varepsilon v$ with $\varepsilon \in \{\pm 1\}$, and if $v \in V(1)$ is an eigenvector for $T_{0,1}$ with eigenvalue λ, then $(\theta \pm \theta')v$ is an eigenvector with eigenvalue $\lambda \pm q^2 + \varepsilon$.

Proof. The first formula follows by adding and subtracting the first two formulas from Proposition 6.3.1, observing that the δ_1 and δ_2 terms are zero at the minimal level. Similarly, (6.28) follows by adding and subtracting (6.20) and (6.21). Equations (6.27) and (6.29) follow also by combining several of the formulas from Proposition 6.3.1. □

6.4 Computation of Hecke Eigenvalues

We already proved uniqueness at the minimal level for non-supercuspidal, irreducible, admissible representations of $\mathrm{GSp}(4, F)$ with trivial central character (Theorem 5.6.1): If such a representation (π, V) is paramodular, and if N_π is the minimal paramodular level, then $\dim V(N_\pi) = 1$. Hence, acting with the two Hecke operators $T_{0,1}$ and $T_{1,0}$ on $V(N_\pi)$, we obtain two eigenvalues λ and μ, respectively. The results of the present section will allow us to compute these eigenvalues for all non-supercuspidal representations, except those of type VII, VIII and IX; for these representations see Corollary 7.4.6. The actual application of these results will happen in Theorem 7.5.2. We start by recalling analogous results for GL(2).

Preliminaries on GL(2)

Let (π, V) be an irreducible, admissible representation of $\mathrm{GL}(2,F)$ with trivial central character. We define the usual Hecke subgroups

$$\Gamma_0(\mathfrak{p}^n) = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2,\mathfrak{o}) : \ c \in \mathfrak{p}^n\},$$

and the Atkin–Lehner element

$$\begin{bmatrix} 0 & 1 \\ -\varpi^n & 0 \end{bmatrix},$$

which normalizes $\Gamma_0(\mathfrak{p}^n)$. If π is infinite-dimensional, there exists a non-negative integer n such that $V_0(n) := \{v \in V : \pi(g)v = v \text{ for all } g \in \Gamma_0(n)\} \neq 0$. If n is minimal with this property, then $V_0(n)$ is one-dimensional. The Hecke operators

$$T_1 := \text{characteristic function of} \quad \Gamma_0(\mathfrak{p}^n)\begin{bmatrix} \varpi & \\ & 1 \end{bmatrix}\Gamma_0(\mathfrak{p}^n)$$

and

$$T_1^* := \text{characteristic function of} \quad \Gamma_0(\mathfrak{p}^n)\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix}\Gamma_0(\mathfrak{p}^n)$$

act on $V_0(n)$. At the minimal level they act by scalars λ and λ^* on the one-dimensional space $V_0(n)$. Since T_1^* is Atkin–Lehner conjugate to T_1, we actually have $\lambda = \lambda^*$. We call this number simply the *Hecke eigenvalue* of the representation π. To compute the Hecke eigenvalues we need the following lemma.

Lemma 6.4.1. *For $K = \mathrm{GL}(2,\mathfrak{o})$ we have disjoint decompositions*

$$\begin{aligned} K\begin{bmatrix} \varpi & \\ & 1 \end{bmatrix}K &= \begin{bmatrix} 1 & \\ & \varpi \end{bmatrix}K \sqcup \bigsqcup_{x \in \mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\begin{bmatrix} \varpi & \\ & 1 \end{bmatrix}K \\ &= \begin{bmatrix} \varpi & \\ & 1 \end{bmatrix}K \sqcup \bigsqcup_{x \in \mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & \\ x & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix}K. \end{aligned}$$

For any $n \geq 1$ we have disjoint decompositions

$$\Gamma_0(\mathfrak{p}^n)\begin{bmatrix} \varpi & \\ & 1 \end{bmatrix}\Gamma_0(\mathfrak{p}^n) = \bigsqcup_{x \in \mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\begin{bmatrix} \varpi & \\ & 1 \end{bmatrix}\Gamma_0(\mathfrak{p}^n)$$

and

$$\Gamma_0(\mathfrak{p}^n)\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix}\Gamma_0(\mathfrak{p}^n) = \bigsqcup_{x \in \mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & \\ x\varpi^n & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix}\Gamma_0(\mathfrak{p}^n).$$

Proof. These are well-known statements. The second decomposition can be quickly proved using the Iwahori factorization

$$\Gamma_0(\mathfrak{p}^n) = \begin{bmatrix} 1 & \mathfrak{o} \\ & 1 \end{bmatrix} \begin{bmatrix} \mathfrak{o}^\times & \\ & \mathfrak{o}^\times \end{bmatrix} \begin{bmatrix} 1 & \\ \mathfrak{p}^n & 1 \end{bmatrix}.$$

The last decomposition follows from the previous one by conjugating with the Atkin–Lehner element. □

Table 6.1 lists the Hecke eigenvalues of all the irreducible, admissible, infinite-dimensional representations of $\mathrm{GL}(2,F)$ with trivial central character. Using the formulas from Lemma 6.4.1, these eigenvalues can be quickly computed provided the local newform in the Kirillov model is known. The explicit form of the newform, as well as the minimal level, is given, for example, in [Sch1].

Table 6.1. Hecke eigenvalues for PGL(2).

representation	parameter	level	Hecke eigenvalue
$\chi \times \chi^{-1}$ (irreducible)	χ unramified	0	$q^{1/2}(\chi(\varpi) + \chi^{-1}(\varpi))$
	χ ramified	$2a(\chi)$	0
$\chi\mathrm{St}_{\mathrm{GL}(2)}$	χ unramified	1	$\chi(\varpi)$
	χ ramified	$2a(\chi)$	0
$\chi\mathbf{1}_{\mathrm{GL}(2)}$	χ unramified	0	$(q+1)\chi(\varpi)$
	χ ramified	—	—
π supercuspidal		≥ 2	0

Eigenvalues for Siegel Induced Representations

Consider an induced representation $\pi \rtimes \sigma$, where (π, V) is an irreducible, admissible, infinite-dimensional representation of $\mathrm{GL}(2,F)$ and σ is a character of $F^\times$ with $\omega_\pi\sigma^2 = 1$. We know by Theorem 5.2.2 that the minimal paramodular level of $\pi \rtimes \sigma$ is $n = a(\sigma\pi) + 2a(\sigma)$, and that the space of $\mathrm{K}(\mathfrak{p}^n)$-invariant vectors is one-dimensional. The following proposition gives the resulting Hecke eigenvalues.

Proposition 6.4.2. *Consider an induced representation $\pi \rtimes \sigma$ as above with $\omega_\pi\sigma^2 = 1$. Let $n = a(\sigma\pi) + 2a(\sigma)$ be the minimal paramodular level. Assume that $n \geq 1$. Let λ_1 be the Hecke eigenvalue of $\sigma\pi$ as listed in Table 6.1. Let f be a non-zero $\mathrm{K}(\mathfrak{p}^n)$-invariant vector in $\pi \rtimes \sigma$.*

i) We have $T_{0,1}f = \lambda f$ *with*

$$\lambda = \begin{cases} q^{3/2}(\sigma(\varpi)+\sigma(\varpi)^{-1}) + (q+1)\lambda_1 & \text{if } \sigma \text{ unramified, } a(\sigma\pi)=1,\\ q^{3/2}(\sigma(\varpi)+\sigma(\varpi)^{-1}) & \text{if } \sigma \text{ unramified, } a(\sigma\pi)\geq 2,\\ q\lambda_1 & \text{if } \sigma \text{ ramified, } a(\sigma\pi)\in\{0,1\},\\ 0 & \text{if } \sigma \text{ ramified, } a(\sigma\pi)\geq 2. \end{cases}$$

ii) We have $T_{1,0}f = \mu f$ *with*

$$\mu = \begin{cases} q^{3/2}(\sigma(\varpi)+\sigma(\varpi)^{-1})\lambda_1 & \text{if } \sigma \text{ unramified, } a(\sigma\pi)=1\\ 0 & \text{if } \sigma \text{ unramified, } a(\sigma\pi)\geq 2,\\ 0 & \text{if } \sigma \text{ ramified, } a(\sigma\pi)=0,\\ -q^2 & \text{if } \sigma \text{ ramified, } a(\sigma\pi)\geq 1. \end{cases}$$

The same formulas hold if $\pi = \chi\mathbf{1}_{\mathrm{GL}(2)}$ *and* $\sigma\chi$ *is unramified (if* $\sigma\chi$ *is ramified, then* $\chi\mathbf{1}_{\mathrm{GL}(2)}\rtimes\sigma$ *has no paramodular vectors); in this case* $\lambda_1 = (q+1)(\sigma\chi)(\varpi)$.

Proof. In the proof of Theorem 5.2.2 we found that if $a(\sigma)=0$, i.e., if σ is unramified, then a $\mathrm{K}(\mathfrak{p}^n)$-invariant function f in the standard model of $\pi\rtimes\sigma$ is supported on $P(F)\mathrm{K}(\mathfrak{p}^n)$ and has the property $f(1)=v$, where $v\in V$ is invariant under $\Gamma_0(\mathfrak{p}^n)$. If $a(\sigma)>0$, then a $\mathrm{K}(\mathfrak{p}^n)$-invariant function f is supported on $P(F)M_{a(\sigma)}$, where

$$M_i = \begin{bmatrix} 1 &&& \\ & 1 && \\ \varpi^i && 1 & \\ & \varpi^i && 1 \end{bmatrix},$$

and we have $f(M_{a(\sigma)}) = v$, where $v\in V$ is invariant under

$$(\sigma\pi)(\begin{bmatrix} \mathfrak{o} & \mathfrak{p}^{-a(\sigma)} \\ \mathfrak{p}^{a(\sigma\pi)+a(\sigma)} & \mathfrak{o} \end{bmatrix});$$

note that $\sigma\pi$ has trivial central character.

i) We shall now compute $T_{0,1}f$, starting with the case that σ is <u>unramified</u>. By Lemma 6.1.2, $(T_{0,1}f)(1) = A+B+C+D$ with

$$A = \sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}} f(\begin{bmatrix} 1 && y & z\varpi^{-n} \\ & 1 & x & y \\ && 1 & \\ &&& 1 \end{bmatrix}\begin{bmatrix} \varpi &&& \\ & \varpi && \\ && 1 & \\ &&& 1 \end{bmatrix}), \tag{6.30}$$

$$B = \sum_{x,z\in\mathfrak{o}/\mathfrak{p}} f(\begin{bmatrix} 1 & x && z\varpi^{-n} \\ & 1 && \\ && 1 & -x \\ &&& 1 \end{bmatrix}\begin{bmatrix} \varpi &&& \\ & 1 && \\ && \varpi & \\ &&& 1 \end{bmatrix}), \tag{6.31}$$

$$C = \sum_{x,y \in \mathfrak{o}/\mathfrak{p}} f(t_n \begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}), \tag{6.32}$$

$$D = \sum_{x \in \mathfrak{o}/\mathfrak{p}} f(t_n \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix}). \tag{6.33}$$

It is immediate that $A = q^{3/2}\sigma(\varpi)^{-1}v$. For B we get

$$B = q \sum_{x \in \mathfrak{o}/\mathfrak{p}} (\sigma\pi)(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} \varpi & \\ & 1 \end{bmatrix})v.$$

By Lemma 6.4.1, this equals qT_1v. Hence, by the values in Table 6.1,

$$B = \begin{cases} q\lambda_1 v & \text{if } a(\sigma\pi) = 1, \\ 0 & \text{if } a(\sigma\pi) \geq 2; \end{cases}$$

note that we are assuming $n \geq 1$ and $a(\sigma) = 0$, so that $a(\sigma\pi) \geq 1$. For C a straightforward computation shows that

$$C = q \sum_{y \in \mathfrak{o}/\mathfrak{p}} (\sigma\pi)(\begin{bmatrix} 1 & \\ y\varpi^n & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & \varpi \end{bmatrix})v.$$

Again by Lemma 6.4.1, this equals qT_1v. As above we conclude that

$$C = \begin{cases} q\lambda_1 v & \text{if } a(\sigma\pi) = 1, \\ 0 & \text{if } a(\sigma\pi) \geq 2. \end{cases}$$

Finally a computation shows that

$$D = q^{3/2}\sigma(\varpi) \sum_{x \in \mathfrak{o}/\mathfrak{p}} f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ x\varpi^{n-1} & & 1 & \\ & x\varpi^{n-1} & & 1 \end{bmatrix}).$$

First consider the case $a(\sigma\pi) \geq 2$. By (5.10), the argument of f is equivalent to M_1 if x is a unit. Hence only the term $x = 0$ survives, and we get $D = q^{3/2}\sigma(\varpi)v$. Now assume $a(\sigma\pi) = 1$. Then $n = 2a(\sigma) + a(\sigma\pi) = 1$. Using the identity

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ x & & 1 & \\ & x & & 1 \end{bmatrix} = \begin{bmatrix} & 1 & & \varpi^{-1} \\ \varpi & & 1 & \\ & & & x\varpi^{-1} \\ & & x & \end{bmatrix} \begin{bmatrix} & & & -x^{-1}\varpi^{-1} \\ & & -x^{-1} & \\ & 1 & & x^{-1} \\ \varpi & & x^{-1}\varpi & \end{bmatrix},$$

one computes

$$D = q^{3/2}\sigma(\varpi)v + (q-1)(\sigma\pi)(\begin{bmatrix} & 1 \\ \varpi & \end{bmatrix})v.$$

The representation $\sigma\pi$ is an unramified twist of $\mathrm{St}_{\mathrm{GL}(2)}$, and the matrix $\begin{bmatrix} & 1 \\ \varpi & \end{bmatrix}$ is the Atkin–Lehner element of level $\mathfrak{p}$. Its eigenvalue on the newform v is $\varepsilon(1/2,\sigma\pi)$, which is also equal to $-\lambda_1$. Hence

$$D = q^{3/2}\sigma(\varpi)v - (q-1)\lambda_1 v.$$

Adding up $A+B+C+D$ proves our assertion about λ for unramified σ.

Next consider the case that σ is ramified. Similarly as above we have $(T_{0,1}f)(M_{a(\sigma)}) = A+B+C+D$ with

$$\begin{aligned} A &= \sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}} f(M_{a(\sigma)} \begin{bmatrix} 1 & & y & z\varpi^{-n} \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}) \\ &= q^{-3/2}\sigma(\varpi)^{-1} \sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}} f(M_{a(\sigma)+1} \begin{bmatrix} 1 & & y\varpi^{-1} & z\varpi^{-n-1} \\ & 1 & x\varpi^{-1} & y\varpi^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix}), \end{aligned}$$

and so on. Writing

$$M_i = \begin{bmatrix} -\varpi^{-i} & & -1 & \\ & -\varpi^{-i} & & -1 \\ & & -\varpi^{i} & \\ & & & -\varpi^{i} \end{bmatrix} s_2s_1s_2 \begin{bmatrix} 1 & \varpi^{-i} & & \\ & 1 & & \varpi^{-i} \\ & & 1 & \\ & & & 1 \end{bmatrix},$$

we compute

$$\begin{aligned} A &= q^{3a(\sigma)+3/2}\sigma(-\varpi)^{-1}\omega_\pi(-\varpi^{-a(\sigma)-1}) \sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}} \\ &\quad f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ (y\varpi^{a(\sigma)}+1)\varpi^{-a(\sigma)-1} & z\varpi^{-n-1} & 1 & \\ x\varpi^{-1} & (y\varpi^{a(\sigma)}+1)\varpi^{-a(\sigma)-1} & & 1 \end{bmatrix} s_2s_1s_2) \\ &= q^{3a(\sigma)+5/2}\sigma(-\varpi)\omega_\pi(-\varpi^{-a(\sigma)}) \sum_{x,z\in\mathfrak{o}/\mathfrak{p}} \\ &\quad f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \varpi^{-a(\sigma)-1} & z\varpi^{-n-1} & 1 & \\ x\varpi^{-1} & \varpi^{-a(\sigma)-1} & & 1 \end{bmatrix} s_2s_1s_2). \end{aligned}$$

Let A_1 be the part of this expression where $z \in \mathfrak{o}^\times$, and let A_2 be the part where $z = 0$. Using the identity

$$\begin{bmatrix} 1 &&& \\ & 1 && \\ \varpi^{-a(\sigma)-1} & z\varpi^{-n-1} & 1 & \\ x\varpi^{-1} & \varpi^{-a(\sigma)-1} && 1 \end{bmatrix} = \begin{bmatrix} 1 &&& \\ -z^{-1}\varpi^{n-a(\sigma)} & \varpi^{n+1} & z^{-1} & \\ && \varpi^{-n-1} & \\ && z^{-1}\varpi^{-a(\sigma)-1} & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 &&& \\ & 1 && \\ \varpi^{n-a(\sigma)} && 1 & \\ (x - z^{-1}\varpi^{n-2a(\sigma)})\varpi^{-1} & \varpi^{n-a(\sigma)} && 1 \end{bmatrix} s_2 \begin{bmatrix} 1 &&& \\ & -z & -\varpi^{n+1} & \\ && -z^{-1} & \\ &&& 1 \end{bmatrix},$$

we get

$$A_1 = q^{3a(\sigma)+5/2}\sigma(-\varpi)\omega_\pi(-\varpi^{-a(\sigma)}) \sum_{x\in\mathfrak{o}/\mathfrak{p}} \sum_{z\in(\mathfrak{o}/\mathfrak{p})^\times} q^{-3(n+1)/2}$$
$$\pi(\begin{bmatrix} 1 & \\ -z^{-1}\varpi^{n-a(\sigma)} & \varpi^{n+1} \end{bmatrix}) f(\begin{bmatrix} 1 &&& \\ & 1 && \\ \varpi^{n-a(\sigma)} && 1 & \\ (x - z^{-1}\varpi^{n-2a(\sigma)})\varpi^{-1} & \varpi^{n-a(\sigma)} && 1 \end{bmatrix} s_1)$$
$$= q^{3a(\sigma)+5/2}\sigma(-\varpi)\omega_\pi(-\varpi^{-a(\sigma)}) \sum_{x\in\mathfrak{o}/\mathfrak{p}} \sum_{z\in(\mathfrak{o}/\mathfrak{p})^\times} q^{-3(n+1)/2}$$
$$\pi(\begin{bmatrix} 1 & \\ -z^{-1}\varpi^{n-a(\sigma)} & \varpi^{n+1} \end{bmatrix}) f(\begin{bmatrix} 1 &&& \\ & 1 && \\ \varpi^{n-a(\sigma)} && 1 & \\ x\varpi^{-1} & \varpi^{n-a(\sigma)} && 1 \end{bmatrix} s_1).$$

If x is a unit, then the argument of f is equal to

$$s_1 \begin{bmatrix} 1 &&& \\ -x^{-1}\varpi^{n-a(\sigma)+1} & \varpi & x^{-1} & \\ && \varpi^{-1} & \\ && x^{-1}\varpi^{n-a(\sigma)} & 1 \end{bmatrix} M_{n-a(\sigma)+1}$$
$$\begin{bmatrix} 1 &&& \\ && -x^{-1} & \\ & x & \varpi & \\ -x^{-1}\varpi^{2n-2a(\sigma)+1} &&& 1 \end{bmatrix},$$

which is not equivalent to $M_{a(\sigma)}$ in $P(F)\backslash\mathrm{GSp}(4,F)/\mathrm{K}(\mathfrak{p}^n)$. Hence only the term $x = 0$ survives, and we get

$$A_1 = q^{3a(\sigma)+5/2}\sigma(-\varpi)\omega_\pi(-\varpi^{-a(\sigma)}) \sum_{z\in(\mathfrak{o}/\mathfrak{p})^\times} q^{-3(n+1)/2}$$

$$\pi(\begin{bmatrix} 1 & \\ -z^{-1}\varpi^{n-a(\sigma)} & \varpi^{n+1} \end{bmatrix})f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \varpi^{n-a(\sigma)} & & 1 & \\ & \varpi^{n-a(\sigma)} & & 1 \end{bmatrix} s_1)$$

$$= q^{3a(\sigma)+5/2}\sigma(-\varpi)\omega_\pi(-\varpi^{-a(\sigma)}) \sum_{z\in(\mathfrak{o}/\mathfrak{p})^\times} q^{-3(n+1)/2}\,\sigma(-1)$$

$$q^{-3a(\sigma)+3n/2}\omega_\pi(-\varpi^{a(\sigma)-n})\pi(\begin{bmatrix} 1 & \\ -z^{-1}\varpi^{n-a(\sigma)} & \varpi^{n+1} \end{bmatrix}\begin{bmatrix} \varpi^n & \\ & 1 \end{bmatrix})f(M_{a(\sigma)})$$

$$= q \sum_{z\in(\mathfrak{o}/\mathfrak{p})^\times} (\sigma\pi)(\begin{bmatrix} 1 & \\ z\varpi^{n-a(\sigma)} & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix})v.$$

Next we compute

$$A_2 = q^{3a(\sigma)+5/2}\sigma(-\varpi)\omega_\pi(-\varpi^{-a(\sigma)}) \sum_{x\in\mathfrak{o}/\mathfrak{p}}$$

$$f(\begin{bmatrix} 1 & & & \\ & 1 & & \\ \varpi^{-a(\sigma)-1} & & 1 & \\ x\varpi^{-1} & \varpi^{-a(\sigma)-1} & & 1 \end{bmatrix} s_2 s_1 s_2).$$

Let A_{21} be the part where $x = 0$, and let A_{22} be the part where $x \in \mathfrak{o}^\times$. To compute A_{21}, note that

$$M_{-a(\sigma)-1}s_2s_1s_2 = \begin{bmatrix} & 1 & & \\ \varpi^n & & & \\ & & & \varpi^{-n} \\ & & 1 & \end{bmatrix} M_{n-a(\sigma)-1}t_n s_2 \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}.$$

This identity shows that $A_{21} = 0$ unless $n - a(\sigma) - 1 = a(\sigma)$. The latter condition is fulfilled if and only if $a(\sigma\pi) = 1$. Assuming this is the case, we have

$$A_{21} = q\sigma(-\varpi)\omega_\pi(-\varpi^{-a(\sigma)})\pi(\begin{bmatrix} & 1 \\ \varpi^n & \end{bmatrix})f(M_{a(\sigma)})$$

$$= q(\sigma\pi)(\begin{bmatrix} & \varpi^{-a(\sigma)} \\ \varpi^{a(\sigma)+1} & \end{bmatrix})v.$$

The matrix appearing here is a conjugate of the Atkin–Lehner involution for the representation $\sigma\pi$, which is an unramified twist of $\mathrm{St}_{\mathrm{GL}(2)}$. The eigenvalue of the Atkin–Lehner element on the newform is the sign of the ε factor, which is equal to $-\lambda_1$. Hence

$$A_{21} = \begin{cases} 0 & \text{if } a(\sigma\pi) \neq 1, \\ -q\lambda_1 v & \text{if } a(\sigma\pi) = 1. \end{cases}$$

Next we compute A_{22}, first considering the case that $a(\sigma\pi) \neq 0$. The identity

$$\begin{bmatrix} 1 &&& \\ & 1 && \\ \varpi^{-a(\sigma)-1} && 1 & \\ x\varpi^{-1} & \varpi^{-a(\sigma)-1} && 1 \end{bmatrix} s_2 s_1 s_2 = \begin{bmatrix} -\varpi^{a(\sigma)+1} & 1 & \varpi & \\ & -x\varpi^{a(\sigma)} && x \\ && \varpi^{-a(\sigma)} & \varpi^{-2a(\sigma)-1} \\ &&& x\varpi^{-a(\sigma)-1} \end{bmatrix}$$

$$M_{a(\sigma)} \begin{bmatrix} -1 &&& -x^{-1}\varpi^{-2a(\sigma)-1} \\ & -x^{-1}\varpi & 1 & \\ & x^{-1} && \\ &&& x^{-1} \end{bmatrix}$$

shows that

$$\begin{aligned} A_{22} &= q^{3a(\sigma)+5/2}\sigma(-\varpi)\omega_\pi(-\varpi^{-a(\sigma)}) \sum_{x\in(\mathfrak{o}/\mathfrak{p})^\times} q^{-3a(\sigma)-3/2}\sigma(-x) \\ &\quad \pi(\begin{bmatrix} -\varpi^{a(\sigma)+1} & 1 \\ & -x\varpi^{a(\sigma)} \end{bmatrix}) f(M_{a(\sigma)}) \\ &= q \sum_{x\in(\mathfrak{o}/\mathfrak{p})^\times} (\sigma\pi)(\begin{bmatrix} 1 & x\varpi^{-a(\sigma)} \\ & 1 \end{bmatrix} \begin{bmatrix} \varpi & \\ & 1 \end{bmatrix})v. \end{aligned}$$

Now consider the case $a(\sigma\pi) = 0$, or equivalently, $n = 2a(\sigma)$. In this case

$$\begin{bmatrix} 1 &&& \\ & 1 && \\ \varpi^{-a(\sigma)-1} && 1 & \\ x\varpi^{-1} & \varpi^{-a(\sigma)-1} && 1 \end{bmatrix} s_2 s_1 s_2 =$$

$$\begin{bmatrix} \varpi^{a(\sigma)+1} && 1 & \\ -x\varpi^{2a(\sigma)+1} & \varpi^{a(\sigma)+1} && 1 \\ && \varpi^{-a(\sigma)-1} & \\ && x\varpi^{-1} & \varpi^{-a(\sigma)-1} \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ -\varpi^{a(\sigma)+1} && 1 & \\ x\varpi^{2a(\sigma)+1} & -\varpi^{a(\sigma)+1} && 1 \end{bmatrix},$$

and this is not equivalent to $M_{a(\sigma)}$. Hence

$$A_{22} = \begin{cases} q \displaystyle\sum_{x\in(\mathfrak{o}/\mathfrak{p})^\times} (\sigma\pi)(\begin{bmatrix} 1 & x\varpi^{-a(\sigma)} \\ & 1 \end{bmatrix} \begin{bmatrix} \varpi & \\ & 1 \end{bmatrix})v & \text{if } a(\sigma\pi) \neq 0, \\ 0 & \text{if } a(\sigma\pi) = 0. \end{cases}$$

Next we compute

$$B = \sum_{x,z\in\mathfrak{o}/\mathfrak{p}} f(M_{a(\sigma)} \begin{bmatrix} 1 & x && z\varpi^{-n} \\ & 1 && \\ && 1 & -x \\ &&& 1 \end{bmatrix} \begin{bmatrix} \varpi &&& \\ & 1 && \\ && \varpi & \\ &&& 1 \end{bmatrix})$$

$$= \sum_{x,z\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix}1&x\\&1\end{bmatrix}) f(M_{a(\sigma)} \begin{bmatrix}1&&&z\varpi^{-n}\\&1&&\\&&1&\\&&&1\end{bmatrix} \begin{bmatrix}\varpi&&&\\&1&&\\&&\varpi&\\&&&1\end{bmatrix}).$$

If z is a unit, then the matrix identity

$$M_{a(\sigma)} \begin{bmatrix}1&&&z\varpi^{-n}\\&1&&\\&&1&\\&&&1\end{bmatrix} = \begin{bmatrix}&-\varpi^{-a(\sigma)}&&z\varpi^{-n}\\z^{-1}\varpi^{n-a(\sigma)}&z^{-1}\varpi^{n-2a(\sigma)}&-1&\\&&&z\varpi^{a(\sigma)-n}\\&&-\varpi^{a(\sigma)}&1\end{bmatrix}$$

$$\begin{bmatrix}1&&&\\&1&&\\z^{-1}\varpi^{n-a(\sigma)}&&1&\\z^{-1}\varpi^{n}&z^{-1}\varpi^{n-a(\sigma)}&&1\end{bmatrix} s_2 \begin{bmatrix}1&&&\\&1&-z^{-1}\varpi^{n-2a(\sigma)}&\\&&1&\\&&&1\end{bmatrix}$$

shows that the argument of f is equivalent to $M_{n-a(\sigma)+1}$ in the double coset space $P(F)\backslash\mathrm{GSp}(4,F)/\mathrm{K}(\mathfrak{p}^n)$, and we get zero. Hence only the term $z=0$ survives, and we get

$$B = \sum_{x\in\mathfrak{o}/\mathfrak{p}} (\sigma\pi)(\begin{bmatrix}1&x\\&1\end{bmatrix}\begin{bmatrix}\varpi&\\&1\end{bmatrix})v = q(\sigma\pi)(\begin{bmatrix}\varpi&\\&1\end{bmatrix})v.$$

Next we compute

$$C = \sum_{x,y\in\mathfrak{o}/\mathfrak{p}} f(M_{a(\sigma)} t_n \begin{bmatrix}1&&y&\\&1&x&y\\&&1&\\&&&1\end{bmatrix} \begin{bmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{bmatrix})$$

$$= \sum_{x,y\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix}1&\\y\varpi^n&1\end{bmatrix}) f(M_{a(\sigma)} \begin{bmatrix}1&&&\\&1&x&\\&&1&\\&&&1\end{bmatrix} \begin{bmatrix}1&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix}).$$

The identity

$$M_{a(\sigma)} \begin{bmatrix}1&&&\\&1&x&\\&&1&\\&&&1\end{bmatrix} \begin{bmatrix}1&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix}$$

$$= \begin{bmatrix}1&\varpi^{-a(\sigma)}&&-\varpi^{1-2a(\sigma)}\\&-x&&\\&&\varpi&-\varpi^{1-a(\sigma)}\\&&&-x\varpi\end{bmatrix} M_{a(\sigma)-1} \begin{bmatrix}1&&&-x^{-1}\varpi^{1-2a(\sigma)}\\&-x^{-1}\varpi&-1&\\&-x^{-1}&&\\&&&-x^{-1}\end{bmatrix}$$

shows that the argument of f is equivalent to $M_{a(\sigma)-1}$ if x is a unit. Hence only the term $x=0$ survives, and we get

$$C = \sum_{y \in \mathfrak{o}/\mathfrak{p}} (\sigma\pi)(\begin{bmatrix} 1 & \\ y\varpi^n & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & \varpi \end{bmatrix})v = q(\sigma\pi)(\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix})v.$$

Finally we compute

$$D = \sum_{x \in \mathfrak{o}/\mathfrak{p}} f(M_{a(\sigma)} t_n \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix}).$$

The argument of f is equal to

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ \varpi^{a(\sigma)}(1+x\varpi^{n-a(\sigma)}) & & 1 & \\ & \varpi^{a(\sigma)}(1+x\varpi^{n-a(\sigma)}) & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} t_n,$$

and therefore equivalent to $M_{a(\sigma)-1}$. Hence $D = 0$.

Now we summarize everything, starting with the case $a(\sigma\pi) = 0$. The non-zero terms are A_1, B and C, and we get

$$\begin{aligned} (T_{0,1}f)(M_{a(\sigma)}) &= A_1 + C + B \\ &= q \sum_{z \in \mathfrak{o}/\mathfrak{p}} (\sigma\pi)(\begin{bmatrix} 1 & \\ z\varpi^{n-a(\sigma)} & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & \varpi \end{bmatrix})v + q(\sigma\pi)(\begin{bmatrix} \varpi & \\ & 1 \end{bmatrix})v. \end{aligned}$$

By Lemma 6.4.1 this equals $q\lambda_1 v$, so we get $\lambda = q\lambda_1$. If $a(\sigma\pi) = 1$, then all of A_1, A_{21}, A_{22}, B and C are non-zero, and we get

$$\begin{aligned} (T_{0,1}f)(M_{a(\sigma)}) &= A_1 + C + A_{21} + A_{22} + B \\ &= q \sum_{z \in \mathfrak{o}/\mathfrak{p}} (\sigma\pi)(\begin{bmatrix} 1 & \\ z\varpi^{n-a(\sigma)} & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & \varpi \end{bmatrix})v - q\lambda_1 v \\ &\quad + q \sum_{x \in \mathfrak{o}/\mathfrak{p}} (\sigma\pi)(\begin{bmatrix} 1 & x\varpi^{-a(\sigma)} \\ & 1 \end{bmatrix} \begin{bmatrix} \varpi & \\ & 1 \end{bmatrix})v. \end{aligned}$$

By Lemma 6.4.1 this equals $q\lambda_1 v - q\lambda_1 v + q\lambda_1 v$, hence $\lambda = q\lambda_1$. Finally, if $a(\sigma\pi) \geq 2$, then the non-zero terms are A_1, A_{22}, B and C, and we get

$$\begin{aligned} (T_{0,1}f)(M_{a(\sigma)}) &= A_1 + C + A_{22} + B \\ &= q \sum_{z \in \mathfrak{o}/\mathfrak{p}} (\sigma\pi)(\begin{bmatrix} 1 & \\ z\varpi^{n-a(\sigma)} & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & \varpi \end{bmatrix})v \\ &\quad + q \sum_{x \in \mathfrak{o}/\mathfrak{p}} (\sigma\pi)(\begin{bmatrix} 1 & x\varpi^{-a(\sigma)} \\ & 1 \end{bmatrix} \begin{bmatrix} \varpi & \\ & 1 \end{bmatrix})v. \end{aligned}$$

By Lemma 6.4.1 this equals $2q\lambda_1 = 0$ (see Table 6.1).

ii) Next we consider μ, first assuming that σ is unramified. By Lemma 6.1.2, $(T_{1,0}f)(1) = A + B$ with

$$A = \sum_{x,y\in\mathfrak{o}/\mathfrak{p}} \sum_{z\in\mathfrak{o}/\mathfrak{p}^2} f(\begin{bmatrix}1&x&&\\&1&&\\&&1&-x\\&&&1\end{bmatrix}\begin{bmatrix}1&&y&z\varpi^{-n}\\&1&&y\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&\varpi&\\&&&1\end{bmatrix}),$$

$$B = \sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}} f(t_n\begin{bmatrix}1&x&&\\&1&&\\&&1&-x\\&&&1\end{bmatrix}\begin{bmatrix}1&&y&z\varpi^{-n+1}\\&1&&y\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&\varpi&\\&&&1\end{bmatrix}).$$

It is immediately computed that

$$A = q^{3/2}\sigma(\varpi)^{-1}\sum_{x\in\mathfrak{o}/\mathfrak{p}}(\sigma\pi)(\begin{bmatrix}1&x\\&1\end{bmatrix}\begin{bmatrix}\varpi&\\&1\end{bmatrix})v = q^{3/2}\sigma(\varpi)^{-1}\lambda_1 v;$$

again, note that $a(\sigma\pi) > 0$ since $a(\sigma) = 0$ and $n > 0$ by assumption. B can be rewritten as

$$\begin{aligned}B &= q^{3/2}\sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&\\y\varpi^n&1\end{bmatrix}\begin{bmatrix}\varpi^{-1}&\\&1\end{bmatrix})f(\begin{bmatrix}1&&&\\&1&&\\x\varpi^{n-1}&&1&\\z\varpi^{n-1}&x\varpi^{n-1}&&1\end{bmatrix})\\&= B_1 + B_2,\end{aligned}$$

where B_1 is the part of the sum where z is a unit, and B_2 is the part where $z = 0$. Using the matrix identity

$$\begin{aligned}&\begin{bmatrix}1&&&\\&1&&\\x\varpi^{n-1}&&1&\\z\varpi^{n-1}&x\varpi^{n-1}&&1\end{bmatrix}\\&=\begin{bmatrix}-z^{-1}\varpi&-z^{-1}x&&-\varpi^{-n}\\&1&&\\&&1&-x\varpi^{-1}\\&&&-z\varpi^{-1}\end{bmatrix}\begin{bmatrix}&&&\varpi^{-n}\\&1&&\\&-x^2z^{-1}\varpi^{n-1}&1&-z^{-1}x\\-\varpi^n&-xz^{-1}\varpi^n&&-z^{-1}\varpi\end{bmatrix},\end{aligned}$$

we get

$$B_1 = q^{3/2}\sum_{\substack{x,y,z\in\mathfrak{o}/\mathfrak{p}\\z\neq0}}\pi(\begin{bmatrix}1&\\y\varpi^n&1\end{bmatrix}\begin{bmatrix}\varpi^{-1}&\\&1\end{bmatrix})f(\begin{bmatrix}-z^{-1}\varpi&-z^{-1}x&&-\varpi^{-n}\\&1&&\\&&1&-x\varpi^{-1}\\&&&-z\varpi^{-1}\end{bmatrix})$$

$$= \sum_{\substack{x,y,z\in\mathfrak{o}/\mathfrak{p}\\ z\neq 0}} \pi(\begin{bmatrix}1&\\ y\varpi^n&1\end{bmatrix}\begin{bmatrix}\varpi^{-1}&\\ &1\end{bmatrix}\begin{bmatrix}-z^{-1}\varpi&-z^{-1}x\\ &1\end{bmatrix})v$$

$$= (q-1)\sum_{x,y\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix}1&\\ y\varpi^n&1\end{bmatrix}\begin{bmatrix}\varpi^{-1}&\\ &1\end{bmatrix}\begin{bmatrix}\varpi&x\\ &1\end{bmatrix})v$$

$$= (q-1)\sum_{x,y\in\mathfrak{o}/\mathfrak{p}} (\sigma\pi)(\begin{bmatrix}1&\\ y\varpi^n&1\end{bmatrix}\begin{bmatrix}\varpi^{-1}&\\ &1\end{bmatrix}\begin{bmatrix}1&x\\ &1\end{bmatrix}\begin{bmatrix}\varpi&\\ &1\end{bmatrix})v.$$

By Lemma 6.4.1, the summation over x amounts to applying T_1 to v. The same is true for the summation over y. We conclude

$$B_1 = \lambda_1^2(q-1)v.$$

Next we compute

$$B_2 = q^{3/2}\sum_{x,y\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix}1&\\ y\varpi^n&1\end{bmatrix}\begin{bmatrix}\varpi^{-1}&\\ &1\end{bmatrix})f(\begin{bmatrix}1&&&\\ &1&&\\ x\varpi^{n-1}&&1&\\ &x\varpi^{n-1}&&1\end{bmatrix}).$$

Let B_{21} be the part where $x\in\mathfrak{o}^\times$, and let B_{22} be the part where $x=0$. We have $B_{21}=0$ if $n\geq 2$, since then the argument of f is not equivalent to 1 in $P(F)\backslash\mathrm{GSp}(4,F)/\mathrm{K}(\mathfrak{p}^n)$. If $n=1$ we use the identity

$$\begin{bmatrix}1&&&\\ &1&&\\ x&&1&\\ &x&&1\end{bmatrix} = \begin{bmatrix}&1&&x^{-1}\varpi^{-1}\\ \varpi&&x^{-1}&\\ &&&\varpi^{-1}\\ &&1&\end{bmatrix}\begin{bmatrix}&&&-x^{-1}\varpi^{-1}\\ &&-x^{-1}&\\ &x&&1\\ x\varpi&&\varpi&\end{bmatrix}$$

and get

$$B_{21} = q^{3/2}\sum_{y\in\mathfrak{o}/\mathfrak{p}}\ \sum_{x\in(\mathfrak{o}/\mathfrak{p})^\times} \pi(\begin{bmatrix}1&\\ y\varpi&1\end{bmatrix}\begin{bmatrix}\varpi^{-1}&\\ &1\end{bmatrix})f(\begin{bmatrix}&1&&x^{-1}\varpi^{-1}\\ \varpi&&x^{-1}&\\ &&&\varpi^{-1}\\ &&1&\end{bmatrix})$$

$$= (q-1)\sum_{y\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix}1&\\ y\varpi&1\end{bmatrix}\begin{bmatrix}\varpi^{-1}&\\ &1\end{bmatrix}\begin{bmatrix}&1\\ \varpi&\end{bmatrix})f(1)$$

$$= (q-1)\sum_{y\in\mathfrak{o}/\mathfrak{p}} (\sigma\pi)(\begin{bmatrix}1&\\ y\varpi&1\end{bmatrix}\begin{bmatrix}\varpi^{-1}&\\ &1\end{bmatrix}\begin{bmatrix}&1\\ \varpi&\end{bmatrix})v.$$

The matrix $\begin{bmatrix}&1\\ \varpi&\end{bmatrix}$ is the Atkin–Lehner element acting on the newform v of the representation $\sigma\pi$, which is an unramified twist of $\mathrm{St}_{\mathrm{GL}(2)}$. The resulting eigenvalue is $-\lambda_1$. Hence

$$B_{21} = -\lambda_1(q-1)\sum_{y\in\mathfrak{o}/\mathfrak{p}}(\sigma\pi)(\begin{bmatrix}1&\\y\varpi&1\end{bmatrix}\begin{bmatrix}1&\\&\varpi\end{bmatrix})v.$$

By Lemma 6.4.1, we obtain

$$B_{21} = \begin{cases}0 & \text{if } n\geq 2,\\ -\lambda_1^2(q-1)v & \text{if } n=1.\end{cases}$$

It remains to compute

$$\begin{aligned}B_{22} &= q^{3/2}\sum_{y\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&\\y\varpi^n&1\end{bmatrix}\begin{bmatrix}\varpi^{-1}&\\&1\end{bmatrix})f(1)\\ &= q^{3/2}\sigma(\varpi)\sum_{y\in\mathfrak{o}/\mathfrak{p}}(\sigma\pi)(\begin{bmatrix}1&\\y\varpi^n&1\end{bmatrix}\begin{bmatrix}1&\\&\varpi\end{bmatrix})v\\ &= q^{3/2}\sigma(\varpi)\lambda_1 v.\end{aligned}$$

Summarizing everything, we get $\mu = q^{3/2}(\sigma(\varpi)+\sigma(\varpi)^{-1})\lambda_1$, since $\lambda_1=0$ for $n\geq 2$. This proves the assertion about μ in the unramified case.

Next we consider the case that σ is <u>ramified</u>. We have by Lemma 6.1.2, $(T_{1,0}f)(M_{a(\sigma)}) = A+B$ with

$$A = \sum_{x,y\in\mathfrak{o}/\mathfrak{p}}\sum_{z\in\mathfrak{o}/\mathfrak{p}^2} f(M_{a(\sigma)}\begin{bmatrix}1&x&&\\&1&&\\&&1&-x\\&&&1\end{bmatrix}\begin{bmatrix}1&y&z\varpi^{-n}&\\&1&&y\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&\varpi&\\&&&1\end{bmatrix}),$$

$$B = \sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}} f(M_{a(\sigma)}t_n\begin{bmatrix}1&x&&\\&1&&\\&&1&-x\\&&&1\end{bmatrix}\begin{bmatrix}1&y&z\varpi^{-n+1}&\\&1&&y\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&\varpi&\\&&&1\end{bmatrix}).$$

One computes

$$\begin{aligned}A &= q^{3a(\sigma)+1}\omega_\pi(-\varpi^{-a(\sigma)})\sum_{x\in\mathfrak{o}/\mathfrak{p}}\sum_{z\in\mathfrak{o}/\mathfrak{p}^2}\\ &\qquad f(s_2s_1s_2\begin{bmatrix}1&x&&\\&1&&\\&&1&-x\\&&&1\end{bmatrix}\begin{bmatrix}1&\varpi^{-a(\sigma)}&z\varpi^{-n}&\\&1&&\varpi^{-a(\sigma)}\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&\varpi&\\&&&1\end{bmatrix})\\ &= q^{3a(\sigma)+1}\omega_\pi(-\varpi^{-a(\sigma)})\sum_{x\in\mathfrak{o}/\mathfrak{p}}\sum_{z\in\mathfrak{o}/\mathfrak{p}^2}\\ &\qquad \pi(\begin{bmatrix}1&x\\&1\end{bmatrix})f(s_2s_1s_2\begin{bmatrix}1&\varpi^{-a(\sigma)}&z\varpi^{-n}&\\&1&&\varpi^{-a(\sigma)}\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&\varpi&\\&&&1\end{bmatrix}).\end{aligned}$$

We have the matrix identity

$$s_2s_1s_2\begin{bmatrix}1&\varpi^{-a(\sigma)}&&z\varpi^{-n}\\&1&&\varpi^{-a(\sigma)}\\&&1&\\&&&1\end{bmatrix}=\begin{bmatrix}&1&&\\-z^{-1}\varpi^{n}&-z^{-1}\varpi^{n-a(\sigma)}&&z^{-1}\\&&&-\varpi^{-n}\\&&z^{-1}&-z^{-1}\varpi^{-a(\sigma)}\end{bmatrix}$$
$$M_{n-a(\sigma)}\begin{bmatrix}1&&&\\&&1&\\&-z&\varpi^{n-2a(\sigma)}&\\\varpi^{n}&&&z\end{bmatrix}. \tag{6.34}$$

It shows that if z is a unit, then the argument of f is equivalent to $M_{n-a(\sigma)+1}$ in $P(F)\backslash\mathrm{GSp}(4,F)/\mathrm{K}(\mathfrak{p}^n)$. Hence

$$A=q^{3a(\sigma)+1}\omega_\pi(-\varpi^{-a(\sigma)})\sum_{x\in\mathfrak{o}/\mathfrak{p}}\sum_{z\in\mathfrak{o}/\mathfrak{p}}$$
$$\pi(\begin{bmatrix}1&x\\&1\end{bmatrix})f(s_2s_1s_2\begin{bmatrix}1&\varpi^{-a(\sigma)}&&z\varpi^{-n+1}\\&1&&\varpi^{-a(\sigma)}\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&\varpi&\\&&&1\end{bmatrix}).$$

Assume first that $a(\sigma\pi)=0$, i.e., $n=2a(\sigma)$. The identity

$$s_2s_1s_2\begin{bmatrix}1&\varpi^{-a(\sigma)}&&z\varpi^{-n+1}\\&1&&\varpi^{-a(\sigma)}\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&\varpi&\\&&&1\end{bmatrix}$$
$$=\begin{bmatrix}-z\varpi^{a(\sigma)+1}&\varpi&z\varpi^2&\\\varpi^{n}&&&\\&&-z\varpi^{2-n+a(\sigma)}&-\varpi^{2-n}\\&&-\varpi&\end{bmatrix}M_{a(\sigma)-1}\begin{bmatrix}&&&\varpi^{-n}\\&-z\varpi&1&\\&1&&\\\varpi^{n}&&&\end{bmatrix}$$

then shows that $A=0$. Assume now that $a(\sigma\pi)\neq 0$. Let A_1 be the part of the above expression for A where $z=0$, and let A_2 be the part where $z\in\mathfrak{o}^\times$. If $z=0$, then the argument of f equals

$$s_1\begin{bmatrix}\varpi^{n}&&&\\&1&&\\&&1&\\&&&\varpi^{-n}\end{bmatrix}\begin{bmatrix}1&&&\\&\varpi&&\\&&-\varpi&\\&&&-\varpi^2\end{bmatrix}M_{n-a(\sigma)-1}\begin{bmatrix}&&&\varpi^{-n}\\&&1&\\&1&&\\\varpi^{n}&&&\end{bmatrix}.$$

If $a(\sigma\pi)\geq 2$, then this is equivalent to $M_{a(\sigma)+1}$, and we get $A_1=0$. But if $a(\sigma\pi)=1$, we get

$$A_1=q^{3a(\sigma)+1}\omega_\pi(-\varpi^{-a(\sigma)})\sum_{x\in\mathfrak{o}/\mathfrak{p}}$$

$$\pi(\begin{bmatrix}1&x\\&1\end{bmatrix})f(s_1\begin{bmatrix}\varpi^n&&&\\&1&&\\&&1&\\&&&\varpi^{-n}\end{bmatrix}\begin{bmatrix}1&&&\\&\varpi&&\\&&-\varpi&\\&&&-\varpi^2\end{bmatrix}M_{a(\sigma)})$$

$$= q^{3a(\sigma)+1}\omega_\pi(-\varpi^{-a(\sigma)})\sigma(-1)\sum_{x\in\mathfrak{o}/\mathfrak{p}}$$

$$\pi(\begin{bmatrix}1&x\\&1\end{bmatrix}\begin{bmatrix}&1\\1&\end{bmatrix})f(\begin{bmatrix}\varpi^n&&&\\&1&&\\&&1&\\&&&\varpi^{-n}\end{bmatrix}\begin{bmatrix}1&&&\\&\varpi&&\\&&\varpi&\\&&&\varpi^2\end{bmatrix}M_{a(\sigma)})$$

$$= q\sum_{x\in\mathfrak{o}/\mathfrak{p}}(\sigma\pi)(\begin{bmatrix}1&x\\&1\end{bmatrix}\begin{bmatrix}\varpi&\\&1\end{bmatrix}\begin{bmatrix}&\varpi^{-a(\sigma)}\\\varpi^{n-a(\sigma)}&\end{bmatrix})v.$$

In the present case $a(\sigma\pi)=1$ we have $\pi=\chi\mathrm{St}_{\mathrm{GL}(2)}$ with $\sigma\chi$ an unramified quadratic character. The matrix $\begin{bmatrix}&\varpi^{-a(\sigma)}\\\varpi^{n-a(\sigma)}&\end{bmatrix}$ is (a conjugate of) the Atkin–Lehner element, whose eigenvalue on the local newform is $-(\sigma\chi)(\varpi)=-\lambda_1$, the sign of the ε-factor (see Table 6.1). Hence

$$A_1=\begin{cases}0 & \text{if } a(\sigma\pi)\neq 1,\\ -q^2\lambda_1(\sigma\pi)(\begin{bmatrix}\varpi&\\&1\end{bmatrix})v & \text{if } a(\sigma\pi)=1.\end{cases}$$

Next we compute A_2. Since we are assuming $a(\sigma\pi)\neq 0$, we can use the identity

$$s_2s_1s_2\begin{bmatrix}1&\varpi^{-a(\sigma)}&z\varpi^{-n+1}&\\&1&&\varpi^{-a(\sigma)}\\&&1&\\&&&1\end{bmatrix}=\begin{bmatrix}&1&&\\\varpi^{n-1}&-z^{-1}\varpi^{n-1-a(\sigma)}&&z^{-1}\\&&&-\varpi^{-n+1}\\&&-1&-z^{-1}\varpi^{-a(\sigma)}\end{bmatrix}$$
$$M_{n-1-a(\sigma)}\begin{bmatrix}-z^{-1}&&&\\&&1&\\&1&-z^{-1}\varpi^{n-1-2a(\sigma)}&\\\varpi^{n-1}&&&z\end{bmatrix}$$

and get

$$A_2=q^{3a(\sigma)+1}\omega_\pi(-\varpi^{-a(\sigma)})\sum_{x\in\mathfrak{o}/\mathfrak{p}}\sum_{z\in(\mathfrak{o}/\mathfrak{p})^\times}q^{-3n/2+3/2}\sigma(-1)$$

$$\pi(\begin{bmatrix}1&x\\&1\end{bmatrix}\begin{bmatrix}&1\\\varpi^{n-1}&-z^{-1}\varpi^{n-1-a(\sigma)}\end{bmatrix})f(M_{n-1-a(\sigma)}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&\varpi&\\&&&1\end{bmatrix})$$

$$= q \sum_{x\in\mathfrak{o}/\mathfrak{p}} \sum_{z\in(\mathfrak{o}/\mathfrak{p})^\times} (\sigma\pi)(\begin{bmatrix}1&x\\&1\end{bmatrix}\begin{bmatrix}\varpi&\\z\varpi^{n-a(\sigma)}&\varpi\end{bmatrix})v$$

$$= q \sum_{x\in\mathfrak{o}/\mathfrak{p}} \sum_{z\in\mathfrak{o}/\mathfrak{p}} (\sigma\pi)(\begin{bmatrix}1&x\\&1\end{bmatrix}\begin{bmatrix}\varpi&\\z\varpi^{n-a(\sigma)}&\varpi\end{bmatrix})v$$

$$- q \sum_{x\in\mathfrak{o}/\mathfrak{p}} (\sigma\pi)(\begin{bmatrix}1&x\\&1\end{bmatrix}\begin{bmatrix}\varpi&\\&\varpi\end{bmatrix})v$$

$$= q \sum_{x\in\mathfrak{o}/\mathfrak{p}} \sum_{z\in\mathfrak{o}/\mathfrak{p}} (\sigma\pi)(\begin{bmatrix}1&x\\&1\end{bmatrix}\begin{bmatrix}\varpi&\\&1\end{bmatrix}\begin{bmatrix}1&\\z\varpi^{n-a(\sigma)}&1\end{bmatrix}\begin{bmatrix}1&\\&\varpi\end{bmatrix})v - q^2v$$

$$= q\lambda_1 \sum_{x\in\mathfrak{o}/\mathfrak{p}} (\sigma\pi)(\begin{bmatrix}1&x\\&1\end{bmatrix}\begin{bmatrix}\varpi&\\&1\end{bmatrix})v - q^2v$$

$$= q^2\lambda_1(\sigma\pi)(\begin{bmatrix}\varpi&\\&1\end{bmatrix})v - q^2v.$$

Since $a(\sigma\pi) \geq 2$ if and only if $\lambda_1 = 0$, we get

$$A = A_1 + A_2 = -q^2v.$$

As for B, it is easily computed that

$$B = q \sum_{y,z\in\mathfrak{o}/\mathfrak{p}} f(M_{a(\sigma)}t_n \begin{bmatrix}1&y&z\varpi^{-n+1}&\\&1&&y\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&\varpi&\\&&&1\end{bmatrix})$$

$$= q \sum_{y,z\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix}1&\\y\varpi^n&1\end{bmatrix})f(M_{a(\sigma)}t_n \begin{bmatrix}1&&z\varpi^{-n+1}&\\&1&&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&\varpi&\\&&&1\end{bmatrix})$$

$$= q \sum_{y,z\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix}1&\\y\varpi^n&1\end{bmatrix})f(M_{a(\sigma)} \begin{bmatrix}&1&&\\&&1&\\&&&1\\z\varpi^{n+1}&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&\varpi&&\\&&\varpi&\\&&&\varpi^2\end{bmatrix}).$$

It follows from (5.8) that the argument of f is equivalent to $M_{a(\sigma)-1}$. Hence $B = 0$. This proves the assertion about μ if σ is ramified. □

Corollary 6.4.3. *Let ξ be the non-trivial, unramified, quadratic character of $F^\times$, and let σ be an unramified, quadratic character. Then the representation $\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$ of type Va has minimal paramodular level $n = 2$. The $T_{0,1}$ eigenvalue on the essentially unique newform is $\lambda = 0$. The $T_{1,0}$ eigenvalue is $\mu = -q^2 - q$.*

Proof. Since $T_{0,1}$ changes its sign and $T_{1,0}$ remains invariant when the representation is twisted with ξ, we may assume that $\sigma = 1$. By (2.10), the representation $L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2})$ of type Vb is a subrepresentation of $\nu^{-1/2}\xi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{1/2}$. Let f be the newform of level $\mathfrak{p}$ in $\nu^{-1/2}\xi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{1/2}$; it lies in the subspace realizing Vb. By Table A.13, $u_1 f = f$. Let f_0 be the newform of level $\mathfrak{p}$ in $\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}$. Evidently, $u_1 f_0 = f_0$. By the remarks following Proposition 5.5.5, the newform in $\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$ is given by $\theta f_0 - \theta' f_0$. By Proposition 6.4.2 we have $T_{0,1} f_0 = (q^2-1) f_0$ and $T_{1,0} f_0 = (-q^2-q) f_0$. Hence the assertion follows from Corollary 6.3.2 ii). □

Corollary 6.4.4. *Let π be a supercuspidal representation of* $\mathrm{PGL}(2,F)$*, and σ an unramified, quadratic character of $F^\times$.*

i) The $T_{0,1}$ eigenvalue on the newform of a representation $\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$ of type XIa is $\lambda = q\sigma(\varpi)$. The $T_{1,0}$ eigenvalue is $\mu = -q^2$.
ii) The $T_{0,1}$ eigenvalue on the newform of a representation $L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$ of type XIb is $\lambda = q(q+1)\sigma(\varpi)$. The $T_{1,0}$ eigenvalue is $\mu = 0$.

Proof. We may assume that $\sigma = 1$. There are exact sequences

$$0 \longrightarrow L(\nu^{1/2}\pi,\nu^{-1/2}) \longrightarrow \nu^{-1/2}\pi \rtimes \nu^{1/2} \longrightarrow \delta(\nu^{1/2}\pi,\nu^{-1/2}) \longrightarrow 0$$

and

$$0 \longrightarrow \delta(\nu^{1/2}\pi,\nu^{-1/2}) \longrightarrow \nu^{1/2}\pi \rtimes \nu^{-1/2} \longrightarrow L(\nu^{1/2}\pi,\nu^{-1/2}) \longrightarrow 0.$$

The induced representations in the middle have a unique newform of level $n = a(\pi)$. By Proposition 5.5.5, this newform lies in the subspace $L(\nu^{1/2}\pi,\nu^{-1/2})$ of $\nu^{-1/2}\pi \rtimes \nu^{1/2}$. Hence ii) is immediate from Proposition 6.4.2. Let f_0 be the newform in $\nu^{1/2}\pi \rtimes \nu^{-1/2}$. By the remarks following Proposition 5.5.5, the newform in $\delta(\nu^{1/2}\pi,\nu^{-1/2})$ is given by $\theta f_0 - \theta' f_0$. Hence i) follows from Proposition 6.4.2 and Corollary 6.3.2 i). □

Eigenvalues for Klingen Induced Representations

Consider an induced representation $\chi \rtimes \sigma\pi$, where χ and σ are characters of $F^\times$ and $\pi = \mathrm{St}_{\mathrm{GSp}(2)}$. We assume $\chi\sigma^2 = 1$, so that the central character of $\chi \rtimes \sigma\pi$ is trivial. Let V be a model for π. By Theorem 5.4.2, the minimal level of $\chi \rtimes \sigma\pi$ is $n = 2a(\sigma\pi)$. We shall concentrate on the case of unramified χ and σ, so that $n = 2$. The double coset space $Q(F)\backslash\mathrm{GSp}(4,F)/\mathrm{K}(\mathfrak{p}^2)$ is represented by $\{1, s_1, L_1\}$; see Proposition 5.1.2. By the proof of Theorem 5.4.2, there is an essentially unique $\mathrm{K}(\mathfrak{p}^2)$-invariant vector f in the standard model of $\chi \rtimes \sigma\pi$, which is supported on $Q(F)L_1\mathrm{K}(\mathfrak{p}^2)$. It has the property that $f(L_1) = v$, where $v \in V$ is the newform for $\sigma\pi$, with the invariance property

$$\pi(\begin{bmatrix} a & b \\ c & d \end{bmatrix})v = v \qquad \text{for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o}^\times \end{bmatrix}. \tag{6.35}$$

Proposition 6.4.5. *Let χ and σ be unramified characters of $F^\times$ with $\chi\sigma^2 = 1$. Then the induced representation $\chi \rtimes \sigma\mathrm{St}_{\mathrm{GL}(2)}$ has minimal level $n = 2$. Let f be a non-zero $\mathrm{K}(\mathfrak{p}^2)$-invariant vector. Then*

$$T_{0,1}f = \lambda f \qquad \text{with} \qquad \lambda = q(\sigma(\varpi) + \sigma(\varpi)^{-1})$$

and

$$T_{1,0}f = \mu f \qquad \text{with} \qquad \mu = -q(q-1).$$

Proof. We have $(T_{0,1}f)(L_1) = A + B + C + D$ with

$$A = \sum_{x,y,z \in \mathfrak{o}/\mathfrak{p}} f(L_1 \begin{bmatrix} 1 & & y & z\varpi^{-n} \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}), \tag{6.36}$$

$$B = \sum_{x,z \in \mathfrak{o}/\mathfrak{p}} f(L_1 \begin{bmatrix} 1 & x & & z\varpi^{-n} \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix}), \tag{6.37}$$

$$C = \sum_{x,y \in \mathfrak{o}/\mathfrak{p}} f(L_1 t_2 \begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}), \tag{6.38}$$

$$D = \sum_{x \in \mathfrak{o}/\mathfrak{p}} f(L_1 t_2 \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix}). \tag{6.39}$$

(compare (6.30) – (6.33)). We compute

$$\begin{aligned} A &= q^2 \sum_{x \in \mathfrak{o}/\mathfrak{p}} f(\begin{bmatrix} 1 & & & \\ & 1 & x & \\ & & 1 & \\ & & & 1 \end{bmatrix} L_1 \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}) \\ &= q^2 \sum_{x \in \mathfrak{o}/\mathfrak{p}} (\sigma\pi)(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}) f(L_1 \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}) \\ &= q\,\chi(\varpi) \sum_{x \in \mathfrak{o}/\mathfrak{p}} (\sigma\pi)(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} \varpi & \\ & 1 \end{bmatrix}) v. \end{aligned}$$

The vector v is the local newform for the representation $\sigma\pi = \sigma\mathrm{St}_{\mathrm{GL}(2)}$, which may not have trivial central character. However, since σ is unramified, v is characterized by the property (6.35). Thus, applying the GL(2) Hecke operator T_1 to v produces a multiple of v. The resulting Hecke eigenvalue can be computed as in the case of trivial central character, and can be read off from Table 6.1. Hence we get

$$A = q\,\chi(\varpi)\sigma(\varpi)v = q\sigma(\varpi)^{-1}v.$$

For each $x, z \in \mathfrak{o}$ we have the matrix identity

$$L_1 \begin{bmatrix} 1 & x & & z\varpi^{-2} \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \varpi u^{-1} & x & zu^{-1} & z\varpi^{-2} \\ & u & zu^{-1}\varpi & z\varpi^{-1} \\ & & \varpi u^{-1} & -x \\ & & & u \end{bmatrix} \begin{bmatrix} 1 & & & \\ u^{-1}\varpi^2 & 1 & & \\ & & 1 & \\ & & -u^{-1}\varpi^2 & 1 \end{bmatrix}, \qquad u = 1 + x\varpi.$$

Since u is a unit, it shows that $B = 0$. We have

$$\begin{aligned} C &= \sum_{x,y \in \mathfrak{o}/\mathfrak{p}} (\sigma\pi)(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}) f(L_1 t_2 \begin{bmatrix} 1 & & y & \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}) \\ &= \sum_{x,y \in \mathfrak{o}/\mathfrak{p}} (\sigma\pi)(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}) f(L_1 \begin{bmatrix} 1 & & & \\ y\varpi^2 & 1 & & \\ & & 1 & \\ & & -y\varpi^2 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}) \\ &= q \sum_{x,y \in \mathfrak{o}/\mathfrak{p}} (\sigma\pi)(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} \varpi & \\ & 1 \end{bmatrix}) f(\begin{bmatrix} 1 & & & \\ 1 + y\varpi & 1 & & \\ & & 1 & \\ & & -1 - y\varpi & 1 \end{bmatrix}). \end{aligned}$$

The argument of f is equivalent to s_1 in $Q(F)\backslash \mathrm{GSp}(4, F)/\mathrm{K}(\mathfrak{p}^2)$, showing that $C = 0$. Finally,

$$D = q \sum_{x \in \mathfrak{o}/\mathfrak{p}} (\sigma\pi)(\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix}) f(L_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ x\varpi & & 1 & \\ & x\varpi & & 1 \end{bmatrix}).$$

Let D_1 be the term of this sum where $x = 0$, and D_2 be the sum over $x \in (\mathfrak{o}/\mathfrak{p})^\times$. Evidently,

$$D_1 = q(\sigma\pi)(\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix})v.$$

As for D_2, the matrix identity

$$L_1M_1 = \begin{bmatrix}1&&&\\&1&&\\&1&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&\varpi^{-1}&\varpi^{-2}\\&1&2&\varpi^{-1}\\&&-1&\\&&&-1\end{bmatrix}L_1\begin{bmatrix}&&&\varpi^{-2}\\&&1&\\&1&-1&\\\varpi^2&&&-1\end{bmatrix}$$

shows that

$$D_2 = q\sum_{x\in(\mathfrak{o}/\mathfrak{p})^\times}(\sigma\pi)(\begin{bmatrix}1&\\&x\varpi\end{bmatrix}\begin{bmatrix}1&\\1&1\end{bmatrix}\begin{bmatrix}1&2\\&-1\end{bmatrix})f(L_1)$$
$$= q\sum_{x\in(\mathfrak{o}/\mathfrak{p})^\times}(\sigma\pi)(\begin{bmatrix}1&\\x\varpi&1\end{bmatrix}\begin{bmatrix}1&\\&\varpi\end{bmatrix})v.$$

Hence

$$D = D_1 + D_2 = q\sum_{x\in\mathfrak{o}/\mathfrak{p}}(\sigma\pi)(\begin{bmatrix}1&\\x\varpi&1\end{bmatrix}\begin{bmatrix}1&\\&\varpi\end{bmatrix})v.$$

Similarly as above, the summation amounts to applying T_1 to the newform v for the representation $\sigma\mathrm{St}_{\mathrm{GL}(2)}$. Hence, by Table 6.1,

$$D = q\sigma(\varpi)v.$$

Summarizing, we get

$$(T_{0,1}f)(L_1) = A + D = q(\sigma(\varpi)+\sigma(\varpi)^{-1})v = q(\sigma(\varpi)+\sigma(\varpi)^{-1})f(L_1),$$

proving the assertion about λ.

Next we compute μ. By Lemma 6.1.2, $(T_{1,0}f)(L_1) = A + B$ with

$$A = \sum_{x,y\in\mathfrak{o}/\mathfrak{p}}\sum_{z\in\mathfrak{o}/\mathfrak{p}^2} f(L_1\begin{bmatrix}1&x&&\\&1&&\\&&1&-x\\&&&1\end{bmatrix}\begin{bmatrix}1&&y&z\varpi^{-2}\\&1&&y\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&\varpi&\\&&&1\end{bmatrix}),$$

$$B = \sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}} f(L_1t_2\begin{bmatrix}1&x&&\\&1&&\\&&1&-x\\&&&1\end{bmatrix}\begin{bmatrix}1&&y&z\varpi^{-2+1}\\&1&&y\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&\varpi&\\&&&1\end{bmatrix}).$$

We compute

$$A = \sum_{x,y\in\mathfrak{o}/\mathfrak{p}}\sum_{z\in\mathfrak{o}/\mathfrak{p}^2} f(\begin{bmatrix}1&&y+z\varpi^{-1}&z\varpi^{-2}\\&1&2\varpi y+z&y+z\varpi^{-1}\\&&1&\\&&&1\end{bmatrix}L_1$$
$$\begin{bmatrix}1&x&&\\&1&&\\&&1&-x\\&&&1\end{bmatrix}\begin{bmatrix}\varpi^2&&&\\&\varpi&&\\&&\varpi&\\&&&1\end{bmatrix})$$

$$= \sum_{x,y\in\mathfrak{o}/\mathfrak{p}} \sum_{z\in\mathfrak{o}/\mathfrak{p}^2} (\sigma\pi)(\begin{bmatrix} 1 & 2\varpi y + z \\ & 1 \end{bmatrix}) f(L_1 \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix}).$$

Now for every $x \in \mathfrak{o}$,

$$\begin{aligned} & L_1 \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} \\ & = \begin{bmatrix} u^{-1}\varpi^2 & x\varpi & & \\ & u\varpi & & \\ & & u^{-1}\varpi & -x \\ & & & u \end{bmatrix} \begin{bmatrix} 1 & & & \\ u^{-1}\varpi^2 & 1 & & \\ & & 1 & \\ & & -u^{-1}\varpi^2 & 1 \end{bmatrix}, \qquad u = 1 + x\varpi. \end{aligned}$$

Since u is a unit, this identity shows that $A = 0$. As for B, we have

$$B = q^2\sigma^2(\varpi) \sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}} f(\begin{bmatrix} 1 & & & \\ 1 + y\varpi & 1 & & \\ x\varpi & & 1 & \\ z\varpi & x\varpi & -1 - y\varpi & 1 \end{bmatrix}).$$

Using the identity (2.8) on the matrix $\begin{bmatrix} 1 & \\ 1 + y\varpi & 1 \end{bmatrix}$ and noting that $1 + y\varpi$ is a unit, we get

$$\begin{aligned} B &= q^2\sigma^2(\varpi) \sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}} f(s_1 \begin{bmatrix} 1 & \frac{1}{1+y\varpi} & & \\ & 1 & & \\ & & 1 & -\frac{1}{1+y\varpi} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ x\varpi & & 1 & \\ z\varpi & x\varpi & & 1 \end{bmatrix}) \\ &= q^2\sigma^2(\varpi) \sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}} f(s_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ (x+zu)\varpi & (2xu + zu^2)\varpi & 1 & \\ z\varpi & (x+zu)\varpi & & 1 \end{bmatrix}) \quad (u = \frac{-1}{1+y\varpi}) \\ &= q^2\sigma^2(\varpi) \sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}} (\sigma\pi)(\begin{bmatrix} 1 & \\ z\varpi & 1 \end{bmatrix}) f(s_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ (x+zu)\varpi & & 1 & \\ & (x+zu)\varpi & & 1 \end{bmatrix}) \\ &= q^3\sigma^2(\varpi) \sum_{z\in\mathfrak{o}/\mathfrak{p}} \sum_{x\in(\mathfrak{o}/\mathfrak{p})^\times} (\sigma\pi)(\begin{bmatrix} 1 & \\ z\varpi & 1 \end{bmatrix}) f(s_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ x\varpi & & 1 & \\ & x\varpi & & 1 \end{bmatrix}). \end{aligned}$$

The identity

$$s_1 \begin{bmatrix} 1 &&& \\ & 1 && \\ x\varpi && 1 & \\ & x\varpi && 1 \end{bmatrix} = \begin{bmatrix} x^{-1}\varpi & 1 && \\ && -x^{-1}\varpi^{-1} & \varpi^{-2} \\ & x\varpi && \\ &&& x\varpi^{-1} \end{bmatrix} \begin{bmatrix} 1 &&& \\ -x^{-1}\varpi & 1 && \\ && 1 & \\ && x^{-1}\varpi & 1 \end{bmatrix} t_2$$

shows that

$$\begin{aligned} B &= q \sum_{z\in\mathfrak{o}/\mathfrak{p}} \sum_{x\in(\mathfrak{o}/\mathfrak{p})^\times} (\sigma\pi)(\begin{bmatrix} 1 & \\ z\varpi & 1 \end{bmatrix} \begin{bmatrix} & -\varpi^{-1} \\ \varpi & \end{bmatrix}) f(\begin{bmatrix} 1 &&& \\ -x^{-1}\varpi & 1 && \\ && 1 & \\ && x^{-1}\varpi & 1 \end{bmatrix}) \\ &= q(q-1) \sum_{z\in\mathfrak{o}/\mathfrak{p}} (\sigma\pi)(\begin{bmatrix} 1 & \\ z\varpi & 1 \end{bmatrix} \begin{bmatrix} \varpi^{-1} & \\ & 1 \end{bmatrix} \begin{bmatrix} & -1 \\ \varpi & \end{bmatrix})v. \\ &= q(q-1)\sigma^2(\varpi^{-1}) \sum_{z\in\mathfrak{o}/\mathfrak{p}} (\sigma\pi)(\begin{bmatrix} 1 & \\ z\varpi & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & \varpi \end{bmatrix} \begin{bmatrix} & 1 \\ \varpi & \end{bmatrix})v. \end{aligned}$$

The Atkin–Lehner element $\begin{bmatrix} & 1 \\ \varpi & \end{bmatrix}$ has eigenvalue $-\sigma(\varpi)$ on the newform v. The summation over z amounts to applying T_1 to the newform, which yields another factor $\sigma(\varpi)$. Hence $B = -q(q-1)v$, proving the assertion about μ. □

Corollary 6.4.6. *Let f be the newform of level $\mathfrak{p}^3$ for $\sigma\mathrm{St}_{\mathrm{GSp}(4)}$, where σ is an unramified, quadratic character. Then*

$$T_{0,1}f = \lambda f \qquad \textit{with} \qquad \lambda = \sigma(\varpi)$$

and

$$T_{1,0}f = \mu f \qquad \textit{with} \qquad \mu = -q^2.$$

Proof. Given the behavior of Hecke operators under twisting, we may assume that $\sigma = 1$. There is an exact sequence

$$0 \longrightarrow \mathrm{St}_{\mathrm{GSp}(4)} \longrightarrow \nu^2 \rtimes \nu^{-1}\mathrm{St}_{\mathrm{GSp}(2)} \longrightarrow L(\nu^2, \nu^{-1}\mathrm{St}_{\mathrm{GSp}(2)}) \longrightarrow 0;$$

see (2.9). Let f_0 be the newform of level $\mathfrak{p}^2$ in $\nu^2 \rtimes \nu^{-1}\mathrm{St}_{\mathrm{GSp}(2)}$. Let p be the projection from $\nu^2 \rtimes \nu^{-1}\mathrm{St}_{\mathrm{GSp}(2)}$ to $L(\nu^2, \nu^{-1}\mathrm{St}_{\mathrm{GSp}(2)})$. Since $\mathrm{St}_{\mathrm{GSp}(4)}$ has minimal level $\mathfrak{p}^3$, we have $p(f_0) \neq 0$, so that $p(f_0)$ is the newform for $L(\nu^2, \nu^{-1}\mathrm{St}_{\mathrm{GSp}(2)})$. By Proposition 5.5.13 we have $\theta p(f_0) = \theta' p(f_0)$. Hence $p(\theta f_0 - \theta' f_0) = 0$. This means that $f_1 := \theta f_0 - \theta' f_0$ lies in the subspace of $\nu^2 \rtimes \nu^{-1}\mathrm{St}_{\mathrm{GSp}(2)}$ realizing the representation $\mathrm{St}_{\mathrm{GSp}(4)}$. Again by Proposition 5.5.13 we have $f_1 \neq 0$, so that f_1 is the newform for $\mathrm{St}_{\mathrm{GSp}(4)}$. Using Corollary 6.3.2 and Proposition 6.4.5, it is now easy to compute $T_{0,1}f_1$ and $T_{1,0}f_1$. □

6.5 Some Consequences of Unitarity

To close this chapter, we prove that if the representation π is unitary, then the Hecke operators $T_{0,1}$ and $T_{1,0}$ are self-adjoint and hence diagonalizable. As mentioned at the beginning of this chapter, this will play an important role in the proof of uniqueness at the minimal level for supercuspidal representations; see Theorem 7.5.1.

Lemma 6.5.1. *Let (π, V) be a smooth representation of $\mathrm{GSp}(4,F)$ for which the center acts trivially. Assume that π is unitary with inner product $\langle\cdot,\cdot\rangle$. Let K be a compact, open subgroup of $\mathrm{GSp}(4,F)$, and fix the Haar measure on $\mathrm{GSp}(4,F)$ that gives K volume one. For $h \in \mathrm{GSp}(4,F)$ let T_h be the characteristic function of KhK. Then*

$$\langle T_h v, w\rangle = \langle v, T_{h^{-1}} w\rangle$$

for $h \in \mathrm{GSp}(4,F)$ and $v, w \in V^K$. That is, $T_{h^{-1}}$ is the adjoint of T_h for $h \in \mathrm{GSp}(4,F)$.

Proof. Let $h \in \mathrm{GSp}(4,F)$. Let $v, w \in V^K$. Then, by (6.2),

$$\begin{aligned}
\mathrm{vol}(K \cap hKh^{-1})\langle T_h v, w\rangle &= \langle \int\limits_K \pi(k)\pi(h)v\,dk, w\rangle \\
&= \int\limits_K \langle \pi(k)\pi(h)v, w\rangle\,dk \\
&= \int\limits_K \langle v, \pi(h^{-1})\pi(k^{-1})w\rangle\,dk \\
&= \int\limits_K \langle v, \pi(h^{-1})w\rangle\,dk \\
&= \int\limits_K \langle \pi(k^{-1})v, \pi(h^{-1})w\rangle\,dk \\
&= \int\limits_K \langle v, \pi(k)\pi(h^{-1})w\rangle\,dk \\
&= \langle v, \int\limits_K \pi(k)\pi(h^{-1})w\,dk\rangle \\
&= \mathrm{vol}(K \cap h^{-1}Kh)\langle v, T_{h^{-1}}w\rangle.
\end{aligned}$$

Since $\mathrm{vol}(K \cap hKh^{-1}) = \mathrm{vol}(K \cap h^{-1}Kh)$, the proof is complete. □

Proposition 6.5.2. *Let (π, V) be a smooth representation of $\mathrm{GSp}(4,F)$ for which the center acts trivially. Assume that π is unitary with inner product $\langle\cdot,\cdot\rangle$. Let $n \geq 0$ be a non-negative integer. Then $T_{0,1} : V(n) \to V(n)$ and $T_{1,0} : V(n) \to V(n)$ are self-adjoint.*

Proof. Let

$$h_1 = \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \quad h_2 = \begin{bmatrix} \varpi^2 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{bmatrix},$$

so that $T_{0,1} = T_{h_1}$ and $T_{1,0} = T_{h_2}$. The assertion of the lemma follows from Lemma 6.5.1, the equalities

$$\mathrm{K}(\mathfrak{p}^n)h_1\mathrm{K}(\mathfrak{p}^n) = \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} \mathrm{K}(\mathfrak{p}^n)h_1^{-1}\mathrm{K}(\mathfrak{p}^n)$$

and

$$\mathrm{K}(\mathfrak{p}^n)h_2\mathrm{K}(\mathfrak{p}^n) = \begin{bmatrix} \varpi^2 & & & \\ & \varpi^2 & & \\ & & \varpi^2 & \\ & & & \varpi^2 \end{bmatrix} \mathrm{K}(\mathfrak{p}^n)h_2^{-1}\mathrm{K}(\mathfrak{p}^n),$$

and the assumption that the center of $\mathrm{GSp}(4, F)$ acts trivially on V. □

Corollary 6.5.3. *Let (π, V) be an admissible representation of* $\mathrm{GSp}(4, F)$ *for which the center acts trivially. Assume that π is unitary with inner product $\langle\cdot,\cdot\rangle$; also, assume that π is paramodular. At any level $n \geq N_\pi$, the operators $T_{0,1}$, $T_{1,0}$ and $T_{1,0}^*$ are self-adjoint, hence diagonalizable. For $n = N_\pi$, they are simultaneously diagonalizable. The (self-dual) operators $T_{0,1}$ and $T_{1,0} + T_{1,0}^*$ are simultaneously diagonalizable for arbitrary n.*

Proof. All of these Hecke operators are diagonalizable by the previous result. At the minimal level, $T_{0,1}$ and $T_{1,0}$ commute by Corollary 6.2.2 (if $N_\pi = 1$, then they commute since $\dim V(N_\pi) = 1$; see Table A.13). The operators $T_{0,1}$ and $T_{1,0} + T_{1,0}^*$ always commute by Corollary 6.2.3. □

7

Proofs of the Main Theorems

In this chapter we prove the main results as described in the introduction. To do so, we use results proved in previous chapters, and an additional new idea: The computation of $Z(s,W)$ for W a Hecke eigenvector in a generic representation at the minimal paramodular level. The formula expresses $Z(s,W)$ in terms of Hecke eigenvalues and $W(1)$, and has multiple applications. One application is the completion of the proof of uniqueness at the minimal paramodular level and the computation of Hecke eigenvalues in the remaining open cases. Uniqueness was previously proved for non-supercuspidal representations; see Theorem 5.6.1. Another application is the computation of $L(s,\pi)$ in terms of the basic invariants of the newform, namely the level, the Atkin–Lehner eigenvalue, and the Hecke eigenvalues; furthermore, this formula for $L(s,\pi)$ for generic π motivates the formula for $L(s,\varphi_\pi)$ for non-generic π, where φ_π is the L-parameter of π.

The computation of $Z(s,W)$ is divided naturally into three parts, $N_\pi = 0$, $N_\pi = 1$ and $N_\pi \geq 2$. The most difficult case is $N_\pi \geq 2$, for which we require an additional auxiliary computation. This is carried out in Sect. 7.3. The final section of this chapter contains the statements and proofs of all of the main results of this work.

7.1 Zeta Integrals: The Unramified Case

In this section we consider irreducible, admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character that contain non-zero vectors fixed under the maximal compact subgroup $K = \mathrm{GSp}(4,\mathfrak{o})$. Such a fixed vector is unique up to scalars. In the generic case we shall compute its zeta integral, using the unramified Hecke algebra $\mathcal{H}(K)$ as a tool. The Hecke operators $T_{0,1}$ and $T_{1,0}$ defined in (6.3) resp. (6.4) (with $n = 0$) act on the space of K-fixed vectors by scalars λ and μ, respectively.

Lemma 7.1.1. *Let χ_1, χ_2, σ be unramified characters of $F^\times$ with $\chi_1\chi_2\sigma^2 = 1$. Let f_0 be the essentially unique non-zero* $\mathrm{GSp}(4,\mathfrak{o})$ *invariant vector in $\chi_1 \times \chi_2 \rtimes \sigma$. Define $\lambda, \mu \in \mathbb{C}$ by $T_{0,1}f_0 = \lambda f_0$ and $T_{1,0}f_0 = \mu f_0$. Then*

$$\lambda = q^{3/2}\sigma(\varpi)\big(1+\chi_1(\varpi)\big)\big(1+\chi_2(\varpi)\big)$$

and

$$\mu = q^2\big(\chi_1(\varpi)+\chi_2(\varpi)+\chi_1(\varpi)^{-1}+\chi_2(\varpi)^{-1}+1-q^{-2}\big).$$

Proof. These formulas follow easily from the coset decompositions (6.5) and (6.6). □

Let π be a generic, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. Assume that π is spherical, so that the ψ_{c_1,c_2} Whittaker model of π contains a non-zero $\mathrm{GSp}(4,\mathfrak{o})$-invariant vector W. The Whittaker function W is determined by the numbers

$$c_{i,j} := W(\begin{bmatrix} \varpi^{2i+j} &&& \\ & \varpi^{i+j} && \\ && \varpi^{i} & \\ &&& 1 \end{bmatrix})$$

for $i,j \in \mathbb{Z}$. Note that $c_{i,j} = 0$ if $i < 0$ or $j < 0$. We define the Hecke eigenvalues λ and μ as above by $T_{0,1}W = \lambda W$ and $T_{1,0}W = \mu W$.

Lemma 7.1.2. *The numbers $c_{i,j}$ satisfy the relations*

$$\begin{aligned} \lambda c_{i,j} &= q^3 c_{i,j+1} + q^2 c_{i+1,j-1} + q c_{i-1,j+1} + c_{i,j-1} \quad (i,j \geq 0),\\ (\mu - q^2+1)c_{i,j} &= q^4 c_{i+1,j} + q^3 c_{i-1,j+2} + q c_{i+1,j-2} + c_{i-1,j} \quad (i \geq 0,\ j \geq 1),\\ (\mu+1)c_{i,0} &= q^4 c_{i+1,0} + q^3 c_{i-1,2} + c_{i-1,0} \qquad (i \geq 0). \end{aligned}$$

Proof. These are easy computations using the coset decompositions (6.5) and (6.6). □

To compute the zeta integral of W, note that by Lemma 4.1.1

$$\begin{aligned} Z(s,W) &= \int\limits_{F^\times} W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix})|a|^{s-3/2}\,d^\times a \\ &= (1-q^{-1})\sum_{j=0}^{\infty} c_{0,j}q^{-j(s-3/2)}. \end{aligned} \tag{7.1}$$

For the analogous sum with $c_{1,j}$ instead of $c_{0,j}$ we have the following result.

Lemma 7.1.3. *We have the formula*

$$(1-q^{-1})\sum_{j=0}^{\infty} c_{1,j}q^{-j(s-3/2)} = \frac{\mu-q^2+1}{q^4(1+q^{-2s})}Z(s,W) + \frac{q-1}{q^3(1+q^{-2s})}c_{0,0}.$$

Proof. Multiplying (7.1) with $\mu - q^2 + 1$ and using Lemma 7.1.2, we compute

$$
\begin{aligned}
(\mu - q^2 + 1)Z(s,W) &= (1 - q^{-1}) \sum_{j=0}^{\infty} (\mu - q^2 + 1) c_{0,j} q^{-j(s-3/2)} \\
&= (1 - q^{-1}) \sum_{j=1}^{\infty} (q^4 c_{1,j} + q c_{1,j-2}) q^{-j(s-3/2)} \\
&\qquad + (1 - q^{-1})(\mu - q^2 + 1) c_{0,0} \\
&= (1 - q^{-1}) q^4 \sum_{j=1}^{\infty} c_{1,j} q^{-j(s-3/2)} + (1 - q^{-1}) q \sum_{j=2}^{\infty} c_{1,j-2} q^{-j(s-3/2)} \\
&\qquad + (1 - q^{-1})(\mu + 1) c_{0,0} - q^2 (1 - q^{-1}) c_{0,0} \\
&= (1 - q^{-1}) q^4 \sum_{j=1}^{\infty} c_{1,j} q^{-j(s-3/2)} + (1 - q^{-1}) q \sum_{j=0}^{\infty} c_{1,j} q^{-(j+2)(s-3/2)} \\
&\qquad + (1 - q^{-1}) q^4 c_{1,0} - q(q-1) c_{0,0} \\
&= (1 - q^{-1}) q^4 \sum_{j=0}^{\infty} c_{1,j} q^{-j(s-3/2)} + (1 - q^{-1}) q^{1-2(s-3/2)} \sum_{j=0}^{\infty} c_{1,j} q^{-j(s-3/2)} \\
&\qquad - q(q-1) c_{0,0} \\
&= (1 - q^{-1}) q^4 (1 + q^{-2s}) \sum_{j=0}^{\infty} c_{1,j} q^{-j(s-3/2)} - q(q-1) c_{0,0}.
\end{aligned}
$$

The assertion follows. □

Proposition 7.1.4. *Let W be the* $\mathrm{GSp}(4, \mathfrak{o})$ *invariant vector in the* ψ_{c_1,c_2} *Whittaker model of a spherical, generic, irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character. Then*

$$
Z(s,W) = \frac{(1 - q^{-1}) W(1)}{1 - q^{-3/2} \lambda q^{-s} + (q^{-2}\mu + 1 + q^{-2}) q^{-2s} - q^{-3/2} \lambda q^{-3s} + q^{-4s}},
$$

where λ and μ are the Hecke eigenvalues defined by $T_{0,1}W = \lambda W$ and $T_{1,0}W = \mu W$.

Proof. We multiply (7.1) with λ and compute, using Lemmas 7.1.2 and 7.1.3,

$$
\begin{aligned}
\lambda Z(s,W) &= (1 - q^{-1}) \sum_{j=0}^{\infty} (q^3 c_{0,j+1} + q^2 c_{1,j-1} + c_{0,j-1}) q^{-j(s-3/2)} \\
&= (1 - q^{-1}) q^3 \sum_{j=0}^{\infty} c_{0,j+1} q^{-j(s-3/2)} \\
&\qquad + (1 - q^{-1}) q^2 \sum_{j=1}^{\infty} c_{1,j-1} q^{-j(s-3/2)}
\end{aligned}
$$

$$
\begin{aligned}
&+ (1-q^{-1})\sum_{j=1}^{\infty} c_{0,j-1}q^{-j(s-3/2)}\\
&= (1-q^{-1})q^3\sum_{j=1}^{\infty} c_{0,j}q^{-(j-1)(s-3/2)}\\
&\quad+ (1-q^{-1})q^2\sum_{j=0}^{\infty} c_{1,j}q^{-(j+1)(s-3/2)}\\
&\quad+ (1-q^{-1})\sum_{j=0}^{\infty} c_{0,j}q^{-(j+1)(s-3/2)}\\
&= (1-q^{-1})q^{s+3/2}\sum_{j=1}^{\infty} c_{0,j}q^{-j(s-3/2)}\\
&\quad+ (1-q^{-1})q^{-s+7/2}\sum_{j=0}^{\infty} c_{1,j}q^{-j(s-3/2)}\\
&\quad+ (1-q^{-1})q^{-s+3/2}\sum_{j=0}^{\infty} c_{0,j}q^{-j(s-3/2)}\\
&= (1-q^{-1})q^{s+3/2}\Big(\sum_{j=0}^{\infty} c_{0,j}q^{-j(s-3/2)} - c_{0,0}\Big)\\
&\quad+ q^{-s+7/2}\Big(\frac{\mu-q^2+1}{q^4(1+q^{-2s})}Z(s,W) + \frac{q-1}{q^3(1+q^{-2s})}c_{0,0}\Big)\\
&\quad+ (1-q^{-1})q^{-s+3/2}\sum_{j=0}^{\infty} c_{0,j}q^{-j(s-3/2)}\\
&= q^{s+3/2}Z(s,W) - (1-q^{-1})q^{s+3/2}c_{0,0}\\
&\quad+ q^{-s+7/2}\Big(\frac{\mu-q^2+1}{q^4(1+q^{-2s})}Z(s,W) + \frac{q-1}{q^3(1+q^{-2s})}c_{0,0}\Big)\\
&\quad+ q^{-s+3/2}Z(s,W).
\end{aligned}
$$

Solving for $Z(s,W)$ proves the asserted formula. □

To formulate the next result we define, as usual, for a character χ of $F^\times$,

$$
L(s,\chi) = \begin{cases} (1-\chi(\varpi)q^{-s})^{-1} & \text{if } \chi \text{ is unramified,}\\ 0 & \text{if } \chi \text{ is ramified.}\end{cases}
$$

Corollary 7.1.5. *Let χ_1, χ_2 and σ be unramified characters of $F^\times$ with $\chi_1\chi_2\sigma^2 = 1$. Assume that the induced representation $\pi = \chi_1 \times \chi_2 \rtimes \sigma$ is irreducible, so that π is a generic, spherical type I representation. Then, after suitable normalization,*

$$
Z(s,W) = L(s,\chi_1\chi_2\sigma)L(s,\chi_1\sigma)L(s,\chi_2\sigma)L(s,\sigma), \tag{7.2}
$$

where W is a non-zero $\mathrm{GSp}(4,\mathfrak{o})$*-invariant vector in the* ψ_{c_1,c_2} *Whittaker model of* π.

Proof. This follows by combining Proposition 7.1.4 with Lemma 7.1.1, once we know that $W(1)\neq 0$. But if $W(1)=c_{0,0}$ were zero, then $c_{0,j}=0$ for all j by Proposition 7.1.4. The formulas in Lemma 7.1.2 would then imply $c_{i,j}=0$ for all i and j. This is impossible since it would mean that $W=0$. □

An alternative way to compute the zeta integral of W is to use the Casselman–Shalika formula for the spherical Whittaker function. Let $\alpha_i=\chi_i(\varpi)$, $i=1,2$, and $\gamma=\sigma(\varpi)$. By [CS], Theorem 5.4, we have, up to a constant,

$$W(\begin{bmatrix}\varpi^a&&&\\&\varpi^b&&\\&&\varpi^{c-b}&\\&&&\varpi^{c-a}\end{bmatrix})$$
$$=q^{-2a-b+3c/2}\,\gamma^c\Big(\alpha_1^{a+3}\alpha_2^{b+2}-\alpha_1^{b+2}\alpha_2^{a+3}-\alpha_1^{a+3}\alpha_2^{c-b+1}+\alpha_1^{b+2}\alpha_2^{c-a}$$
$$+\alpha_1^{c-b+1}\alpha_2^{a+3}-\alpha_1^{c-a}\alpha_2^{b+2}-\alpha_1^{c-b+1}\alpha_2^{c-a}+\alpha_1^{c-a}\alpha_2^{c-b+1}\Big), \tag{7.3}$$

for all integers a,b,c such that $a\geq b$ and $2b\geq c$ (these conditions on a,b,c are the ones defining the set A^- in [CS]). If these conditions are not satisfied, then W takes the value zero; see Lemma 4.1.2. Using (7.3) and Lemma 4.1.1, a straightforward calculation gives the same result for $Z(s,W)$ as in (7.2). Note that the formula (7.3), and consequently the result (7.2), hold for the spherical function in any unramified type I representation, not only those with trivial central character. Hence, the zeta integral of the newform in type I representations always computes the L-factor of the representation.

7.2 Zeta Integrals: The Level $\mathfrak{p}$ Case

In this section we shall compute the zeta integral of the newform for a generic, irreducible, admissible representation with minimal paramodular level $\mathfrak{p}$. Being Iwahori-spherical, such a representation is a subquotient of a representation induced from an unramified character of the Borel subgroup. See Table A.13 for a list of all such representations. The level $\mathfrak{p}$ representations in Table A.13 are those of type IIa, IVc, Vb, Vc, and VIc, and for each of these the space of $\mathrm{K}(\mathfrak{p})$ invariant vectors is one-dimensional. The only generic level $\mathfrak{p}$ representations are those of type IIa. Let $\pi=\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$ be such a representation, with unramified characters χ and σ such that $\chi^2\neq\nu^{\pm1}$ and $\chi\neq\nu^{\pm3/2}$. Let $V=\mathcal{W}(\pi,\psi_{c_1,c_2})$ be the Whittaker model of π, with $c_1,c_2\in\mathfrak{o}^\times$ as usual.

Let $W\in V(1)$ be the newform of π, unique up to scalars. In Sect. 5.1 we proved that $B(F)\backslash\mathrm{GSp}(4,F)/\mathrm{K}(\mathfrak{p})$ is represented by the two elements 1 and s_1. It follows that W is determined by the numbers

$$c_{i,j} := W(\begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix}) \tag{7.4}$$

and

$$c'_{i,j} := W(\begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix} s_1). \tag{7.5}$$

Note that $c_{i,j} = 0$ if $i < 0$ or $j < 0$, and that $c'_{i,j} = 0$ if $i < -1$ or $j < 1$.

Lemma 7.2.1. *Let ε be the Atkin–Lehner eigenvalue of W. Then*

$$c'_{i,j} = \varepsilon c_{i+1,j-1} \qquad \text{for all } i, j \in \mathbb{Z}.$$

Proof. The Atkin–Lehner element of level $\mathfrak{p}$ is

$$u_1 = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -\varpi & \\ & & & \varpi \end{bmatrix} s_2 s_1 s_2,$$

and ε is defined by $\pi(u_1)W = \varepsilon W$. We compute

$$\begin{aligned} c'_{i,j} &= W(\begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix} s_1) \\ &= W(\begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix} s_1 \begin{bmatrix} \varpi^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix} s_1 s_2 s_1) \\ &= W(\begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \varpi^{-1} & & \\ & & \varpi & \\ & & & 1 \end{bmatrix} s_2 s_1 s_2) \\ &= W(\begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \varpi^{-1} & & \\ & & 1 & \\ & & & \varpi^{-1} \end{bmatrix} u_1) \\ &= \varepsilon W(\begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \varpi & & & \\ & 1 & & \\ & & \varpi & \\ & & & 1 \end{bmatrix}). \end{aligned}$$

This proves the lemma. □

We further define

$$c''_{i,j} = \sum_{x,y,z \in \mathfrak{o}/\mathfrak{p}} W(\begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^i & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ x & 1 & & \\ y & & 1 & \\ z & y & -z & 1 \end{bmatrix}).$$

Lemma 7.2.2. *For all $i, j \geq 0$ we have*

$$c_{i,j} + q^3 c_{i+1,j} + q c'_{i,j} + q^2 c'_{i-1,j+2} = 0 \tag{7.6}$$

and

$$c'_{i,j} + q c'_{i-1,j+2} + q^2 c_{i+1,j} + c''_{i,j} = 0. \tag{7.7}$$

Consequently, $c_{i,j} = q c''_{i,j}$ for $i, j \geq 0$. Furthermore,

$$c''_{-1,j} = -c'_{-1,j} = -\varepsilon c_{0,j-1} \tag{7.8}$$

for all $j \geq 0$.

Proof. Let δ_1 be the level lowering operator $V(1) \to V(0)$ introduced in Sect. 3.3. Since $\mathfrak{p}$ is the minimal level, we have $\delta_1 W = 0$. Formula (7.6) is therefore an easy consequence of (3.21). Similarly (7.7) and (7.8) follow from (3.20) (and Lemma 7.2.1). □

Lemma 7.2.3. *Define $\lambda \in \mathbb{C}$ by $T_{0,1} W = \lambda W$. Then we have for all $i \geq 1$ and $j \geq 0$,*

$$\lambda c_{i,j} = q^3 c_{i,j+1} + q^2 c_{i+1,j-1} + q c_{i-1,j+1} + c_{i,j-1} + (q^2-1) c'_{i-1,j+1}. \tag{7.9}$$

For all $j \geq 0$ we have

$$\lambda c_{0,j} = q^3 c_{0,j+1} + q^2 c_{1,j-1} + c_{0,j-1} - c'_{-1,j+1}. \tag{7.10}$$

Proof. This is a straightforward calculation using Lemma 6.1.2. □

Lemma 7.2.4. *Define $\mu \in \mathbb{C}$ by $T_{1,0} W = \mu W$. Then we have for all $i \geq 1$ and $j \geq 0$*

$$\mu c_{i,j} = q^4 c_{i+1,j} + q^{-1} c_{i-1,j} \tag{7.11}$$

and

$$\mu c_{0,j} = q^4 c_{1,j} - \varepsilon c_{0,j-1}. \tag{7.12}$$

Here ε is defined by $\pi(u_1) W = \varepsilon W$.

Proof. A straightforward calculation using Lemma 6.1.2 gives

$$\mu c_{i,j} = q^4 c_{i+1,j} + c''_{i-1,j}$$

for $i, j \geq 0$ (and $c''_{i,j} = 0$ for $i < -1$). Using Lemma 7.2.2, the assertion follows. □

Proposition 7.2.5. *Let π be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character, for which the minimal paramodular level is* $\mathfrak{p}$ *(such a representation is necessarily of type IIa). Then* $\dim V(1)=1$, *and for $W \in V(1)$ in the ψ_{c_1,c_2} Whittaker model of π we have*

$$Z(s,W)=\frac{(1-q^{-1})W(1)}{1-q^{-3/2}(\lambda+\varepsilon)q^{-s}+(\mu q^{-2}+1)q^{-2s}+\varepsilon q^{-1/2}q^{-3s}}.$$

Here $\lambda,\mu\in\mathbb{C}$ are defined by $T_{0,1}W=\lambda W$ and $T_{1,0}W=\mu W$, and ε is the Atkin–Lehner eigenvalue of W, defined by $\pi(u_1)W=\varepsilon W$.

Proof. Since $\mathfrak{p}$ is the minimal level, π is an Iwahori-spherical representation. As such it can be realized as a subrepresentation of a representation induced from an unramified character of the Borel subgroup. A look at Table A.13 shows that IIa is the only generic level $\mathfrak{p}$ representation, and that $\dim V(1) = 1$.

It follows easily from (7.12) that

$$(\mu+\varepsilon q^{-s+3/2})Z(s,W)=q^4(1-q^{-1})\sum_{j=0}^{\infty}c_{1,j}q^{-j(s-3/2)}.$$

By (7.10) and Lemma 7.2.1 we get

$$(\lambda+\varepsilon)c_{0,j}=q^3c_{0,j+1}+q^2c_{1,j-1}+c_{0,j-1}.$$

Hence

$$\begin{aligned}(\lambda+\varepsilon)Z(s,W)&=(1-q^{-1})\sum_{j=0}^{\infty}(\lambda+\varepsilon)c_{0,j}q^{-j(s-3/2)}\\&=(1-q^{-1})\sum_{j=0}^{\infty}\big(q^3c_{0,j+1}+q^2c_{1,j-1}+c_{0,j-1}\big)q^{-j(s-3/2)}\\&=(1-q^{-1})q^3\sum_{j=0}^{\infty}c_{0,j+1}q^{-j(s-3/2)}\\&\qquad+(1-q^{-1})q^2\sum_{j=1}^{\infty}c_{1,j-1}q^{-j(s-3/2)}\\&\qquad+(1-q^{-1})\sum_{j=1}^{\infty}c_{0,j-1}q^{-j(s-3/2)}\\&=(1-q^{-1})q^3\sum_{j=1}^{\infty}c_{0,j}q^{-(j-1)(s-3/2)}\\&\qquad+(1-q^{-1})q^2\sum_{j=0}^{\infty}c_{1,j}q^{-(j+1)(s-3/2)}\end{aligned}$$

$$
\begin{aligned}
&\quad + (1-q^{-1}) \sum_{j=0}^{\infty} c_{0,j} q^{-(j+1)(s-3/2)} \\
&= (1-q^{-1}) q^{s+3/2} \sum_{j=1}^{\infty} c_{0,j} q^{-j(s-3/2)} \\
&\quad + (1-q^{-1}) q^{-s+7/2} \sum_{j=0}^{\infty} c_{1,j} q^{-j(s-3/2)} \\
&\quad + (1-q^{-1}) q^{-s+3/2} \sum_{j=0}^{\infty} c_{0,j} q^{-j(s-3/2)} \\
&= (1-q^{-1}) q^{s+3/2} \big(\sum_{j=0}^{\infty} c_{0,j} q^{-j(s-3/2)} - c_{0,0} \big) \\
&\quad + q^{-s+7/2} q^{-4} (\mu + \varepsilon q^{-s+3/2}) Z(s,W) \\
&\quad + q^{-s+3/2} Z(s,W) \\
&= q^{s+3/2} Z(s,W) - (1-q^{-1}) q^{s+3/2} c_{0,0} \\
&\quad + q^{-s-1/2} (\mu + \varepsilon q^{-s+3/2}) Z(s,W) \\
&\quad + q^{-s+3/2} Z(s,W).
\end{aligned}
$$

Solving for $Z(s,W)$ gives the result. □

Corollary 7.2.6. *Let χ and σ be unramified characters of $F^\times$ with $\chi^2\sigma^2 = 1$ and such that $\chi^2 \neq \nu^{\pm 1}$ and $\chi \neq \nu^{\pm 3/2}$. Then $\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$ is a generic, irreducible, admissible representation of type IIa with trivial central character. Let W be a non-zero $\mathrm{K}(\mathfrak{p})$-invariant vector in the ψ_{c_1,c_2} Whittaker model of π. Such a vector is unique up to multiples, and upon suitable normalization we have*

$$Z(s,W) = L(s,\sigma) L(s,\sigma^{-1}) L(s,\nu^{1/2}\chi\sigma).$$

Hence, the zeta integral of the newform represents the L-factor of the representation.

Proof. We will first prove that $Z(s,W) \neq 0$. By Proposition 7.2.5 this is equivalent to $W(1) \neq 0$. Assume that $W(1) = 0$. Then, from Proposition 7.2.5, $c_{0,j} = 0$ for all $j \in \mathbb{Z}$. The formulas in Lemma 7.2.4 further imply that $c_{i,j} = 0$ for all $i, j \in \mathbb{Z}$. By Lemma 7.2.1, $c'_{i,j} = 0$ for all $i, j \in \mathbb{Z}$. As noted above, it was proved in Sect. 5.1 that $B(F)\backslash \mathrm{GSp}(4,F)/\mathrm{K}(\mathfrak{p})$ is represented by the two elements 1 and s_1, implying that W is determined by the $c_{i,j}$ and $c'_{i,j}$ for all $i, j \in \mathbb{Z}$. Hence we obtain $W = 0$, a contradiction. This shows that $Z(s,W) \neq 0$.

In Proposition 6.4.2 we determined the Hecke eigenvalues λ and μ by direct computations in induced models. The result is

$$\lambda = q^{3/2}(\sigma(\varpi) + \sigma(\varpi)^{-1}) + (q+1)\sigma(\varpi)\chi(\varpi)$$

and

$$\mu = q^{3/2}(\sigma(\varpi) + \sigma(\varpi)^{-1})\sigma(\varpi)\chi(\varpi).$$

Furthermore, by Table A.13, the Atkin–Lehner eigenvalue of the newform is $\varepsilon = -\sigma(\varpi)\chi(\varpi)$ for representations of type IIa. Substituting all these values into the formula in Proposition 7.2.5 proves the result about $Z(s, W)$. Note that this is the L-function by Table A.8. □

Remark: Between the values λ, μ and ε there is the relation

$$\lambda\varepsilon + \mu + q + 1 = 0$$

for representations of type IIa.

7.3 The Operator *R*

This section computes the values of a certain vector $\pi(s_2)RW$ on matrices of the form $\mathrm{diag}(\varpi^k, \varpi^k, 1, 1)$. These values will be used in the next section, which considers the case of representations of paramodular level $N_\pi \geq 2$. Let (π, V) be a generic, irreducible, admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character, where $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. For $W \in V$ we define $RW \in V$ by

$$RW = q \int\limits_{\mathfrak{o}} \pi(\begin{bmatrix} 1 & & & \\ \lambda\varpi^{n-1} & 1 & & \\ & & 1 & \\ & & -\lambda\varpi^{n-1} & 1 \end{bmatrix})W \, d\lambda. \tag{7.13}$$

Here, the Haar measure on F gives $\mathfrak{o}$ measure 1. We will primarily apply R to elements W of $V(n)$, in which case we can write

$$RW = \sum_{\lambda \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & & & \\ \lambda\varpi^{n-1} & 1 & & \\ & & 1 & \\ & & -\lambda\varpi^{n-1} & 1 \end{bmatrix})W.$$

For $W \in V$ denote by $Z_N(s, W)$ the simplified zeta integral occurring in Lemma 4.1.1, i.e.,

$$Z_N(s, W) = \int\limits_{F^\times} W(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix})|a|^{s-3/2} \, d^\times a. \tag{7.14}$$

This converges for $\mathrm{Real}(s) > 3/2 - \min(u_1, \ldots, u_N)$, where $u_1, \ldots, u_N$ are as in Lemma 2.6.1. The argument is as in the proof of Proposition 2.6.3.

Lemma 7.3.1. *Let $n \geq 2$ and W a Klingen vector of level $\mathfrak{p}^n$. Then*

$$Z_N(s, \pi(s_2)RW) = Z(s, \pi(s_2)RW).$$

In particular, this holds for $W \in V(n)$.

Proof. It will suffice to show that $\pi(s_2)RW$ is invariant under the elements from Lemma 4.1.1. We have the following identities for $\lambda, \mu \in \mathfrak{o}$:

$$\begin{bmatrix}1&\mu&&\\&1&\mu&\\&&1&\\&&&1\end{bmatrix} s_2 \begin{bmatrix}1&&&\\\lambda\varpi^{n-1}&1&&\\&&1&\\&&-\lambda\varpi^{n-1}&1\end{bmatrix} = s_2 \begin{bmatrix}1&&&\\\lambda\varpi^{n-1}&1&&\\&&1&\\&&-\lambda\varpi^{n-1}&1\end{bmatrix}$$
$$\times \begin{bmatrix}1-\lambda\mu\varpi^{n-1}&-\mu&&\\\lambda^2\mu\varpi^{2n-2}&1+\lambda\mu\varpi^{n-1}&&\\&&1-\lambda\mu\varpi^{n-1}&\mu\\&&-\lambda^2\mu\varpi^{2n-2}&1+\lambda\mu\varpi^{n-1}\end{bmatrix},$$

$$\begin{bmatrix}1&&&\\&1&\mu\varpi&\\&&1&\\&&&1\end{bmatrix} s_2 \begin{bmatrix}1&&&\\\lambda\varpi^{n-1}&1&&\\&&1&\\&&-\lambda\varpi^{n-1}&1\end{bmatrix}$$
$$= s_2 \begin{bmatrix}1&&&\\\lambda\varpi^{n-1}&1&&\\&&1&\\&&-\lambda\varpi^{n-1}&1\end{bmatrix} \begin{bmatrix}1&&&\\&1&&\\-\lambda\mu\varpi^{n}&-\varpi\mu&1&\\-\lambda^2\mu\varpi^{2n-1}&-\lambda\mu\varpi^{n}&&1\end{bmatrix},$$

$$\begin{bmatrix}1&&&\\&1&&\\&\mu&1&\\&&&1\end{bmatrix} s_2 \begin{bmatrix}1&&&\\\lambda\varpi^{n-1}&1&&\\&&1&\\&&-\lambda\varpi^{n-1}&1\end{bmatrix}$$
$$= s_2 \begin{bmatrix}1&&&\\\lambda\varpi^{n-1}&1&&\\&&1&\\&&-\varpi^{n-1}x&1\end{bmatrix} \begin{bmatrix}1&&&\\&1&-\mu&\\&&1&\\&&&1\end{bmatrix}.$$

Since the rightmost elements of each identity are contained in $\mathrm{Kl}(\mathfrak{p}^n)$, our claim follows. □

Proposition 7.3.2. *Let $n \geq 2$ and let $W \in V(n)$. Then*

$$Z_N(s, \pi(s_2)RW) = Z(s, W).$$

Proof. For typesetting reasons we will abbreviate the γ-factor $\gamma(s, \pi)$ by $\gamma(s)$. By Lemma 7.3.1 and the functional equation (2.60),

$$Z_N(s, \pi(s_2)RW) = Z(s, \pi(s_2)RW)$$

$$= \gamma(s)^{-1} Z(1-s, \pi(s_2 s_1 s_2 \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} s_2) RW)$$
$$= \gamma(s)^{-1} Z(1-s, \pi(s_2 s_1) RW)$$
$$= \gamma(s)^{-1} Z(1-s, \pi(s_2 s_1) R\pi(s_1 s_2 s_2^{-1} s_1^{-1}) W)$$
$$= \gamma(s)^{-1} q \int\limits_{\mathfrak{o}} \int\limits_{F^\times} \int\limits_{F} W(\begin{bmatrix} a & & & \\ & a & & \\ & x & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & \mu\varpi^{n-1} & & \\ & 1 & & \mu\varpi^{n-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2^{-1} s_1^{-1}) |a|^{(1-s)-3/2} \, dx \, d^\times a \, d\mu.$$

Now

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & x & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & \mu\varpi^{n-1} & & \\ & 1 & & \mu\varpi^{n-1} \\ & & 1 & \\ & & & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -x\mu\varpi^{n-1} & & x\mu^2\varpi^{2n-2} \\ & 1 & & \\ & & 1 & x\mu\varpi^{n-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & \mu\varpi^{n-1} & & \\ & 1 & & \mu\varpi^{n-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & x & 1 & \\ & & & 1 \end{bmatrix}.$$

So

$$Z_N(s, \pi(s_2) RW)$$
$$= \gamma(s)^{-1} q \int\limits_{\mathfrak{o}} \int\limits_{F^\times} \int\limits_{F} \psi(-c_1 x\mu\varpi^{n-1}) W(\begin{bmatrix} a & & & \\ & a & & \\ & x & 1 & \\ & & & 1 \end{bmatrix} s_2 s_1) |a|^{-s-1/2} \, dx \, d^\times a \, d\mu$$
$$= \gamma(s)^{-1} q \int\limits_{F^\times} \int\limits_{F} (\int\limits_{\mathfrak{o}} \psi(-c_1 x\mu\varpi^{n-1}) \, d\mu) W(\begin{bmatrix} a & & & \\ & a & & \\ & x & 1 & \\ & & & 1 \end{bmatrix} s_2 s_1) |a|^{-s-1/2} \, dx \, d^\times a$$
$$= \gamma(s)^{-1} q \int\limits_{F^\times} \int\limits_{v(x) \geq 1-n} W(\begin{bmatrix} a & & & \\ & a & & \\ & x & 1 & \\ & & & 1 \end{bmatrix} s_2 s_1 s_2) |a|^{-s-1/2} \, dx \, d^\times a$$
$$= \gamma(s)^{-1} q \int\limits_{F^\times} \int\limits_{v(x) \geq 1-n} W(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2 s_1 s_2 \begin{bmatrix} 1 & & x & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}) |a|^{-s-1/2} \, dx \, d^\times a$$

$$= \gamma(s)^{-1} q\Big(\int_{v(x)\geq 1-n} dx\Big) \int_{F^\times} W\left(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2 s_1 s_2\right) |a|^{-s-1/2}\, d^\times a$$

$$= \gamma(s)^{-1} q |\varpi|^{1-n} \int_{F^\times} W\left(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2 s_1 s_2\right) |a|^{-s-1/2}\, d^\times a$$

$$= \gamma(s)^{-1} |\varpi|^{-n} \int_{F^\times} W\left(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi^{-n} & \\ & & & \varpi^{-n} \end{bmatrix} u_n\right) |a|^{-s-1/2}\, d^\times a$$

$$= \gamma(s)^{-1} |\varpi|^{-n} \int_{F^\times} W\left(\begin{bmatrix} \varpi^n a & & & \\ & \varpi^n a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} u_n\right) |a|^{-s-1/2}\, d^\times a$$

$$= \gamma(s)^{-1} |\varpi|^{-n} \int_{F^\times} W\left(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} u_n\right) |\varpi^{-n} a|^{-s-1/2}\, d^\times a$$

$$= \gamma(s)^{-1} |\varpi|^{-n+n(s+1/2)} Z(1-s, \pi(u_n)W).$$

By the functional equation (2.61), the last expression is equal to $Z(s,W)$. This completes the computation. □

Corollary 7.3.3. *Let $n \geq 2$ and $W \in V(n)$. Then*

$$(\pi(s_2)RW)\left(\begin{bmatrix} \varpi^k & & & \\ & \varpi^k & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) = W\left(\begin{bmatrix} \varpi^k & & & \\ & \varpi^k & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\right), \quad k \in \mathbb{Z}.$$

In particular,

$$(\pi(s_2)RW)\left(\begin{bmatrix} \varpi^k & & & \\ & \varpi^k & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) = 0 \qquad \textit{for} \quad k < 0.$$

Proof. By Proposition 7.3.2 and Lemma 4.1.1, we have

$$Z_N(s, \pi(s_2)RW) = Z(s,W) = Z_N(s,W).$$

Evidently,

$$Z_N(s,\pi(s_2)RW)=\sum_{k=-\infty}^{\infty}(1-q^{-1})(\pi(s_2)RW)(\begin{bmatrix}\varpi^k&&&\\&\varpi^k&&\\&&1&\\&&&1\end{bmatrix})|\varpi^k|^{s-3/2}$$

and, by Lemma 4.1.2,

$$Z_N(s,W)=\sum_{k=0}^{\infty}(1-q^{-1})W(\begin{bmatrix}\varpi^k&&&\\&\varpi^k&&\\&&1&\\&&&1\end{bmatrix})|\varpi^k|^{s-3/2}.$$

The corollary follows. □

7.4 Zeta Integrals: The Higher Level Case

In this section we prove the analogues of Proposition 7.1.4 and Proposition 7.2.5 from the first two sections of this chapter for representations of minimal paramodular level $N_\pi \geq 2$. As a corollary we obtain the Hecke eigenvalues for an eigenvector at the minimal paramodular level for all representations with $L(s,\pi)=1$; these include the generic supercuspidal representations, and representations of type VIIa, VIIIa and IXa.

In preparation for Hecke operator calculations we begin this section with a result on a certain Klingen vector of level one below the minimal paramodular level. We shall come back to this vector at the end of the section.

A Certain Klingen Vector

Let (π,V) be a generic, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. Let $n\geq 2$ be an integer and $W\in V(n)$. We define

$$W':=\sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}}\pi(\begin{bmatrix}1&&&\\y\varpi^{n-1}&1&&\\x\varpi^{n-1}&&1&\\z\varpi^{n-1}&x\varpi^{n-1}&-y\varpi^{n-1}&1\end{bmatrix})W.$$

It follows from the Iwahori factorization (2.7) that W' is a Klingen vector of level $n-1$. Let

$$c_{i,j}:=W(\begin{bmatrix}\varpi^{2i+j}&&&\\&\varpi^{i+j}&&\\&&\varpi^i&\\&&&1\end{bmatrix}),\quad c'_{i,j}:=W'(\begin{bmatrix}\varpi^{2i+j}&&&\\&\varpi^{i+j}&&\\&&\varpi^i&\\&&&1\end{bmatrix}).\quad(7.15)$$

The following lemma shows that, if n is the minimal level, then the values of W' on diagonal matrices are determined by the values of W on diagonal matrices.

Lemma 7.4.1. *With the above notations we have*

$$c'_{i,j} = 0 \qquad \text{for } i < 0.$$

If in addition n is the minimal paramodular level of π, then

$$c'_{i,j} = -q^2 c_{i+1,j} \qquad \text{for } i \geq 0.$$

Proof. A computation shows that

$$\begin{bmatrix} 1 &&& \\ -y\varpi^{n-1} & 1 && \\ -(x+\lambda z)\varpi^{n-1} && 1 & \\ -z\varpi^{n-1} & -(x+\lambda z)\varpi^{n-1} & y\varpi^{n-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\lambda && \\ & 1 && \\ && 1 & \lambda \\ &&& 1 \end{bmatrix}$$
$$\times \begin{bmatrix} 1 &&& \\ y\varpi^{n-1} & 1 && \\ x\varpi^{n-1} && 1 & \\ z\varpi^{n-1} & x\varpi^{n-1} & -y\varpi^{n-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda && \\ & 1 && \\ && 1 & -\lambda \\ &&& 1 \end{bmatrix} \in \mathrm{K}(\mathfrak{p}^n)$$

for $\lambda, x, y, z \in \mathfrak{o}$. Hence, if we abbreviate $d = \mathrm{diag}(\varpi^{2i+j}, \varpi^{i+j}, \varpi^i, 1)$ and let $\lambda \in \mathfrak{o}$, then

$$\begin{aligned} c'_{i,j} &= \sum_{x,y,z \in \mathfrak{o}/\mathfrak{p}} W(d \begin{bmatrix} 1 &&& \\ y\varpi^{n-1} & 1 && \\ x\varpi^{n-1} && 1 & \\ z\varpi^{n-1} & x\varpi^{n-1} & -y\varpi^{n-1} & 1 \end{bmatrix}) \\ &= \sum_{x,y,z \in \mathfrak{o}/\mathfrak{p}} W(d \begin{bmatrix} 1 &&& \\ y\varpi^{n-1} & 1 && \\ x\varpi^{n-1} && 1 & \\ z\varpi^{n-1} & x\varpi^{n-1} & -y\varpi^{n-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda && \\ & 1 && \\ && 1 & -\lambda \\ &&& 1 \end{bmatrix}) \\ &= \sum_{x,y,z \in \mathfrak{o}/\mathfrak{p}} W(d \begin{bmatrix} 1 & \lambda && \\ & 1 && \\ && 1 & -\lambda \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ y\varpi^{n-1} & 1 && \\ (x+\lambda z)\varpi^{n-1} && 1 & \\ z\varpi^{n-1} & (x+\lambda z)\varpi^{n-1} & -y\varpi^{n-1} & 1 \end{bmatrix}) \\ &= \psi(c_1\lambda\varpi^i) \sum_{x,y,z \in \mathfrak{o}/\mathfrak{p}} W(d \begin{bmatrix} 1 &&& \\ y\varpi^{n-1} & 1 && \\ x\varpi^{n-1} && 1 & \\ z\varpi^{n-1} & x\varpi^{n-1} & -y\varpi^{n-1} & 1 \end{bmatrix}). \end{aligned}$$

Since $\lambda \in \mathfrak{o}$ is arbitrary and $c_1 \in \mathfrak{o}^\times$, we conclude that $c_{i,j} = 0$ for $i < 0$. Now assume that n is the minimal paramodular level. Then

$$\sum_{g \in \mathrm{K}(\mathfrak{p}^{n-1})/\mathrm{Kl}(\mathfrak{p}^{n-1})} \pi(g)W' = 0.$$

Hence, by Lemma 3.3.1,

$$\sum_{u\in\mathfrak{o}/\mathfrak{p}^{n-1}} W'(g\begin{bmatrix}1&&u\varpi^{-n+1}&\\&1&&\\&&1&\\&&&1\end{bmatrix})+\sum_{v\in\mathfrak{o}/\mathfrak{p}^{n-2}} W'(gt_{n-1}\begin{bmatrix}1&&v\varpi^{-n+2}&\\&1&&\\&&1&\\&&&1\end{bmatrix})=0$$

for all $g\in \mathrm{GSp}(4,F)$. Choose

$$g=\begin{bmatrix}\varpi^{2i+j}&&&\\&\varpi^{i+j}&&\\&&\varpi^{i}&\\&&&1\end{bmatrix},$$

and let A be the first sum and B be the second sum. Evidently,

$$A=q^{n-1}\,W'(\begin{bmatrix}\varpi^{2i+j}&&&\\&\varpi^{i+j}&&\\&&\varpi^{i}&\\&&&1\end{bmatrix})=q^{n-1}\,c'_{i,j}.$$

By definition,

$$B=\sum_{v\in\mathfrak{o}/\mathfrak{p}^{n-2}}\ \sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}} W(gt_{n-1}\begin{bmatrix}1&&v\varpi^{-n+2}&\\&1&&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\y\varpi^{n-1}&1&&\\x\varpi^{n-1}&&1&\\z\varpi^{n-1}&x\varpi^{n-1}&-y\varpi^{n-1}&1\end{bmatrix}).$$

It is easily checked that the commutator of the two matrices occurring in the argument lies in $\mathrm{K}(\mathfrak{p}^n)$. Hence

$$\begin{aligned}
B&=q^{n-2}\sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}} W(gt_{n-1}\begin{bmatrix}1&&&\\y\varpi^{n-1}&1&&\\x\varpi^{n-1}&&1&\\z\varpi^{n-1}&x\varpi^{n-1}&-y\varpi^{n-1}&1\end{bmatrix})\\
&=q^{n}\sum_{x\in\mathfrak{o}/\mathfrak{p}} W(\begin{bmatrix}\varpi^{2i+j}&&&\\&\varpi^{i+j}&&\\&&\varpi^{i}&\\&&&1\end{bmatrix}t_{n-1}\begin{bmatrix}1&&&\\&1&&\\x\varpi^{n-1}&&1&\\&x\varpi^{n-1}&&1\end{bmatrix})\\
&=q^{n}\sum_{x\in\mathfrak{o}/\mathfrak{p}} W(\begin{bmatrix}\varpi^{2i+j+1}&&&\\&\varpi^{i+j}&&\\&&\varpi^{i}&\\&&&\varpi^{-1}\end{bmatrix}t_{n}\begin{bmatrix}1&&&\\&1&&\\x\varpi^{n-1}&&1&\\&x\varpi^{n-1}&&1\end{bmatrix})\\
&=q^{n}\sum_{x\in\mathfrak{o}/\mathfrak{p}} W(\begin{bmatrix}\varpi^{2(i+1)+j}&&&\\&\varpi^{i+1+j}&&\\&&\varpi^{i+1}&\\&&&1\end{bmatrix}\begin{bmatrix}1&x\varpi^{-1}&&\\&1&&\\&&1&-x\varpi^{-1}\\&&&1\end{bmatrix})
\end{aligned}$$

$$= q^n \sum_{x \in \mathfrak{o}/\mathfrak{p}} \psi(c_1 x \varpi^i)\, W(\begin{bmatrix} \varpi^{2(i+1)+j} &&& \\ & \varpi^{i+1+j} && \\ && \varpi^{i+1} & \\ &&& 1 \end{bmatrix}).$$

Therefore $B = 0$ if $i < 0$, and $B = q^{n+1} c_{i+1,j}$ if $i \geq 0$. The assertion follows from $A + B = 0$. □

Recursion Formulas

For the following lemmas let (π, V) be a generic, irreducible, admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character. We shall eventually prove that the space of paramodular vectors at the minimal level is one-dimensional. Consequently, the Hecke operators $T_{0,1}$ and $T_{1,0}$ act on this space as scalars. However, some of the following recursion formulas hold for Hecke eigenvectors at an arbitrary level.

Lemma 7.4.2. *Let $n \geq 2$, and let $W \in V(n)$ be an eigenvector for $T_{0,1}$ with eigenvalue λ. Then the numbers $c_{i,j}$ defined in (7.15) satisfy the relation*

$$\lambda c_{0,j} = q^3 c_{0,j+1} + q^2 c_{1,j-1} + c_{0,j-1} \qquad \textit{for all } j \geq 0. \tag{7.16}$$

Proof. Evaluating $T_{0,1} W = \lambda W$ at the above diagonal matrix, we compute, using Lemma 6.1.2 i), for $i, j \geq 0$:

$$\begin{aligned}
\lambda c_{i,j} &= \sum_{x,y,z \in \mathfrak{o}/\mathfrak{p}} W(\begin{bmatrix} \varpi^{2i+j} &&& \\ & \varpi^{i+j} && \\ && \varpi^i & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 & & y & z\varpi^{-n} \\ & 1 & x & y \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} \varpi &&& \\ & \varpi && \\ && 1 & \\ &&& 1 \end{bmatrix}) \\
&\quad + \sum_{x,z \in \mathfrak{o}/\mathfrak{p}} W(\begin{bmatrix} \varpi^{2i+j} &&& \\ & \varpi^{i+j} && \\ && \varpi^i & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 & x & & z\varpi^{-n} \\ & 1 && \\ && 1 & -x \\ &&& 1 \end{bmatrix} \begin{bmatrix} \varpi &&& \\ & 1 && \\ && \varpi & \\ &&& 1 \end{bmatrix}) \\
&\quad + \sum_{x,y \in \mathfrak{o}/\mathfrak{p}} W(\begin{bmatrix} \varpi^{2i+j} &&& \\ & \varpi^{i+j} && \\ && \varpi^i & \\ &&& 1 \end{bmatrix} t_n \begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} \varpi &&& \\ & \varpi && \\ && 1 & \\ &&& 1 \end{bmatrix}) \\
&\quad + \sum_{x \in \mathfrak{o}/\mathfrak{p}} W(\begin{bmatrix} \varpi^{2i+j} &&& \\ & \varpi^{i+j} && \\ && \varpi^i & \\ &&& 1 \end{bmatrix} t_n \begin{bmatrix} 1 & x && \\ & 1 && \\ && 1 & -x \\ &&& 1 \end{bmatrix} \begin{bmatrix} \varpi &&& \\ & 1 && \\ && \varpi & \\ &&& 1 \end{bmatrix}) \\
&= q^3 W(\begin{bmatrix} \varpi^{2i+j} &&& \\ & \varpi^{i+j} && \\ && \varpi^i & \\ &&& 1 \end{bmatrix} \begin{bmatrix} \varpi &&& \\ & \varpi && \\ && 1 & \\ &&& 1 \end{bmatrix})
\end{aligned}$$

$$+\,q^2 W(\begin{bmatrix}\varpi^{2i+j}&&&\\&\varpi^{i+j}&&\\&&\varpi^{i}&\\&&&1\end{bmatrix}\begin{bmatrix}\varpi&&&\\&1&&\\&&\varpi&\\&&&1\end{bmatrix})$$

$$+\,q\sum_{y\in\mathfrak{o}/\mathfrak{p}} W(\begin{bmatrix}\varpi^{2i+j}&&&\\&\varpi^{i+j}&&\\&&\varpi^{i}&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\y\varpi^{n}&1&&\\&&1&\\&&-y\varpi^{n}&1\end{bmatrix}\begin{bmatrix}1&&&\\&\varpi&&\\&&1&\\&&&\varpi\end{bmatrix})$$

$$+\sum_{x\in\mathfrak{o}/\mathfrak{p}} W(\begin{bmatrix}\varpi^{2i+j}&&&\\&\varpi^{i+j}&&\\&&\varpi^{i}&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\x\varpi^{n}&&1&\\&x\varpi^{n}&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&&\varpi&\\&&&\varpi\end{bmatrix})$$

$$= q^3 c_{i,j+1} + q^2 c_{i+1,j-1}$$

$$+\,q\sum_{y\in\mathfrak{o}/\mathfrak{p}} W(\begin{bmatrix}\varpi^{2i+j-1}&&&\\&\varpi^{i+j}&&\\&&\varpi^{i-1}&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\y\varpi^{n-1}&1&&\\&&1&\\&&-y\varpi^{n-1}&1\end{bmatrix})$$

$$+\sum_{x\in\mathfrak{o}/\mathfrak{p}} W(\begin{bmatrix}\varpi^{2i+j-1}&&&\\&\varpi^{i+j-1}&&\\&&\varpi^{i}&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\x\varpi^{n-1}&&1&\\&x\varpi^{n-1}&&1\end{bmatrix}).$$

We claim that the first sum is zero if $i = 0$. Indeed, for $u \in \mathfrak{o}$ we have

$$\begin{aligned}&\begin{bmatrix}1&\\y\varpi^{n-1}&1\end{bmatrix}\begin{bmatrix}1&u\\&1\end{bmatrix}\\&=\begin{bmatrix}1&u\\&1\end{bmatrix}\begin{bmatrix}1&\\y\varpi^{n-1}&1\end{bmatrix}\begin{bmatrix}1-uy\varpi^{n-1}&-u^2y\varpi^{n-1}\\uy^2\varpi^{2n-2}&1+uy\varpi^{n-1}+u^2y^2\varpi^{2n-2}\end{bmatrix},\end{aligned}\tag{7.17}$$

and therefore, by our assumption that $n \geq 2$,

$$W(\begin{bmatrix}\varpi^{2i+j-1}&&&\\&\varpi^{i+j}&&\\&&\varpi^{i-1}&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\y\varpi^{n-1}&1&&\\&&1&\\&&-y\varpi^{n-1}&1\end{bmatrix})$$

$$= W(\begin{bmatrix}\varpi^{2i+j-1}&&&\\&\varpi^{i+j}&&\\&&\varpi^{i-1}&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\y\varpi^{n-1}&1&&\\&&1&\\&&-y\varpi^{n-1}&1\end{bmatrix}\begin{bmatrix}1&u&&\\&1&&\\&&1&-u\\&&&1\end{bmatrix})$$

$$= W(\begin{bmatrix}\varpi^{2i+j-1}&&&\\&\varpi^{i+j}&&\\&&\varpi^{i-1}&\\&&&1\end{bmatrix}\begin{bmatrix}1&u&&\\&1&&\\&&1&-u\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\y\varpi^{n-1}&1&&\\&&1&\\&&-y\varpi^{n-1}&1\end{bmatrix})$$

$$= \psi(c_1\varpi^{i-1}u)W(\begin{bmatrix}\varpi^{2i+j-1}&&&\\&\varpi^{i+j}&&\\&&\varpi^{i-1}&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\y\varpi^{n-1}&1&&\\&&1&\\&&-y\varpi^{n-1}&1\end{bmatrix}).$$

Hence this expression is zero for $i = 0$. By the above we get

$$\lambda c_{0,j} = q^3 c_{0,j+1} + q^2 c_{1,j-1} + \sum_{x\in\mathfrak{o}/\mathfrak{p}} W(\begin{bmatrix}\varpi^{j-1}&&&\\&\varpi^{j-1}&&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\x\varpi^{n-1}&&1&\\&x\varpi^{n-1}&&1\end{bmatrix}).$$

The sum can be rewritten as

$$\sum_{x\in\mathfrak{o}/\mathfrak{p}} W(\begin{bmatrix}\varpi^{j-1}&&&\\&\varpi^{j-1}&&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\x\varpi^{n-1}&&1&\\&x\varpi^{n-1}&&1\end{bmatrix}s_2)$$

$$= \sum_{x\in\mathfrak{o}/\mathfrak{p}} W(\begin{bmatrix}\varpi^{j-1}&&&\\&\varpi^{j-1}&&\\&&1&\\&&&1\end{bmatrix}s_2\begin{bmatrix}1&&&\\x\varpi^{n-1}&1&&\\&&1&\\&&-x\varpi^{n-1}&1\end{bmatrix})$$

$$= q\int\limits_{\mathfrak{o}} W(\begin{bmatrix}\varpi^{j-1}&&&\\&\varpi^{j-1}&&\\&&1&\\&&&1\end{bmatrix}s_2\begin{bmatrix}1&&&\\x\varpi^{n-1}&1&&\\&&1&\\&&-x\varpi^{n-1}&1\end{bmatrix})\,dx$$

$$= (\pi(s_2)RW)(\begin{bmatrix}\varpi^{j-1}&&&\\&\varpi^{j-1}&&\\&&1&\\&&&1\end{bmatrix}),$$

where R is the operator defined in Sect. 7.3. By Corollary 7.3.3,

$$(\pi(s_2)RW)(\begin{bmatrix}\varpi^{k}&&&\\&\varpi^{k}&&\\&&1&\\&&&1\end{bmatrix}) = W(\begin{bmatrix}\varpi^{k}&&&\\&\varpi^{k}&&\\&&1&\\&&&1\end{bmatrix}) \qquad \text{for all } k\in\mathbb{Z}.$$

Consequently we obtain the asserted formula. □

To get information about the $c_{1,j-1}$ term in this formula, we consider the other Hecke operator $T_{1,0}$. We define the numbers $c_{i,j}$ and $c'_{i,j}$ as in (7.15).

Lemma 7.4.3. *Let $n \geq 2$, and let $W \in V(n)$ be an eigenvector for $T_{1,0}$ with eigenvalue μ. Then*

$$\mu c_{i,j} = q^4 c_{i+1,j} + c'_{i-1,j} \qquad \textit{for all } i,j \geq 0.$$

For $i = 0$ *we have* $c'_{i-1,j} = 0$*, so that*

$$\mu c_{0,j} = q^4 c_{1,j} \qquad \textit{for all } j \in \mathbb{Z}.$$

Proof. By Lemma 6.1.2 ii) we get for $i, j \geq 0$,

$$\mu c_{i,j} = \sum_{x,y \in \mathfrak{o}/\mathfrak{p}} \sum_{z \in \mathfrak{o}/\mathfrak{p}^2} W(\begin{bmatrix} \varpi^{2i+j} &&& \\ & \varpi^{i+j} && \\ && \varpi^i & \\ &&& 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & x && \\ & 1 && \\ && 1 & -x \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 && y & z\varpi^{-n} \\ & 1 && y \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} \varpi^2 &&& \\ & \varpi && \\ && \varpi & \\ &&& 1 \end{bmatrix})$$
$$+ \sum_{x,y,z \in \mathfrak{o}/\mathfrak{p}} W(\begin{bmatrix} \varpi^{2i+j} &&& \\ & \varpi^{i+j} && \\ && \varpi^i & \\ &&& 1 \end{bmatrix}$$
$$t_n \begin{bmatrix} 1 & x && \\ & 1 && \\ && 1 & -x \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 && y & z\varpi^{-n+1} \\ & 1 && y \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} \varpi^2 &&& \\ & \varpi && \\ && \varpi & \\ &&& 1 \end{bmatrix})$$
$$= q^4 \, W(\begin{bmatrix} \varpi^{2i+j} &&& \\ & \varpi^{i+j} && \\ && \varpi^i & \\ &&& 1 \end{bmatrix} \begin{bmatrix} \varpi^2 &&& \\ & \varpi && \\ && \varpi & \\ &&& 1 \end{bmatrix})$$
$$+ \sum_{x,y,z \in \mathfrak{o}/\mathfrak{p}} W(\begin{bmatrix} \varpi^{2i+j} &&& \\ & \varpi^{i+j} && \\ && \varpi^i & \\ &&& 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 &&& \\ y\varpi^n & 1 && \\ x\varpi^n && 1 & \\ z\varpi^{n+1} & x\varpi^n & -y\varpi^n & 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & \varpi && \\ && \varpi & \\ &&& \varpi^2 \end{bmatrix}),$$

hence

$$\mu c_{i,j} = q^4 \, c_{i+1,j} + \sum_{x,y,z \in \mathfrak{o}/\mathfrak{p}} W(\begin{bmatrix} \varpi^{2i+j-2} &&& \\ & \varpi^{i+j-1} && \\ && \varpi^{i-1} & \\ &&& 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ y\varpi^{n-1} & 1 & & \\ x\varpi^{n-1} & & 1 & \\ z\varpi^{n-1} & x\varpi^{n-1} & -y\varpi^{n-1} & 1 \end{bmatrix}). \qquad (7.18)$$

This proves the first equation. For the second one note that $c'_{i,j} = 0$ for $i < 0$ by Lemma 7.4.1. □

Lemma 7.4.4. *Assume that $N_\pi \geq 2$ is the minimal paramodular level of the irreducible, admissible, generic representation (π, V) of $\mathrm{GSp}(4, F)$ with trivial central character. Assume that $W \in V(N_\pi)$ satisfies $T_{0,1}W = \lambda W$ and $T_{1,0}W = \mu W$ with $\lambda, \mu \in \mathbb{C}$. Then the numbers $c_{i,j}$ defined in (7.15) satisfy the following recursion formulas.*

$$\mu c_{0,j} = q^4\, c_{1,j} \qquad \text{for all } j \in \mathbb{Z}. \qquad (7.19)$$

$$(\mu + q^2)c_{i,j} = q^4\, c_{i+1,j} \qquad \text{for all } i \geq 1,\ j \in \mathbb{Z}. \qquad (7.20)$$

$$\lambda c_{0,j} = q^3 c_{0,j+1} + (q^{-2}\mu + 1)c_{0,j-1} \qquad \text{for all } j \geq 0. \qquad (7.21)$$

Proof. The first formula has already been proven in Lemma 7.4.3. Substituting it into (7.16) we obtain (7.21). The second formula follows by combining Lemma 7.4.3 with the result $c'_{i-1,j} = -q^2 c_{i,j}$ $(i \geq 1)$ from Lemma 7.4.1. □

Proposition 7.4.5. *Let (π, V) be a generic, irreducible, admissible representation of $\mathrm{GSp}(4.F)$ with trivial central character, where $V = \mathcal{W}(\psi_{c_1,c_2})$. Assume that for the minimal paramodular level we have $N_\pi \geq 2$. Let $W \in V(N_\pi)$ be a local newform that is an eigenvector for both Hecke operators $T_{0,1}$ and $T_{1,0}$,*

$$T_{0,1}W = \lambda W, \qquad T_{1,0}W = \mu W, \qquad \lambda, \mu \in \mathbb{C}.$$

Then:

i) $W = 0$ if and only if $W(1) = 0$.
ii) The zeta integral of W is given by

$$Z(s, W) = \frac{(1 - q^{-1})W(1)}{1 - q^{-3/2}\lambda q^{-s} + (q^{-2}\mu + 1)q^{-2s}}. \qquad (7.22)$$

In particular, $Z(s, W)$ is non-zero if W is non-zero.

Proof. i) Assume that $W(1) = 0$. By the definition above, $c_{0,0} = W(1)$. Since we always have $c_{i,j} = 0$ if $i < 0$ or $j < 0$, it follows from (7.21) that $c_{0,j} = 0$ for all j. By (7.19) we get that also $c_{1,j} = 0$ for all j. By (7.20) we then conclude that $c_{i,j} = 0$ for all i and j. This means that W vanishes on all diagonal elements. By Corollary 4.3.8 it follows that $W = 0$.

ii) By Lemma 4.1.1,

$$Z(s,W) = \int\limits_{F^\times} W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix})|a|^{s-3/2}\,d^\times a$$
$$= (1-q^{-1})\sum_{j=0}^{\infty} c_{0,j}q^{-j(s-3/2)}.$$

Multiplying with λ and using the recursion formula (7.21), we get

$$\lambda\, Z(s,W) = (1-q^{-1})\sum_{j=0}^{\infty}\big(q^3c_{0,j+1} + (q^{-2}\mu+1)c_{0,j-1}\big)q^{-j(s-3/2)}$$
$$= (1-q^{-1})q^{s+3/2}\big(\frac{1}{1-q^{-1}}Z(s,w) - c_{0,0}\big)$$
$$+ (q^{-2}\mu+1)q^{-s+3/2}Z(s,W).$$

Solving for the zeta integral gives

$$Z(s,W) = \frac{(1-q^{-1})c_{0,0}}{1-q^{-3/2}\lambda q^{-s} + (q^{-2}\mu+1)q^{-2s}}.$$

The result follows since $c_{0,0} = W(1)$. □

Proposition 7.4.5 can be used to compute the Hecke eigenvalues for a class of representations that might be called the "Siegel-cuspidal" representations.

Corollary 7.4.6. *Let (π, V) be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4.F)$ *with trivial central character, where $V = \mathcal{W}(\psi_{c_1,c_2})$. Assume that $L(s,\pi) = 1$, and that for the minimal paramodular level we have $N_\pi \geq 2$ (in particular, this is satisfied for representations of type VII, VIIIa and IXa, and for generic supercuspidal representations). Let $W \in V(N_\pi)$ be a non-zero eigenvector for both Hecke operators $T_{0,1}$ and $T_{1,0}$. Then*

$$T_{0,1}W = 0 \qquad \text{and} \qquad T_{1,0}W = -q^2W.$$

The zeta integral of W is constant,

$$Z(s,W) = (1-q^{-1})W(1).$$

In particular, if we normalize W by $W(1) = (1-q^{-1})^{-1}$, then $Z(s,W) = 1 = L(s,\pi)$. The precise shape of this normalized newform is

$$W(1) = (1-q^{-1})^{-1}, \qquad W(\begin{bmatrix} \varpi^2 &&& \\ & \varpi && \\ && \varpi & \\ &&& 1 \end{bmatrix}) = -q^{-2}(1-q^{-1})^{-1},$$

and $W(\mathrm{diag}(\varpi^{2i+j}, \varpi^{i+j}, \varpi^i, 1)) = 0$ if $(i,j) \neq (0,0)$ and $(i,j) \neq (1,0)$.

Proof. Since $L(s,\pi)$ is by definition a generator of the fractional ideal $I(\pi)$ of $\mathbb{C}[q^{-s},q^s]$ consisting of all zeta integrals (see Proposition 2.6.4), the hypothesis $L(s,\pi)=1$ implies that $Z(s,W)$ is a polynomial in q^{-s} and q^s. But $Z(s,W)$ is given by the formula (7.22), implying that $\lambda=0$ and $\mu=-q^2$. It then follows from Lemma 7.4.4 that $c_{1,0}=-q^{-2}c_{0,0}\neq 0$, and that $c_{i,j}=0$ for all other values of i,j. □

The Shadow Vector

Let (π,V) be an irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. Assume that π is paramodular and that the minimal paramodular level N_π is ≥ 2. Let $W\in V(N_\pi)$, $W\neq 0$, and assume that $T_{0,1}W=\lambda W$ and $T_{1,0}W=\mu W$ for some $\lambda,\mu\in\mathbb{C}$. In the beginning of this section we defined the vector

$$W' := \sum_{x,y,z\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 &&& \\ y\varpi^{N_\pi-1} & 1 && \\ x\varpi^{N_\pi-1} && 1 & \\ z\varpi^{N_\pi-1} & x\varpi^{N_\pi-1} & -y\varpi^{N_\pi-1} & 1 \end{bmatrix})W. \tag{7.23}$$

This is a Klingen vector of level $N_\pi-1$; we call it the *shadow* of the newform. Although we have no further need for it, the shadow vector may be important in other contexts. Thus we include the following observations.

Lemma 7.4.7. *The vector W' is invariant under the group*

$$\begin{bmatrix} 1 && \mathfrak{p}^{-(N_\pi-2)} & \\ & 1 && \\ && 1 & \\ &&& 1 \end{bmatrix}.$$

Proof. This is a straightforward calculation. □

Assuming that π is generic, we define the numbers $c_{i,j}$ and $c'_{i,j}$ as in (7.15).

Proposition 7.4.8. *Let (π,V) be a generic, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character, and let the notations be as above. Then the following are equivalent.*

i) W' vanishes on the Klingen parabolic $Q(F)$.
ii) $Z(s,W')=0$.
iii) $c_{1,j}=0$ for all $j\in\mathbb{Z}$.
iv) $c_{i,j}=0$ for all $i,j\in\mathbb{Z}$ with $i\geq 1$.
v) The Hecke eigenvalue μ is zero, i.e., $T_{1,0}W=0$.

In any case we have $Z(s,W')=-q^{-2}\mu Z(s,W)$.

Proof. i) ⇒ ii) This is trivial.

ii) ⇒ iii) follows from Lemma 7.4.1.

iii) ⇒ iv) follows from (7.20) in Lemma 7.4.4.

iv) ⇒ v) Since we know $c_{0,j} \neq 0$ for some j, this follows from (7.19) in Lemma 7.4.4.

v) ⇒ i) follows from (7.19) combined with Lemma 7.4.1.

Lemma 7.4.1 and (7.19) show that $c'_{0,j} = -q^{-2}\mu c_{0,j}$. Hence we get the last assertion. □

A look at Table A.14, which will be proven in the next section, shows that many representations have $T_{1,0}$ eigenvalue $\mu = -q^2$. For these representations we have $Z(s, W') = Z(s, W)$. Hence, if the newform is normalized so that its zeta integral computes the L-factor, the zeta integral of the shadow vector does as well.

7.5 Main Results

Now we can finally prove the main results of this monograph as described in the introduction. The order in which the results are presented here differs from the introduction, because of logical dependencies. The first result, uniqueness at the minimal level, allows us to speak of *the* newform of a paramodular representation.

Theorem 7.5.1 (Uniqueness at the Minimal Level). *Let* (π, V) *be an irreducible, admissible representation of* $\mathrm{GSp}(4.F)$ *with trivial central character. Assume that* π *is paramodular, and let* N_π *be the minimal paramodular level. Then* $\dim V(N_\pi) = 1$.

Proof. For non-supercuspidal π the statement has already been proven in Theorem 5.6.1. Assume that π is supercuspidal. If π is non-generic, then π has no paramodular vectors by Theorem 3.4.3. Assume that π is generic. As a supercuspidal representation with trivial central character, π is unitarizable. By Corollary 6.2.2 and Corollary 6.5.3, the operators $T_{0,1}$ and $T_{1,0}$ act as commuting and diagonalizable endomorphism on $V(N_\pi)$. Hence there exists a basis of $V(N_\pi)$ consisting of simultaneous eigenvectors for $T_{0,1}$ and $T_{1,0}$. Each such eigenvector has the form given in Corollary 7.4.6. Since paramodular vectors are determined by their values on diagonal elements (Corollary 4.3.8), it follows that $\dim V(N_\pi) = 1$. □

Given a paramodular representation (π, V) with minimal level N_π, we call any non-zero vector in $V(N_\pi)$ a *newform*. By Theorem 7.5.1, newforms are unique up to scalars.

It follows from Theorem 7.5.1 that every paramodular representation comes with two complex numbers λ and μ attached to it, namely the eigenvalues of the two Hecke operators $T_{0,1}$ and $T_{1,0}$ on a newform. We have all the information needed to tabulate these numbers for all paramodular representations. We stress the fact that the two Hecke eigenvalues λ and μ attached to a paramodular representation are independent of the model in which the representation is given.

Theorem 7.5.2 (Hecke Eigenvalues). *Let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character. Assume that π has non-zero paramodular vectors, and let N_π be the minimal paramodular level. Let $v \in V(N_\pi)$ be non-zero, and define $\lambda, \mu \in \mathbb{C}$ by*

$$T_{0,1}v = \lambda v, \qquad T_{1,0}v = \mu v.$$

Then λ and μ are given as in Table A.14 on page 294.

Proof. For group I representations the eigenvalues are given in Lemma 7.1.1 for the unramified case, and follow from Proposition 6.4.2 otherwise. The eigenvalues for representations of type IIa and IIb also follow from Proposition 6.4.2, except for the unramified IIb case, where they follow from Lemma 7.1.1.

The eigenvalues for the unramified IIIa case follow from Proposition 6.4.5, and can be deduced from Proposition 6.4.2 for the ramified IIIa case. For the unramified IIIb case they follow from Lemma 7.1.1.

The eigenvalues for an unramified twist of the Steinberg representation $\mathrm{St}_{\mathrm{GSp}(4)}$ were determined in Corollary 6.4.6. In the unramified IVb case they follow from Proposition 6.4.5, and in all the remaining cases of group IV they can be determined using Proposition 6.4.2.

Most of the eigenvalues for group V representations follow from Proposition 6.4.2, except for the following: The unramified Vd case follows from Lemma 7.1.1, and the eigenvalues for the unramified Va case from Corollary 6.4.3.

The eigenvalues for the unramified VIa case follow from Proposition 6.4.5 (using the fact from Theorem 3.4.3 that VIb never has paramodular vectors). For the unramified VId case they follow from Lemma 7.1.1. For all other type VI representations the eigenvalues can be deduced from Proposition 6.4.2.

The eigenvalues for representations of type VII, VIIIa and IXa were determined in Corollary 7.4.6. Representations of type VIIIb and IXb are not paramodular by Theorem 3.4.3.

Eigenvalues for group X are immediate from Proposition 6.4.2. For XIa,b and unramified σ, see Corollary 6.4.4. For XIa and ramified σ, Proposition 6.4.2 applies, since in this case XIb has no paramodular vectors by Theorem 3.4.3.

Hecke eigenvalues for generic, supercuspidal representations were determined in Corollary 7.4.6. □

The following theorem shows that the newform, via its level, Atkin–Lehner eigenvalue and Hecke eigenvalues, computes the L-function of a generic, paramodular representation.

Theorem 7.5.3 (Hecke Eigenvalues and L-functions). *Let (π, V) be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character. Let W be a newform of π, i.e., a non-zero element of the one-dimensional space $V(N_\pi)$. Let*

$$T_{0,1}W = \lambda_\pi W, \quad T_{1,0}W = \mu_\pi W,$$

where λ_π and μ_π are complex numbers.

i) Assume $N_\pi = 0$, so that π is unramified. Then

$$L(s,\pi)=\frac{1}{1-q^{-3/2}\lambda_\pi q^{-s}+(q^{-2}\mu_\pi+1+q^{-2})q^{-2s}-q^{-3/2}\lambda_\pi q^{-3s}+q^{-4s}}.$$

ii) Assume $N_\pi = 1$, and let $\pi(u_1)W = \varepsilon_\pi W$, where $\varepsilon_\pi = \pm 1$ is the Atkin–Lehner eigenvalue of W. Then

$$L(s,\pi) = \frac{1}{1-q^{-3/2}(\lambda_\pi+\varepsilon_\pi)q^{-s}+(q^{-2}\mu_\pi+1)q^{-2s}+\varepsilon_\pi q^{-1/2}q^{-3s}}.$$

iii) Assume $N_\pi \geq 2$. Then

$$L(s,\pi) = \frac{1}{1-q^{-3/2}\lambda_\pi q^{-s}+(q^{-2}\mu_\pi+1)q^{-2s}}.$$

Proof. This is obtained by evaluating both sides of each formula. The factors $L(s,\pi)$ have been evaluated in [Tak]; via Proposition 2.4.4 they appear in Table A.8. The right hand sides are evaluated by substituting the values for λ_π and μ_π from Table A.14. □

The following theorem summarizes our main results for generic representations. As already pointed out in the introduction, for this class of representations the theorem is analogous to the corresponding theorem in the GL(2) theory.

Theorem 7.5.4 (Generic Main Theorem). *Let (π, V) be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character. Then the following statements hold:*

i) There exists an n such that $V(n) \neq 0$, i.e., π is paramodular.
ii) If N_π is the minimal n such that $V(n) \neq 0$, then $\dim V(N_\pi) = 1$.
iii) Assume $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. There exists $W_\pi \in V(N_\pi)$ such that

$$Z(s, W_\pi) = L(s,\pi).$$

Proof. i) was proved in Theorem 4.4.1.

ii) was proved in Theorem 7.5.1.

iii) Let W be a non-zero element of $V(N_\pi)$. Let λ, μ be the Hecke eigenvalues of W, defined by

$$T_{0,1}W = \lambda W, \qquad T_{1,0}W = \mu W.$$

Then we have the following formulas for the zeta integral of W. If $N_\pi = 0$ (the unramified case),

$$Z(s,W) = \frac{(1-q^{-1})W(1)}{1 - q^{-3/2}\lambda q^{-s} + (q^{-2}\mu + 1 + q^{-2})q^{-2s} - q^{-3/2}\lambda q^{-3s} + q^{-4s}} \tag{7.24}$$

by Proposition 7.1.4. If $N_\pi = 1$, then

$$Z(s,W) = \frac{(1-q^{-1})W(1)}{1 - q^{-3/2}(\lambda+\varepsilon)q^{-s} + (\mu q^{-2} + 1)q^{-2s} + \varepsilon q^{-1/2}q^{-3s}} \tag{7.25}$$

by Proposition 7.2.5. Here ε is the Atkin–Lehner eigenvalue of W, defined by $u_1 W = \varepsilon W$. If $N_\pi \geq 2$, then

$$Z(s,W) = \frac{(1-q^{-1})W(1)}{1 - q^{-3/2}\lambda q^{-s} + (q^{-2}\mu + 1)q^{-2s}} \tag{7.26}$$

by Proposition 7.4.5. By Theorem 4.3.7, $Z(s,W) \neq 0$, therefore $W(1) \neq 0$. By normalizing W we may assume that $(1-q^{-1})W(1) = 1$. Then the asserted equality follows from Theorem 7.5.3. □

As a consequence of the generic main theorem, we obtain a calculation of the ε-factor $\varepsilon(s,\pi)$ in terms of basic invariants N_π and ε_π of the newform in a generic representation π. This result is also analogous to the GL(2) theory.

Corollary 7.5.5 (ε-factors of Generic Representations). *Let (π, V) be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character. Let N_π be the paramodular level of π as in Theorem 7.5.4, and let ε_π be the eigenvalue of the Atkin–Lehner involution $\pi(u_{N_\pi})$ on the one-dimensional space $V(N_\pi)$. Then*

$$\varepsilon(s,\pi) = \varepsilon_\pi q^{-N_\pi(s-1/2)}.$$

Proof. This follows from iii) of Theorem 7.5.4 and (2.61) with $W = W_\pi$ and $n = N_\pi$. □

The following two theorems show that in any paramodular representation the oldforms are obtained by applying the level raising operators θ, θ', η to the newform and taking linear combinations. The first theorem focuses on the generic case, where the results are more specific. We point out that other specific results exist for other classes of representations; see Sections 5.3 and 5.5.

Theorem 7.5.6 (Generic Oldforms Theorem). *Let (π, V) be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character. Let N_π be the paramodular level of π and let W_π be the newform as in Theorem 7.5.4. Then, for any integer $n \geq N_\pi$,*

$$\dim V(n) = \left[\frac{(n-N_\pi+2)^2}{4}\right].$$

For $n \geq N_\pi$, the space $V(n)$ is spanned by the linearly independent vectors

$$\theta'^i\theta^j\eta^k W_\pi, \quad i,j,k \geq 0,\ i+j+2k = n-N_\pi. \tag{7.27}$$

In particular, all oldforms are obtained by applying level raising operators to the newform and taking linear combinations.

Proof. We already know that $\dim V(n) \geq \left[\frac{(n-N_\pi+2)^2}{4}\right]$ and that the vectors (7.27) are linearly independent by Proposition 4.3.9. We need to prove that if $W \in V(n)$, $n \geq N_\pi$, then W is in the span of the vectors (7.27). We prove this by induction on n. If $n = N_\pi$, then this follows from ii) of Theorem 7.5.4. Assume that $n > N_\pi$ and that the assertion holds for all m such that $N_\pi \leq m < n$. Let $W \in V(n)$. By iii) of Proposition 4.1.4, $Z(s,W)/L(s,\pi)$ is a polynomial in q^{-s} of degree at most $n-N_\pi$. Note that the number N appearing in Proposition 4.1.4 is N_π by Corollary 7.5.5. By Proposition 4.1.3, there exists a W' in the span of the vectors (7.27) such that $Z(s,W') = Z(s,W)$. Hence $W-W'$ is degenerate, and the η Principle Theorem 4.3.7 implies that $W-W'=0$, or that $n \geq 2$ and $W-W' = \eta W''$ for some $W'' \in V(n-2)$. Applying the induction hypothesis completes the proof. □

Theorem 7.5.7 (Oldforms Principle). *Let (π, V) be an irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character. Assume that π is paramodular. If v is a non-zero element of the one-dimensional space $V(N_\pi)$ and $n \geq N_\pi$, then the space $V(n)$ is spanned by the (not necessarily linearly independent) vectors*

$$\theta'^i\theta^j\eta^k v, \quad i,j,k \geq 0,\ i+j+2k = n-N_\pi.$$

In other words, all oldforms can be obtained from the newform v by applying level raising operators and taking linear combinations.

Proof. In the non-supercuspidal case this was proven in Theorem 5.6.1 iv). If π is supercuspidal, then, as a paramodular representation, π is generic; see Theorem 3.4.3. Hence the result follows from Theorem 7.5.6. □

One can ask about the relationship between paramodular representations and L-packets: Should every conjectural L-packet have at most one paramodular representation? The following theorem implies that this is true, because one can deduce from the desiderata of the conjectural local Langlands correspondence for $\mathrm{GSp}(4)$ that any conjectural L-packet with more than one element must be tempered and contains a unique generic element.

Theorem 7.5.8 (Tempered Representations). *Let π be an irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character. Assume π is tempered. Then π is paramodular if and only if π is generic.*

Proof. If π is supercuspidal, this follows from Theorem 3.4.3 and Theorem 4.4.1. Now assume π is non-supercuspidal. If π is not generic, then, by Table A.1, π is of type VIb or VIIIb. In this case, Theorem 3.4.3 implies that π is not paramodular. If π is generic, then π is paramodular by Theorem 4.4.1. □

Our final result shows that the newform, via its invariants N_π, ε_π, λ_π and μ_π, computes the L- and ε-factor of the L-parameter of any non-supercuspidal, paramodular representation.

Theorem 7.5.9 (Non-supercuspidal Newforms and L- and ε-factors). *Let (π, V) be a paramodular, non-supercuspidal, irreducible, admissible representation of* $\mathrm{GSp}(4, F)$ *with trivial central character. Let $\varphi_\pi : W_F' \to$* $\mathrm{GSp}(4, \mathbb{C})$ *be the L-parameter assigned to π as in Sect. 2.4. Let N_π be the minimal paramodular level of π, and let $v \in V(N_\pi)$ be a non-zero vector. Let ε_π be the Atkin–Lehner eigenvalue of v, and let λ_π and μ_π be the Hecke eigenvalues of v, defined by $T_{0,1}v = \lambda_\pi v$ and $T_{1,0}v = \mu_\pi v$. Then*

$$\varepsilon(s, \varphi_\pi) = \varepsilon_\pi q^{-N_\pi(s-1/2)}.$$

i) Assume $N_\pi = 0$, so that π is unramified. Then

$$L(s, \varphi_\pi) = \frac{1}{1 - q^{-3/2}\lambda_\pi q^{-s} + (q^{-2}\mu_\pi + 1 + q^{-2})q^{-2s} - q^{-3/2}\lambda_\pi q^{-3s} + q^{-4s}}.$$

ii) Assume $N_\pi = 1$. Then

$$L(s, \varphi_\pi) = \frac{1}{1 - q^{-3/2}(\lambda_\pi + \varepsilon_\pi)q^{-s} + (q^{-2}\mu_\pi + 1)q^{-2s} + \varepsilon_\pi q^{-1/2}q^{-3s}}.$$

iii) Assume $N_\pi \geq 2$. Then

$$L(s, \varphi_\pi) = \frac{1}{1 - q^{-3/2}\lambda_\pi q^{-s} + (q^{-2}\mu_\pi + 1)q^{-2s}}.$$

Proof. The statement $\varepsilon(s, \varphi_\pi) = \varepsilon_\pi q^{-N_\pi(s-1/2)}$ was proved in Theorem 5.7.3. It is easy to compute the L-factors (as defined in (2.48)) of the L-parameters; the results are listed in Table A.8. The right hand sides of the asserted formulas can be evaluated by using Table A.14 (for λ_π and μ_π) and Table A.12 (for ε_π). A case-by-case comparison now shows that the formulas hold. □

Since any paramodular representation is either generic or non-supercuspidal, we conclude that the newform computes the relevant L- and ε-factors of a paramodular representation.

A

Tables for Representations of GSp(4)

A.1 Non-supercuspidal Representations

Table A.1 below gives a list of all irreducible, admissible, non-supercuspidal representations of $\mathrm{GSp}(4,F)$. We have organized these representations into eleven groups. Groups I to VI contain representations supported in the minimal parabolic subgroup B; groups VII to IX contain representations supported in the Klingen parabolic subgroup Q; and groups X and XI contain representations supported in the Siegel parabolic subgroup P.

All the information in Table A.1, as well as the notations, are taken from [ST]. A more detailed description of the representations listed can be found in Sect. 2.2. The "tempered" column shows the conditions on the inducing data under which a representation is tempered. The "ess. L^2" column indicates the essentially square-integrable representations, i.e., those representations that become square-integrable after suitable twisting. Finally, the rightmost column indicates the generic representations (see Sect. 2.1 for the precise definition).

Table A.1. Non-supercuspidal representations of $\mathrm{GSp}(4,F)$

		constituent of	representation	tempered	ess. L^2	generic
I		$\chi_1\times\chi_2\rtimes\sigma$ (irreducible)		χ_i,σ unit.		•
II	a	$\nu^{1/2}\chi\times\nu^{-1/2}\chi\rtimes\sigma$	$\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$	χ,σ unit.		•
	b	$(\chi^2\neq\nu^{\pm1},\chi\neq\nu^{\pm3/2})$	$\chi\mathbf{1}_{\mathrm{GL}(2)}\rtimes\sigma$			
III	a	$\chi\times\nu\rtimes\nu^{-1/2}\sigma$	$\chi\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	χ,σ unit.		•
	b	$(\chi\notin\{1,\nu^{\pm2}\})$	$\chi\rtimes\sigma\mathbf{1}_{\mathrm{GSp}(2)}$			
IV	a	$\nu^2\times\nu\rtimes\nu^{-3/2}\sigma$	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	σ unit.	•	•
	b		$L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$			
	c		$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma)$			
	d		$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$			
V	a	$\nu\xi\times\xi\rtimes\nu^{-1/2}\sigma$	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	σ unit.	•	•
	b	$(\xi^2=1,\ \xi\neq1)$	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$			
	c		$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\xi\nu^{-1/2}\sigma)$			
	d		$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$			
VI	a	$\nu\times1_{F^\times}\rtimes\nu^{-1/2}\sigma$	$\tau(S,\nu^{-1/2}\sigma)$	σ unit.		•
	b		$\tau(T,\nu^{-1/2}\sigma)$	σ unit.		
	c		$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$			
	d		$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$			
VII		$\chi\rtimes\pi$ (irreducible)		χ,π unit.		•
VIII	a	$1_{F^\times}\rtimes\pi$	$\tau(S,\pi)$	π unit.		•
	b		$\tau(T,\pi)$	π unit.		
IX	a	$\nu\xi\rtimes\nu^{-1/2}\pi$	$\delta(\nu\xi,\nu^{-1/2}\pi)$	π unit.	•	•
	b	$(\xi\neq1,\ \xi\pi=\pi)$	$L(\nu\xi,\nu^{-1/2}\pi)$			
X		$\pi\rtimes\sigma$ (irreducible)		π,σ unit.		•
XI	a	$\nu^{1/2}\pi\rtimes\nu^{-1/2}\sigma$	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	π,σ unit.	•	•
	b	$(\omega_\pi=1)$	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$			

A.2 Unitary Representations

Table A.2 below lists all the irreducible, admissible, unitarizable representations of $\mathrm{GSp}(4,F)$. The table represents a reformulation of Theorem 4.4, Proposition 4.7 and Proposition 4.9 of [ST]. We include this table for completeness only; the unitary property is largely irrelevant for the paramodular newform theory.

Table A.2 uses the notation e for the *exponent* of an essentially square integrable representation. We only need the following two special cases. If χ is a character of $F^\times$, then $e(\chi)$ is defined by $|\chi(x)| = |x|^{e(\chi)}$ for $x \in F^\times$. If π is a supercuspidal representation of $\mathrm{GL}(2,F)$, then $e(\pi)$ is defined by the condition that $\nu^{-e(\pi)}\pi$ is unitarizable.

Table A.2. Unitary representations of $\mathrm{GSp}(4,F)$

		representation	conditions for unitarity
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	$e(\chi_1) = e(\chi_2) = e(\sigma) = 0$
			$\chi_1 = \nu^\beta\chi,\ \chi_2 = \nu^\beta\chi^{-1},\ e(\sigma) = -\beta,$ $e(\chi) = 0,\ \chi^2 \neq 1,\ 0 < \beta < 1/2$
			$\chi_1 = \nu^\beta,\ e(\chi_2) = 0,\ e(\sigma) = -\beta/2,$ $\chi_2 \neq 1,\ 0 < \beta < 1$
			$\chi_1 = \nu^{\beta_1}\chi,\ \chi_2 = \nu^{\beta_2}\chi,\ e(\sigma) = (-\beta_1 - \beta_2)/2,$ $\chi^2 = 1,\ 0 \leq \beta_2 \leq \beta_1,\ 0 < \beta_1 < 1,\ \beta_1 + \beta_2 < 1$
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$e(\sigma) = e(\chi) = 0$
			$\chi = \xi\nu^\beta,\ e(\sigma) = -\beta,\ \xi^2 = 1,\ 0 < \beta < 1/2$
	b	$\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$	$e(\sigma) = e(\chi) = 0$
			$\chi = \xi\nu^\beta,\ e(\sigma) = -\beta,\ \xi^2 = 1,\ 0 < \beta < 1/2$
III	a	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$e(\sigma) = e(\chi) = 0$
	b	$\chi \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}$	$e(\sigma) = e(\chi) = 0$
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$e(\sigma) = 0$
	b	$L(\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	never unitary
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GSp}(2)}, \nu^{-3/2}\sigma)$	never unitary
	d	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$	$e(\sigma) = 0$

		representation	conditions for unitarity
V	a	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	$e(\sigma)=0$
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$e(\sigma)=0$
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\xi\nu^{-1/2}\sigma)$	$e(\sigma)=0$
	d	$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$	$e(\sigma)=0$
VI	a	$\tau(S,\nu^{-1/2}\sigma)$	$e(\sigma)=0$
	b	$\tau(T,\nu^{-1/2}\sigma)$	$e(\sigma)=0$
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$e(\sigma)=0$
	d	$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$	$e(\sigma)=0$
VII		$\chi\rtimes\pi$ (irreducible)	$e(\chi)=e(\pi)=0$
			$\chi=\nu^\beta\xi,\ \pi=\nu^{-\beta/2}\rho,\ 0<\beta<1,$ $\xi^2=1,\ \xi\neq1,\ e(\rho)=0,\ \xi\rho=\rho$
VIII	a	$\tau(S,\pi)$	$e(\pi)=0$
	b	$\tau(T,\pi)$	$e(\pi)=0$
IX	a	$\delta(\nu\xi,\nu^{-1/2}\pi)$	$e(\pi)=0$
	b	$L(\nu\xi,\nu^{-1/2}\pi)$	$e(\pi)=0$
X		$\pi\rtimes\sigma$ (irreducible)	$e(\sigma)=e(\pi)=0$
			$\pi=\nu^\beta\rho,\ e(\sigma)=-\beta,\ 0<\beta<1/2,\ \omega_\rho=1$
XI	a	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$e(\sigma)=e(\pi)=0$
	b	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$e(\sigma)=e(\pi)=0$
π supercuspidal			$e(\omega_\pi)=0$

A.3 Jacquet Modules

The two tables in this section list the semisimplifications of the normalized Jacquet modules of all non-supercuspidal representations with respect to the unipotent radical of the Siegel and Klingen parabolic subgroups. The Jacquet modules with respect to the unipotent radical of the Siegel parabolic are representations of $\mathrm{GL}(2,F) \times F^\times$, and the Jacquet modules with respect to the unipotent radical of the Klingen parabolic are representations of $F^\times \times \mathrm{GSp}(2,F)$. Note that $\mathrm{GSp}(2,F) = \mathrm{GL}(2,F)$; to translate into standard GL(2) notation, use the formula $\chi \rtimes \sigma = \chi\sigma \times \sigma$. The last column lists the number of irreducible constituents. These Jacquet modules were computed using Section 2 of [ST], pages 93–94.

Table A.3. Jacquet modules: The Siegel parabolic

		representation	semisimplification	#
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	$(\chi_1 \times \chi_2) \otimes \sigma + (\chi_1^{-1} \times \chi_2^{-1}) \otimes \chi_1\chi_2\sigma$ $+(\chi_1 \times \chi_2^{-1}) \otimes \chi_2\sigma + (\chi_2 \times \chi_1^{-1}) \otimes \chi_1\sigma$	4
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$\chi\mathrm{St}_{\mathrm{GL}(2)} \otimes \sigma + \chi^{-1}\mathrm{St}_{\mathrm{GL}(2)} \otimes \chi^2\sigma$ $+(\chi\nu^{1/2} \times \chi^{-1}\nu^{1/2}) \otimes \chi\nu^{-1/2}\sigma$	3
	b	$\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$	$\chi\mathbf{1}_{\mathrm{GL}(2)} \otimes \sigma + \chi^{-1}\mathbf{1}_{\mathrm{GL}(2)} \otimes \chi^2\sigma$ $+(\chi\nu^{-1/2} \times \chi^{-1}\nu^{-1/2}) \otimes \chi\nu^{1/2}\sigma$	3
III	a	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$(\chi \times \nu) \otimes \sigma\nu^{-1/2} + (\nu \times \chi^{-1}) \otimes \chi\sigma\nu^{-1/2}$	2
	b	$\chi \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}$	$(\chi \times \nu^{-1}) \otimes \sigma\nu^{1/2} + (\nu^{-1} \times \chi^{-1}) \otimes \chi\sigma\nu^{1/2}$	2
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)} \otimes \nu^{-3/2}\sigma$	1
	b	$L(\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	$\nu^{3/2}\mathbf{1}_{\mathrm{GL}(2)} \otimes \nu^{-3/2}\sigma + (\nu \times \nu^{-2}) \otimes \nu^{1/2}\sigma$	2
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$	$\nu^{-3/2}\mathrm{St}_{\mathrm{GL}(2)} \otimes \nu^{3/2}\sigma + (\nu^2 \times \nu^{-1}) \otimes \nu^{-1/2}\sigma$	2
	d	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$	$\nu^{-3/2}\mathbf{1}_{\mathrm{GL}(2)} \otimes \nu^{3/2}\sigma$	1

		representation	semisimplification	#
V	a	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	$\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}\otimes\nu^{-1/2}\sigma$ $+\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}\otimes\xi\nu^{-1/2}\sigma$	2
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$\nu^{-1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}\otimes\nu^{1/2}\sigma$ $+\nu^{1/2}\xi\mathbf{1}_{\mathrm{GL}(2)}\otimes\xi\nu^{-1/2}\sigma$	2
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\xi\nu^{-1/2}\sigma)$	$\nu^{-1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}\otimes\xi\nu^{1/2}\sigma$ $+\nu^{1/2}\xi\mathbf{1}_{\mathrm{GL}(2)}\otimes\nu^{-1/2}\sigma$	2
	d	$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$	$\nu^{-1/2}\xi\mathbf{1}_{\mathrm{GL}(2)}\otimes\xi\nu^{1/2}\sigma$ $+\nu^{-1/2}\xi\mathbf{1}_{\mathrm{GL}(2)}\otimes\nu^{1/2}\sigma$	2
VI	a	$\tau(S,\nu^{-1/2}\sigma)$	$2\cdot(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}\otimes\nu^{-1/2}\sigma)$ $+\nu^{1/2}\mathbf{1}_{\mathrm{GL}(2)}\otimes\nu^{-1/2}\sigma$	3
	b	$\tau(T,\nu^{-1/2}\sigma)$	$\nu^{1/2}\mathbf{1}_{\mathrm{GL}(2)}\otimes\nu^{-1/2}\sigma$	1
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$\nu^{-1/2}\mathrm{St}_{\mathrm{GL}(2)}\otimes\nu^{1/2}\sigma$	1
	d	$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$	$2\cdot(\nu^{-1/2}\mathbf{1}_{\mathrm{GL}(2)}\otimes\nu^{1/2}\sigma)$ $+\nu^{-1/2}\mathrm{St}_{\mathrm{GL}(2)}\otimes\nu^{1/2}\sigma$	3
VII		$\chi\rtimes\pi$	0	0
VIII	a	$\tau(S,\pi)$	0	0
	b	$\tau(T,\pi)$	0	0
IX	a	$\delta(\nu\xi,\nu^{-1/2}\pi)$	0	0
	b	$L(\nu\xi,\nu^{-1/2}\pi)$	0	0
X		$\pi\rtimes\sigma$	$\pi\otimes\sigma+\tilde{\pi}\otimes\omega_\pi\sigma$	2
XI	a	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$\nu^{1/2}\pi\otimes\nu^{-1/2}\sigma$	1
	b	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$\nu^{-1/2}\pi\otimes\nu^{1/2}\sigma$	1

Table A.4. Jacquet modules: The Klingen parabolic

		representation	semisimplification	#
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	$\chi_1 \otimes (\chi_2 \rtimes \sigma) + \chi_2 \otimes (\chi_1 \rtimes \sigma)$ $+\chi_2^{-1} \otimes (\chi_1 \rtimes \chi_2\sigma) + \chi_1^{-1} \otimes (\chi_2 \rtimes \chi_1\sigma)$	4
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$\chi\nu^{1/2} \otimes (\chi\nu^{-1/2} \rtimes \sigma)$ $+\chi^{-1}\nu^{1/2} \otimes (\chi\nu^{1/2} \rtimes \chi\nu^{-1/2}\sigma)$	2
	b	$\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$	$\chi\nu^{-1/2} \otimes (\chi\nu^{1/2} \rtimes \sigma)$ $+\chi^{-1}\nu^{-1/2} \otimes (\chi\nu^{-1/2} \rtimes \chi\nu^{1/2}\sigma)$	2
III	a	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$\chi \otimes \sigma\mathrm{St}_{\mathrm{GSp}(2)} + \chi^{-1} \otimes \chi\sigma\mathrm{St}_{\mathrm{GSp}(2)}$ $+\nu \otimes (\chi \rtimes \sigma\nu^{-1/2})$	3
	b	$\chi \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}$	$\chi \otimes \sigma\mathbf{1}_{\mathrm{GSp}(2)} + \chi^{-1} \otimes \chi\sigma\mathbf{1}_{\mathrm{GSp}(2)}$ $+\nu^{-1} \otimes (\chi \rtimes \sigma\nu^{1/2})$	3
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$\nu^2 \otimes \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	1
	b	$L(\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	$\nu^{-2} \otimes \nu\sigma\mathrm{St}_{\mathrm{GSp}(2)} + \nu \otimes (\nu^2 \rtimes \nu^{-3/2}\sigma)$	2
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$	$\nu^2 \otimes \nu^{-1}\sigma\mathbf{1}_{\mathrm{GSp}(2)} + \nu^{-1} \otimes (\nu^2 \rtimes \nu^{-1/2}\sigma)$	2
	d	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$	$\nu^{-2} \otimes \nu\sigma\mathbf{1}_{\mathrm{GSp}(2)}$	1
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	$\nu\xi \otimes (\xi \rtimes \nu^{-1/2}\sigma)$	1
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	$\xi \otimes (\nu\xi \rtimes \xi\nu^{-1/2}\sigma)$	1
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$	$\xi \otimes (\nu\xi \rtimes \nu^{-1/2}\sigma)$	1
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	$\nu^{-1/2}\xi \otimes (\xi \rtimes \nu^{1/2}\sigma)$	1
VI	a	$\tau(S, \nu^{-1/2}\sigma)$	$\nu \otimes (1_{F^\times} \rtimes \nu^{-1/2}\sigma) + 1_{F^\times} \otimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	2
	b	$\tau(T, \nu^{-1/2}\sigma)$	$1_{F^\times} \otimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	1
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	$1_{F^\times} \otimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}$	1
	d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$	$1_{F^\times} \otimes \sigma\mathbf{1}_{\mathrm{GSp}(2)} + \nu^{-1} \otimes (1_{F^\times} \rtimes \nu^{1/2}\sigma)$	2

		representation	semisimplification	#
VII		$\chi \rtimes \pi$	$\chi \otimes \pi + \chi^{-1} \otimes \chi\pi$	2
VIII	a	$\tau(S, \pi)$	$1_{F^\times} \otimes \pi$	1
	b	$\tau(T, \pi)$	$1_{F^\times} \otimes \pi$	1
IX	a	$\delta(\nu\xi, \nu^{-1/2}\pi)$	$\nu\xi \otimes \nu^{-1/2}\pi$	1
	b	$L(\nu\xi, \nu^{-1/2}\pi)$	$\nu^{-1}\xi \otimes \nu^{1/2}\pi$	1
X		$\pi \rtimes \sigma$	0	0
XI	a	$\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	0	0
	b	$L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	0	0

A.4 The P_3-Filtration

Let V be an irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character, and let $0 \subset V_2 \subset V_1 \subset V_0 = V_{Z^J}$ be the P_3-filtration from Theorem 2.5.3. The tables in this section list the semisimplifications of the quotient V_0/V_1 and V_1/V_2. The last column lists the number of irreducible constituents. These semisimplifications are obtained from the semisimplifications of the Jacquet modules of V with respect to the Siegel and Klingen parabolic subgroups from Section A.3, using $V_0/V_1 \cong \tau_{\mathrm{GL}(2)}^{P_3}(V_{N_Q})$ and $V_1/V_2 \cong \tau_{\mathrm{GL}(1)}^{P_3}(V_{U,\psi_{-1,0}})$. See Theorem 2.5.3. Note that the factor $\nu^{3/2}$ appearing in all the entries in Table A.6 is a consequence of the fact that the Jacquet modules in Table A.3 are normalized, while the P_3-filtration involves no normalizations.

Table A.5. P_3-filtration: V_0/V_1

		representation	s.s.(V_0/V_1)	#
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	$\tau_{\mathrm{GL}(2)}^{P_3}(\nu(\chi_1\chi_2\sigma \times \chi_1\sigma))$ $+\tau_{\mathrm{GL}(2)}^{P_3}(\nu(\chi_1\chi_2\sigma \times \chi_2\sigma))$ $+\tau_{\mathrm{GL}(2)}^{P_3}(\nu(\chi_1\sigma \times \sigma))$ $+\tau_{\mathrm{GL}(2)}^{P_3}(\nu(\chi_2\sigma \times \sigma))$	4
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$\tau_{\mathrm{GL}(2)}^{P_3}(\nu(\chi^2\sigma \times \nu^{1/2}\chi\sigma))$ $+\tau_{\mathrm{GL}(2)}^{P_3}(\nu(\nu^{1/2}\chi\sigma \times \sigma))$	2
	b	$\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$	$\tau_{\mathrm{GL}(2)}^{P_3}(\nu(\chi^2\sigma \times \nu^{-1/2}\chi\sigma))$ $+\tau_{\mathrm{GL}(2)}^{P_3}(\nu(\nu^{-1/2}\chi\sigma \times \sigma))$	2
III	a	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$\tau_{\mathrm{GL}(2)}^{P_3}(\nu\chi\sigma\mathrm{St}_{\mathrm{GL}(2)})$ $+\tau_{\mathrm{GL}(2)}^{P_3}(\nu\sigma\mathrm{St}_{\mathrm{GL}(2)})$ $+\tau_{\mathrm{GL}(2)}^{P_3}(\nu(\nu^{1/2}\chi\sigma \times \nu^{1/2}\sigma))$	3
	b	$\chi \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}$	$\tau_{\mathrm{GL}(2)}^{P_3}(\nu(\chi\sigma\mathbf{1}_{\mathrm{GL}(2)}))$ $+\tau_{\mathrm{GL}(2)}^{P_3}(\nu\sigma\mathbf{1}_{\mathrm{GL}(2)})$ $+\tau_{\mathrm{GL}(2)}^{P_3}(\nu(\nu^{-1/2}\chi\sigma \times \nu^{-1/2}\sigma))$	3

		representation	s.s.(V_0/V_1)	#
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$\tau^{P_3}_{\mathrm{GL}(2)}(\nu^2\sigma\mathrm{St}_{\mathrm{GL}(2)})$	1
IV	b	$L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	$\tau^{P_3}_{\mathrm{GL}(2)}(\sigma\mathrm{St}_{\mathrm{GL}(2)})$ $+\tau^{P_3}_{\mathrm{GL}(2)}(\nu(\nu^{3/2}\sigma\times\nu^{-1/2}\sigma))$	2
IV	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma)$	$\tau^{P_3}_{\mathrm{GL}(2)}(\nu^2\sigma\mathbf{1}_{\mathrm{GL}(2)})$ $+\tau^{P_3}_{\mathrm{GL}(2)}(\nu(\nu^{1/2}\sigma\times\nu^{-3/2}\sigma))$	2
IV	d	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$	$\tau^{P_3}_{\mathrm{GL}(2)}(\sigma\mathbf{1}_{\mathrm{GL}(2)})$	1
V	a	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	$\tau^{P_3}_{\mathrm{GL}(2)}(\nu(\nu^{1/2}\sigma\times\nu^{1/2}\xi\sigma))$	1
V	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$\tau^{P_3}_{\mathrm{GL}(2)}(\nu(\nu^{1/2}\xi\sigma\times\nu^{-1/2}\sigma))$	1
V	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\xi\nu^{-1/2}\sigma)$	$\tau^{P_3}_{\mathrm{GL}(2)}(\nu(\nu^{1/2}\sigma\times\nu^{-1/2}\xi\sigma))$	1
V	d	$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$	$\tau^{P_3}_{\mathrm{GL}(2)}(\nu(\sigma\times\xi\sigma))$	1
VI	a	$\tau(S,\nu^{-1/2}\sigma)$	$\tau^{P_3}_{\mathrm{GL}(2)}(\nu(\nu^{1/2}\sigma\times\nu^{1/2}\sigma))$ $+\tau^{P_3}_{\mathrm{GL}(2)}(\nu\sigma\mathrm{St}_{\mathrm{GL}(2)})$	2
VI	b	$\tau(T,\nu^{-1/2}\sigma)$	$\tau^{P_3}_{\mathrm{GL}(2)}(\nu\sigma\mathrm{St}_{\mathrm{GL}(2)})$	1
VI	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$\tau^{P_3}_{\mathrm{GL}(2)}(\nu\sigma\mathbf{1}_{\mathrm{GL}(2)})$	1
VI	d	$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$	$\tau^{P_3}_{\mathrm{GL}(2)}(\nu\sigma\mathbf{1}_{\mathrm{GL}(2)})$ $+\tau^{P_3}_{\mathrm{GL}(2)}(\nu(\nu^{-1/2}\sigma\times\nu^{-1/2}\sigma))$	2
VII		$\chi\rtimes\pi$	$\tau^{P_3}_{\mathrm{GL}(2)}(\nu\chi\pi)+\tau^{P_3}_{\mathrm{GL}(2)}(\nu\pi)$	2
VIII	a	$\tau(S,\pi)$	$\tau^{P_3}_{\mathrm{GL}(2)}(\nu\pi)$	1
VIII	b	$\tau(T,\pi)$	$\tau^{P_3}_{\mathrm{GL}(2)}(\nu\pi)$	1
IX	a	$\delta(\nu\xi,\nu^{-1/2}\pi)$	$\tau^{P_3}_{\mathrm{GL}(2)}(\nu^{3/2}\xi\pi)$	1
IX	b	$L(\nu\xi,\nu^{-1/2}\pi)$	$\tau^{P_3}_{\mathrm{GL}(2)}(\nu^{1/2}\xi\pi)$	1
X		$\pi\rtimes\sigma$	0	0
XI	a	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	0	0
XI	b	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	0	0

Table A.6. P_3-filtration: V_1/V_2

		representation	s.s.(V_1/V_2)	#
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	$\tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\chi_1\chi_2\sigma) + \tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\sigma)$ $+\tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\chi_1\sigma) + \tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\chi_2\sigma)$	4
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$\tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\chi^2\sigma) + \tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\sigma)$ $+\tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\nu^{1/2}\chi\sigma)$	3
	b	$\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$	$\tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\nu^{-1/2}\chi\sigma)$	1
III	a	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$\tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\nu^{1/2}\chi\sigma) + \tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\nu^{1/2}\sigma)$	2
	b	$\chi \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}$	$\tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\nu^{-1/2}\chi\sigma) + \tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\nu^{-1/2}\sigma)$	2
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$\tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\nu^{3/2}\sigma)$	1
	b	$L(\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	$\tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\nu^{-1/2}\sigma)$	1
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$	$\tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\nu^{-3/2}\sigma) + \tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\nu^{1/2}\sigma)$	2
	d	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$	0	0
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	$\tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\nu^{1/2}\sigma) + \tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\nu^{1/2}\xi\sigma)$	2
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	$\tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\nu^{-1/2}\sigma)$	1
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$	$\tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\nu^{-1/2}\xi\sigma)$	1
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	0	0
VI	a	$\tau(S, \nu^{-1/2}\sigma)$	$2\tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\nu^{1/2}\sigma)$	2
	b	$\tau(T, \nu^{-1/2}\sigma)$	0	0
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	$\tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\nu^{-1/2}\sigma)$	1
	d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$	$\tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\nu^{-1/2}\sigma)$	1
VII		$\chi \rtimes \pi$	0	0
VIII	a	$\tau(S, \pi)$	0	0
	b	$\tau(T, \pi)$	0	0
IX	a	$\delta(\nu\xi, \nu^{-1/2}\pi)$	0	0
	b	$L(\nu\xi, \nu^{-1/2}\pi)$	0	0
X		$\pi \rtimes \sigma$	$\tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\omega_\pi\sigma) + \tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\sigma)$	2
XI	a	$\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	$\tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\nu^{1/2}\sigma)$	1
	b	$L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	$\tau^{P_3}_{\mathrm{GL}(1)}(\nu^{3/2}\nu^{-1/2}\sigma)$	1

A.5 L-Parameters

Table A.7 gives the L-parameters $\varphi = (\rho, N)$ of each non-supercuspidal representation of $\mathrm{GSp}(4, F)$, as defined in Sect. 2.4. For groups I – VI, an entry $\tau_1, \ldots, \tau_4$ in the "ρ" column stands for the map $W_F \ni w \mapsto \mathrm{diag}(\tau_1(w), \ldots, \tau_4(w)) \in \mathrm{GSp}(4, \mathbb{C})$. For groups VII – XI, let π be the supercuspidal representation of $\mathrm{GL}(2, F)$ as in Table A.1. The symbol φ_π stands for the L-parameter $W_F \to \mathrm{GL}(2, \mathbb{C})$ of π, and φ'_π is defined in (2.1). The character ω_π is the central character of π, identified with a character of W_F. Alternatively, $\omega_\pi(w) = \det(\varphi_\pi(w))$. The entries in the ρ column are to be read in diagonal block notation for groups VII – XI. The nilpotent elements listed in the "N" column are defined as follows.

$$N_1 = \begin{bmatrix} 0 & & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{bmatrix}, \qquad N_2 = \begin{bmatrix} 0 & & & 1 \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}, \qquad N_3 = \begin{bmatrix} 0 & & & 1 \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{bmatrix},$$

$$N_4 = \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & -1 \\ & & & 0 \end{bmatrix}, \qquad N_5 = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & -1 \\ & & & 0 \end{bmatrix}.$$

To define N_6, let S be the symmetric matrix from Lemma 2.4.1. Then

$$N_6 = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}, \qquad \text{where} \qquad B = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} S$$

(see 2.45). Finally, the last column lists the number of elements of

$$\mathcal{C}(\varphi) = \mathrm{Cent}(\varphi)/\mathrm{Cent}(\varphi)^0\, \mathbb{C}^\times,$$

where $\mathrm{Cent}(\varphi)$ denotes the centralizer of the image of φ, where $\mathrm{Cent}(\varphi)^0$ denotes its identity component, and where $\mathbb{C}^\times$ stands for the center of $\mathrm{GSp}(4, \mathbb{C})$.

Table A.7. L-parameters

		representation	ρ	N	$\#\mathcal{C}$
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	$\chi_1\chi_2\sigma,\ \chi_1\sigma,\ \chi_2\sigma,\ \sigma$	0	1
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$\chi^2\sigma,\ \nu^{1/2}\chi\sigma,\ \nu^{-1/2}\chi\sigma,\ \sigma$	N_1	1
	b	$\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$		0	1
III	a	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$\nu^{1/2}\chi\sigma,\ \nu^{-1/2}\chi\sigma,\ \nu^{1/2}\sigma,\ \nu^{-1/2}\sigma$	N_4	1
	b	$\chi \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}$		0	1
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$\nu^{3/2}\sigma,\ \nu^{1/2}\sigma,\ \nu^{-1/2}\sigma,\ \nu^{-3/2}\sigma$	N_5	1
	b	$L(\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$		N_4	1
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$		N_1	1
	d	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$		0	1
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	$\nu^{1/2}\sigma,\ \nu^{1/2}\xi\sigma,\ \nu^{-1/2}\xi\sigma,\ \nu^{-1/2}\sigma$	N_3	2
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$		N_1	1
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$		N_2	1
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$		0	1
VI	a	$\tau(S, \nu^{-1/2}\sigma)$	$\nu^{1/2}\sigma,\ \nu^{1/2}\sigma,\ \nu^{-1/2}\sigma,\ \nu^{-1/2}\sigma$	N_3	2
	b	$\tau(T, \nu^{-1/2}\sigma)$			
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$		N_1	1
	d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$		0	1
VII		$\chi \rtimes \pi$	$\chi\omega_\pi\varphi'_\pi,\ \varphi_\pi$	0	1
VIII	a	$\tau(S, \pi)$	$\omega_\pi\varphi'_\pi,\ \varphi_\pi$	0	2
	b	$\tau(T, \pi)$			
IX	a	$\delta(\nu\xi, \nu^{-1/2}\pi)$	$\xi\nu^{1/2}\omega_\pi\varphi'_\pi,\ \nu^{-1/2}\varphi_\pi$	N_6	1
	b	$L(\nu\xi, \nu^{-1/2}\pi)$		0	1
X		$\pi \rtimes \sigma$	$\sigma\omega_\pi,\ \sigma\varphi_\pi,\ \sigma$	0	1
XI	a	$\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	$\nu^{1/2}\sigma,\ \sigma\varphi_\pi,\ \nu^{-1/2}\sigma$	N_2	2
	b	$L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$		0	1

A.6 L- and ε-factors (degree 4)

The following Table A.8 lists the L-factors $L(s,\varphi)$ (degree 4) for the L-parameters φ of the non-supercuspidal, irreducible, admissible representations of $\mathrm{GSp}(4,F)$ (not necessarily with trivial central character). For a character χ of $F^\times$, the symbol $L(s,\chi)$ in the tables below has the usual meaning:

$$L(s,\chi) = \begin{cases} (1-\chi(\varpi)q^{-s})^{-1} & \text{if } \chi \text{ is unramified,} \\ 1 & \text{if } \chi \text{ is ramified.} \end{cases}$$

Table A.9 lists the ε-factors $\varepsilon(s,\varphi)$ for the L-parameters φ of the non-supercuspidal, irreducible, admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character. See Sect. 2.4, in particular (2.48) and (2.49), for the definitions.

Table A.8. L-factors $L(s,\varphi)$ (degree 4)

		representation	$L(s,\varphi)$
I		$\chi_1\times\chi_2\rtimes\sigma$ (irreducible)	$L(s,\chi_1\chi_2\sigma)L(s,\sigma)L(s,\chi_1\sigma)L(s,\chi_2\sigma)$
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$	$L(s,\chi^2\sigma)L(s,\sigma)L(s,\nu^{1/2}\chi\sigma)$
	b	$\chi\mathbf{1}_{\mathrm{GL}(2)}\rtimes\sigma$	$L(s,\chi^2\sigma)L(s,\sigma)L(s,\nu^{1/2}\chi\sigma)L(s,\nu^{-1/2}\chi\sigma)$
III	a	$\chi\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$L(s,\nu^{1/2}\chi\sigma)L(s,\nu^{1/2}\sigma)$
	b	$\chi\rtimes\sigma\mathbf{1}_{\mathrm{GSp}(2)}$	$L(s,\nu^{1/2}\chi\sigma)L(s,\nu^{1/2}\sigma)L(s,\nu^{-1/2}\chi\sigma)L(s,\nu^{-1/2}\sigma)$
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$L(s,\nu^{3/2}\sigma)$
	b	$L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	$L(s,\nu^{3/2}\sigma)L(s,\nu^{-1/2}\sigma)$
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma)$	$L(s,\nu^{3/2}\sigma)L(s,\nu^{1/2}\sigma)L(s,\nu^{-3/2}\sigma)$
	d	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$	$L(s,\nu^{3/2}\sigma)L(s,\nu^{1/2}\sigma)L(s,\nu^{-1/2}\sigma)L(s,\nu^{-3/2}\sigma)$
V	a	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)L(s,\nu^{1/2}\xi\sigma)$
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)L(s,\nu^{1/2}\xi\sigma)L(s,\nu^{-1/2}\sigma)$
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\xi\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)L(s,\nu^{1/2}\xi\sigma)L(s,\nu^{-1/2}\xi\sigma)$
	d	$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)L(s,\nu^{1/2}\xi\sigma)L(s,\nu^{-1/2}\sigma)L(s,\nu^{-1/2}\xi\sigma)$
VI	a	$\tau(S,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)^2$
	b	$\tau(T,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)^2$
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)^2L(s,\nu^{-1/2}\sigma)$
	d	$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)^2L(s,\nu^{-1/2}\sigma)^2$
VII		$\chi\rtimes\pi$	1
VIII	a	$\tau(S,\pi)$	1
	b	$\tau(T,\pi)$	1
IX	a	$\delta(\nu\xi,\nu^{-1/2}\pi)$	1
	b	$L(\nu\xi,\nu^{-1/2}\pi)$	1
X		$\pi\rtimes\sigma$	$L(s,\sigma)L(s,\omega_\pi\sigma)$
XI	a	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)$
	b	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)L(s,\nu^{-1/2}\sigma)$

Table A.9. ε-factors $\varepsilon(s,\varphi)$ (degree 4)

		inducing data	$a(\varphi)$	$\varepsilon(1/2,\varphi)$
I			$a(\chi_1\sigma)+a(\chi_2\sigma)+2a(\sigma)$	$\chi_1(-1)\quad(=\chi_2(-1))$
II	a	$\sigma\chi$ unr.	$2a(\sigma)+1$	$-\sigma(-1)(\sigma\chi)(\varpi)$
		$\sigma\chi$ ram.	$2a(\chi\sigma)+2a(\sigma)$	$\chi(-1)$
	b	$\sigma\chi$ unr.	$2a(\sigma)$	$\chi(-1)$
		$\sigma\chi$ ram.	$2a(\chi\sigma)+2a(\sigma)$	$\chi(-1)$
III	a	σ unr.	2	1
		σ ram.	$4a(\sigma)$	1
	b	σ unr.	0	1
		σ ram.	$4a(\sigma)$	1
IV	a	σ unr.	3	$-\sigma(\varpi)$
		σ ram.	$4a(\sigma)$	1
	b	σ unr.	2	1
		σ ram.	$4a(\sigma)$	1
	c	σ unr.	1	$-\sigma(\varpi)$
		σ ram.	$4a(\sigma)$	1
	d	σ unr.	0	1
		σ ram.	$4a(\sigma)$	1
V	a	σ, ξ unr.	2	-1
		σ unr., ξ ram.	$2a(\xi)+1$	$-\sigma(\varpi)\xi(-1)$
		σ ram., $\sigma\xi$ unr.	$2a(\sigma)+1$	$-\sigma(-1)(\sigma\xi)(\varpi)$
		σ, $\sigma\xi$ ram.	$2a(\xi\sigma)+2a(\sigma)$	$\xi(-1)$
	b	σ, ξ unr.	1	$\sigma(\varpi)$
		σ unr., ξ ram.	$2a(\xi)$	$\xi(-1)$
		σ ram., $\sigma\xi$ unr.	$2a(\sigma)+1$	$-\xi(-1)(\sigma\xi)(\varpi)$
		σ, $\sigma\xi$ ram.	$2a(\xi\sigma)+2a(\sigma)$	$\xi(-1)$

		inducing data	$a(\varphi)$	$\varepsilon(1/2,\varphi)$
V	c	σ, ξ unr.	1	$-\sigma(\varpi)$
		σ unr., ξ ram.	$2a(\xi)+1$	$-\sigma(\varpi)$
		σ ram., $\sigma\xi$ unr.	$2a(\sigma)$	$\xi(-1)$
		σ, $\sigma\xi$ ram.	$2a(\xi\sigma)+2a(\sigma)$	$\xi(-1)$
	d	σ,ξ unr.	0	1
		σ or ξ ram.	$2a(\xi\sigma)+2a(\sigma)$	$\xi(-1)$
VI	a	σ unr.	2	1
		σ ram.	$4a(\sigma)$	1
	b	σ unr.	2	1
		σ ram.	$4a(\sigma)$	1
	c	σ unr.	1	$-\sigma(\varpi)$
		σ ram.	$4a(\sigma)$	1
	d	σ unr.	0	1
		σ ram.	$4a(\sigma)$	1
VII			$2a(\pi)$	$\chi(-1)\quad(=\omega_\pi(-1))$
VIII	a		$2a(\pi)$	1
	b		$2a(\pi)$	1
IX	a		$2a(\pi)$	$\xi(-1)$
	b		$2a(\pi)$	$\xi(-1)$
X			$a(\sigma\pi)+2a(\sigma)$	$\sigma(-1)\varepsilon(1/2,\sigma\pi)$
XI	a	σ unr.	$a(\sigma\pi)+1$	$-\sigma(\varpi)\varepsilon(1/2,\sigma\pi)$
		σ ram.	$a(\sigma\pi)+2a(\sigma)$	$\sigma(-1)\varepsilon(1/2,\sigma\pi)$
	b	σ unr.	$a(\sigma\pi)$	$\varepsilon(1/2,\sigma\pi)$
		σ ram.	$a(\sigma\pi)+2a(\sigma)$	$\sigma(-1)\varepsilon(1/2,\sigma\pi)$

A.7 L- and ε-factors (degree 5)

Below we describe a homomorphism $\rho_5 : \mathrm{GSp}(4,\mathbb{C}) \to \mathrm{SO}(5,\mathbb{C})$. If $\varphi : W_F' \to \mathrm{GSp}(4,\mathbb{C})$ is an L-parameter, then the composition $\rho_5 \circ \varphi$ is a 5-dimensional representation of W_F'. The following Table A.10 lists the resulting L-factors $L(s, \rho_5 \circ \varphi)$ (degree 5) for the L-parameters of the non-supercuspidal, irreducible, admissible representations of $\mathrm{GSp}(4,F)$ (not necessarily with trivial central character). Table A.11 lists the ε-factors $\varepsilon(s, \rho_5 \circ \varphi)$ for the L-parameters of the non-supercuspidal, irreducible, admissible representations of $\mathrm{GSp}(4,F)$ (not necessarily with trivial central character). See Sect. 2.4, in particular (2.48) and (2.49), for the definitions.

GSp(4) and SO(5)

It is known that the projective group PGSp(4) is isomorphic to

$$\mathrm{SO}(5) = \{g \in \mathrm{SL}(5) : {}^tgJ_5g = J_5\}, \qquad J_5 = \begin{bmatrix} &&&&1\\ &&&1&\\ &&1&&\\ &1&&&\\ 1&&&& \end{bmatrix}, \tag{A.1}$$

as algebraic groups. Over a field k of characteristic not equal to 2 this isomorphism can be realized as follows. Let $V = k^4$ be the space of column vectors of length 4 over k, and let e_1, e_2, e_3, e_4 be the standard basis of V. The group $\mathrm{GSp}(4,k)$ acts on V by matrix multiplication from the left, and then also on the 16-dimensional tensor product space $V \otimes V$. Let us denote by ρ this action on $V \otimes V$ twisted with the inverse of the multiplier homomorphisms, i.e., $\rho(g)(v \otimes w) = \lambda(g)^{-1}(gv) \otimes (gw)$. Then ρ is trivial on the center of $\mathrm{GSp}(4,k)$, and we get an action of $\mathrm{PGSp}(4,k)$. We introduce on V the symplectic form

$$(v, v') := {}^tv \begin{bmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{bmatrix} v',$$

and on the tensor product $V \otimes V$ the symmetric bilinear form $\langle v \otimes w, v' \otimes w' \rangle = (v, v')(w, w')$. Both bilinear forms are obviously invariant under the action of $\mathrm{Sp}(4,k)$, and one checks easily that $\langle\, ,\, \rangle$ is even preserved by the action ρ of $\mathrm{GSp}(4,k)$. Now consider the embedding

$$V \wedge V \longrightarrow V \otimes V, \qquad v \wedge w \longmapsto \frac{1}{2}(v \otimes w - w \otimes v).$$

The restriction of $\langle\, ,\, \rangle$ to $V \wedge V$ is given by

$$\langle v \wedge w,\, v' \wedge w' \rangle = \frac{1}{2}\big((v, v')(w, w') - (v, w')(w, v')\big).$$

Let X be the image of the 5-dimensional subspace spanned by

$$\mathbf{x}_1 = e_1 \wedge e_2, \quad \mathbf{x}_2 = 2e_1 \wedge e_3, \quad \mathbf{x}_3 = e_1 \wedge e_4 - e_2 \wedge e_3,$$
$$\mathbf{x}_4 = e_2 \wedge e_4, \quad \mathbf{x}_5 = 2e_4 \wedge e_3.$$

One computes that the matrix of $\langle \, , \, \rangle$ with respect to this basis is J_5 as in (A.1). A computation shows that X is invariant under the action ρ of $\mathrm{GSp}(4,k)$. Since $\langle \, , \, \rangle$ is preserved by this action, we get a homomorphism $\rho_5 : \mathrm{GSp}(4,k) \to \mathrm{SO}(5,k)$. On the Siegel parabolic subgroup P the homomorphism ρ_5 is explicitly given as follows. Let

$$\begin{aligned} \mathrm{Ad}: \ &\mathrm{GL}(2,k) \longrightarrow \mathrm{SO}(3,k), \\ &\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \frac{1}{ad-bc} \begin{bmatrix} a^2 & -ab & -b^2/2 \\ -2ac & ad+bc & bd \\ -2c^2 & 2cd & d^2 \end{bmatrix}. \end{aligned}$$

Then Ad induces an isomorphism $\mathrm{PGL}(2,k) \cong \mathrm{SO}(3,k)$. On the standard Levi component of the Siegel parabolic subgroup we have

$$\rho_5(\begin{bmatrix} A & \\ & uA' \end{bmatrix}) = \begin{bmatrix} u^{-1}\det(A) & & \\ & \mathrm{Ad}(A) & \\ & & u\det(A)^{-1} \end{bmatrix}, \tag{A.2}$$

while on the unipotent radical ρ_5 is given by

$$\rho_5(\begin{bmatrix} 1 & & x & z \\ & 1 & y & x \\ & & 1 & \\ & & & 1 \end{bmatrix}) = \begin{bmatrix} 1 & 2y & 2x & -z & 2(yz-x^2) \\ & 1 & & & z \\ & & 1 & & -2x \\ & & & 1 & -2y \\ & & & & 1 \end{bmatrix}. \tag{A.3}$$

On the Levi component of the Klingen parabolic subgroup we have

$$\rho_5(\begin{bmatrix} y & & & \\ & a & b & \\ & c & d & \\ & & & y^{-1}(ad-bc) \end{bmatrix}) = \begin{bmatrix} \frac{ya}{ad-bc} & \frac{2yb}{ad-bc} & & & \\ \frac{yc}{2(ad-bc)} & \frac{yd}{ad-bc} & & & \\ & & 1 & & \\ & & & a/y & -2b/y \\ & & & -c/(2y) & d/y \end{bmatrix}, \tag{A.4}$$

and on the unipotent radical

$$\rho_5(\begin{bmatrix} 1 & x & y & z \\ & 1 & & y \\ & & 1 & -x \\ & & & 1 \end{bmatrix}) = \begin{bmatrix} 1 & & 2y & xy-z & -2y^2 \\ & 1 & -x & -x^2/2 & xy+z \\ & & 1 & x & -2y \\ & & & 1 & \\ & & & & 1 \end{bmatrix}. \tag{A.5}$$

The map ρ_5 is clearly surjective. Its kernel is the center of $\mathrm{GSp}(4,k)$, so we get an isomorphism $\mathrm{PGSp}(4,k) \cong \mathrm{SO}(5,k)$.

Table A.10. L-factors $L(s,\rho_5\circ\varphi)$ (degree 5)

		representation	$L(s,\rho_5\circ\varphi)$
I		$\chi_1\times\chi_2\rtimes\sigma$ (irreducible)	$L(s,\chi_1)L(s,\chi_1^{-1})L(s,\chi_2)L(s,\chi_2^{-1})L(s,1_{F^\times})$
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$	$L(s,\nu^{1/2}\chi)L(s,\nu^{1/2}\chi^{-1})L(s,1_{F^\times})$
	b	$\chi\mathbf{1}_{\mathrm{GL}(2)}\rtimes\sigma$	$L(s,\nu^{1/2}\chi)L(s,\nu^{-1/2}\chi)$ $L(s,\nu^{1/2}\chi^{-1})L(s,\nu^{-1/2}\chi^{-1})L(s,1_{F^\times})$
III	a	$\chi\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$L(s,\chi)L(s,\chi^{-1})L(s,\nu)$
	b	$\chi\rtimes\sigma\mathbf{1}_{\mathrm{GSp}(2)}$	$L(s,\chi)L(s,\chi^{-1})L(s,\nu)L(s,\nu^{-1})L(s,1_{F^\times})$
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$L(s,\nu^2)$
	b	$L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	$L(s,\nu^2)L(s,\nu)L(s,\nu^{-2})$
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma)$	$L(s,\nu^2)L(s,1_{F^\times})L(s,\nu^{-1})$
	d	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$	$L(s,\nu^2)L(s,\nu)L(s,1_{F^\times})L(s,\nu^{-1})L(s,\nu^{-2})$
V	a	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	$L(s,\nu\xi)L(s,\xi)L(s,1_{F^\times})$
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$L(s,\nu\xi)L(s,\xi)L(s,1_{F^\times})$
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\xi\nu^{-1/2}\sigma)$	$L(s,\nu\xi)L(s,\xi)L(s,1_{F^\times})$
	d	$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$	$L(s,\nu\xi)L(s,\nu^{-1}\xi)L(s,\xi)^2L(s,1_{F^\times})$
VI	a	$\tau(S,\nu^{-1/2}\sigma)$	$L(s,\nu)L(s,1_{F^\times})^2$
	b	$\tau(T,\nu^{-1/2}\sigma)$	
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$L(s,\nu)L(s,1_{F^\times})^2$
	d	$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$	$L(s,\nu)L(s,\nu^{-1})L(s,1_{F^\times})^3$
VII		$\chi\rtimes\pi$	$L(s,\chi)L(s,\chi^{-1})L(s,\mathrm{Ad}\circ\mu)$
VIII	a	$\tau(S,\pi)$	$L(s,1_{F^\times})^2L(s,\mathrm{Ad}\circ\mu)$
	b	$\tau(T,\pi)$	
IX	a	$\delta(\nu\xi,\nu^{-1/2}\pi)$	$L(s,\nu\xi)L(s,\mathrm{Ad}\circ\mu)L(s,\xi)^{-1}$
	b	$L(\nu\xi,\nu^{-1/2}\pi)$	$L(s,\nu\xi)L(s,\nu^{-1}\xi)L(s,\mathrm{Ad}\circ\mu)$
X		$\pi\rtimes\sigma$	$L(s,\mu)L(s,\det(\mu)^{-1}\mu)L(s,1_{F^\times})$
XI	a	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\mu)L(s,1_{F^\times})$
	b	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\mu)L(s,\nu^{-1/2}\mu)L(s,1_{F^\times})$

Table A.11. ε-factors $\varepsilon(s,\rho_5\circ\varphi)$ (degree 5)

		representation	$a(\rho_5\circ\varphi)$	$\varepsilon(1/2,\rho_5\circ\varphi)$
I		$\chi_1\times\chi_2\rtimes\sigma$ (irreducible)	$2a(\chi_1)+2a(\chi_2)$	$\chi_1(-1)\chi_2(-1)$
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$	χ unr. : 2, χ ram. : $4a(\chi)$	1
	b	$\chi\mathbf{1}_{\mathrm{GL}(2)}\rtimes\sigma$	$4a(\chi)$	1
III	a	$\chi\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$2a(\chi)+2$	$\chi(-1)$
	b	$\chi\rtimes\sigma\mathbf{1}_{\mathrm{GSp}(2)}$	$2a(\chi)$	$\chi(-1)$
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	4	1
	b	$L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	2	1
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma)$	2	1
	d	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$	0	1
V	a	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	ξ unr. : 2, ξ ram. : $4a(\xi)$	1
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	ξ unr. : 2, ξ ram. : $4a(\xi)$	1
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\xi\nu^{-1/2}\sigma)$	ξ unr. : 2, ξ ram. : $4a(\xi)$	1
	d	$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$	$4a(\xi)$	1
VI	a	$\tau(S,\nu^{-1/2}\sigma)$	2	1
	b	$\tau(T,\nu^{-1/2}\sigma)$		
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	2	1
	d	$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$	0	1
VII		$\chi\rtimes\pi$	$2a(\chi)+a(\mathrm{Ad}\circ\mu)$	$\chi(-1)\varepsilon(\frac12,\mathrm{Ad}\circ\mu)$
VIII	a	$\tau(S,\pi)$	$a(\mathrm{Ad}\circ\mu)$	$\varepsilon(\frac12,\mathrm{Ad}\circ\mu)$
	b	$\tau(T,\pi)$		
IX	a	$\delta(\nu\xi,\nu^{-1/2}\pi)$	ξ unr. : $a(\mathrm{Ad}\circ\mu)+2$	$\xi(-1)\varepsilon(\frac12,\mathrm{Ad}\circ\mu)$
			ξ ram. : $2a(\xi)+a(\mathrm{Ad}\circ\mu)$	
	b	$L(\nu\xi,\nu^{-1/2}\pi)$	$2a(\xi)+a(\mathrm{Ad}\circ\mu)$	$\xi(-1)\varepsilon(\frac12,\mathrm{Ad}\circ\mu)$
X		$\pi\rtimes\sigma$	$2a(\mu)$	$\det(\mu)(-1)$
XI	a	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$2a(\mu)$	1
	b	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$2a(\mu)$	1

A.8 Paramodular Dimensions and Atkin–Lehner Eigenvalues

Table A.12 below contains the dimensions of the spaces $V(m)$ of $\mathrm{K}(\mathfrak{p}^m)$ invariant vectors for each irreducible, admissible, non-supercuspidal representation V of $\mathrm{GSp}(4,F)$ with trivial central character. The "N" column gives the minimal paramodular level of the representation, provided the representation is paramodular; a "—" indicates the representation is not paramodular. The dimensions listed in the "$\dim V(m)$" column hold for any $m \geq N$. If $m < N$, then the dimension of $V(m)$ is zero. The last column of the table gives, for the paramodular representations, the eigenvalue ε of the Atkin–Lehner involution u_N on the local newform (the essentially unique paramodular vector at level $\mathfrak{p}^N$).

See Theorem 5.6.1 and Theorem 5.7.2 for proofs of the statements made in Table A.12.

Iwahori-spherical representations

The dimension information given in Table A.13 below is already contained in Table A.12. Listed are the Iwahori-spherical representations of $\mathrm{GSp}(4,F)$ with trivial central character; thus, all the characters in the inducing data are assumed to be unramified. The column named "$V(k)$" contains the dimension of the space $V(k)$ of $\mathrm{K}(\mathfrak{p}^k)$ invariant vectors. For $k = 0,\ldots,3$ we indicated under the dimension the eigenvalues of the Atkin–Lehner involution u_k. These eigenvalues are correct if one assumes that

- in group II, where the central character is $\chi^2\sigma^2$, the character $\chi\sigma$ is trivial.
- in groups IV, V and VI, where the central character is σ^2, the character σ itself is trivial.

If these assumptions are not met, then one has to interchange the plus and minus signs in the $V(1)$ and the $V(3)$ column.

The "a" column gives the conductor of the local parameter attached to the representation; see Sect. 2.4. Except for VIb, which shares an L-packet with VIa, this number coincides with the minimal paramodular level. Finally, the column "$\varepsilon(1/2,\varphi)$" gives the value of the ε-factor at $s = 1/2$ of the L-parameter of each representation. In each case, this sign coincides with the eigenvalue of the Atkin–Lehner involution on the newform.

It is not hard to obtain the information contained in Table A.13 by direct computations. See Theorem 3.2.9 for details.

Table A.12. Paramodular dimensions and Atkin–Lehner eigenvalues

		inducing data	N	$\dim V(m)$	ε
I			$a(\chi_1\sigma)+a(\chi_2\sigma)+2a(\sigma)$	$[\frac{(m-N+2)^2}{4}]$	$\chi_1(-1) \quad (=\chi_2(-1))$
II	a	$\sigma\chi$ unr.	$2a(\sigma)+1$	$[\frac{(m-N+2)^2}{4}]$	$-\sigma(-1)(\sigma\chi)(\varpi)$
		$\sigma\chi$ ram.	$2a(\chi\sigma)+2a(\sigma)$		$\chi(-1)$
	b	$\sigma\chi$ unr.	$2a(\sigma)$	$[\frac{m-N+2}{2}]$	$\chi(-1)$
		$\sigma\chi$ ram.	—	0	—
III	a	σ unr.	2	$[\frac{(m-N+2)^2}{4}]$	1
		σ ram.	$4a(\sigma)$		
	b	σ unr.	0	$m+1$	1
		σ ram.	—	0	—
IV	a	σ unr.	3	$[\frac{(m-N+2)^2}{4}]$	$-\sigma(\varpi)$
		σ ram.	$4a(\sigma)$		1
	b	σ unr.	2	$[\frac{m}{2}]$	1
		σ ram.	—	0	—
	c	σ unr.	1	m	$-\sigma(\varpi)$
		σ ram.	—	0	—
	d	σ unr.	0	1	1
		σ ram.	—	0	—
V	a	σ, ξ unr.	2	$[\frac{(m-N+2)^2}{4}]$	-1
		σ unr., ξ ram.	$2a(\xi)+1$		$-\sigma(\varpi)\xi(-1)$
		σ ram., $\sigma\xi$ unr.	$2a(\sigma)+1$		$-\sigma(-1)(\sigma\xi)(\varpi)$
		σ, $\sigma\xi$ ram.	$2a(\xi\sigma)+2a(\sigma)$		$\xi(-1)$
	b	σ, ξ unr.	1	$[\frac{m+1}{2}]$	$\sigma(\varpi)$
		σ unr., ξ ram.	$2a(\xi)$	$[\frac{m-N+2}{2}]$	$\xi(-1)$
		σ ram., $\sigma\xi$ unr.	—	0	—
		σ, $\sigma\xi$ ram.	—	0	—

		inducing data	N	$\dim V(m)$	ε
V	c	σ, ξ unr.	1	$[\frac{m+1}{2}]$	$-\sigma(\varpi)$
		σ unr., ξ ram.	—	0	—
		σ ram., $\sigma\xi$ unr.	$2a(\sigma)$	$[\frac{m-N+2}{2}]$	$\xi(-1)$
		σ, $\sigma\xi$ ram.	—	0	—
	d	σ, ξ unr.	0	$\frac{1+(-1)^m}{2}$	1
		σ or ξ ram.	—	0	—
VI	a	σ unr.	2	$[\frac{(m-N+2)^2}{4}]$	1
		σ ram.	$4a(\sigma)$		
	b	σ unr.	—	0	—
		σ ram.	—	0	—
	c	σ unr.	1	$[\frac{m+1}{2}]$	$-\sigma(\varpi)$
		σ ram.	—	0	—
	d	σ unr.	0	$[\frac{m+2}{2}]$	1
		σ ram.	—	0	—
VII			$2a(\pi)$	$[\frac{(m-N+2)^2}{4}]$	$\chi(-1) \quad (= \omega_\pi(-1))$
VIII	a		$2a(\pi)$	$[\frac{(m-N+2)^2}{4}]$	1
	b		—	0	—
IX	a		$2a(\pi)$	$[\frac{(m-N+2)^2}{4}]$	$\xi(-1)$
	b		—	0	—
X			$a(\sigma\pi) + 2a(\sigma)$	$[\frac{(m-N+2)^2}{4}]$	$\sigma(-1)\varepsilon(1/2, \sigma\pi)$
XI	a	σ unr.	$a(\sigma\pi) + 1$	$[\frac{(m-N+2)^2}{4}]$	$-\sigma(\varpi)\varepsilon(1/2, \sigma\pi)$
		σ ram.	$a(\sigma\pi) + 2a(\sigma)$		$\sigma(-1)\varepsilon(1/2, \sigma\pi)$
	b	σ unr.	$a(\sigma\pi)$	$[\frac{m-N+2}{2}]$	$\varepsilon(1/2, \sigma\pi)$
		σ ram.	—	0	—

Table A.13. Iwahori-spherical representations of $\mathrm{GSp}(4, F)$

		representation	a	$\varepsilon(1/2, \varphi)$	$V(0)$	$V(1)$	$V(2)$	$V(3)$	$V(n)$
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	0	1	$\mathbf{1}$ $+$	2 $+-$	4 $+++-$	6 $+++$ $---$	$\left[\frac{(n+2)^2}{4}\right]$
II	a	$\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	1	$-(\sigma\chi)(\varpi)$	0	$\mathbf{1}$ $-$	2 $+-$	4 $++--$	$\left[\frac{(n+1)^2}{4}\right]$
	b	$\chi \mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$	0	1	$\mathbf{1}$ $+$	1 $+$	2 $++$	2 $++$	$\left[\frac{n+2}{2}\right]$
III	a	$\chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(2)}$	2	1	0	0	$\mathbf{1}$ $+$	2 $+-$	$\left[\frac{n^2}{4}\right]$
	b	$\chi \rtimes \sigma \mathbf{1}_{\mathrm{GSp}(2)}$	0	1	$\mathbf{1}$ $+$	2 $+-$	3 $++-$	4 $++--$	$n+1$
IV	a	$\sigma \mathrm{St}_{\mathrm{GSp}(4)}$	3	$-\sigma(\varpi)$	0	0	0	$\mathbf{1}$ $-$	$\left[\frac{(n-1)^2}{4}\right]$
	b	$L(\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GSp}(2)})$	2	1	0	0	$\mathbf{1}$ $+$	1 $+$	$\left[\frac{n}{2}\right]$
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$	1	$-\sigma(\varpi)$	0	$\mathbf{1}$ $-$	2 $+-$	3 $+--$	n
	d	$\sigma \mathbf{1}_{\mathrm{GSp}(4)}$	0	1	$\mathbf{1}$ $+$	1 $+$	1 $+$	1 $+$	1
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	2	-1	0	0	$\mathbf{1}$ $-$	2 $+-$	$\left[\frac{n^2}{4}\right]$
	b	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	1	$\sigma(\varpi)$	0	$\mathbf{1}$ $+$	1 $+$	2 $++$	$\left[\frac{n+1}{2}\right]$
	c	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$	1	$-\sigma(\varpi)$	0	$\mathbf{1}$ $-$	1 $+$	2 $--$	$\left[\frac{n+1}{2}\right]$
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	0	1	$\mathbf{1}$ $+$	0	1 $+$	0	$\frac{1+(-1)^n}{2}$
VI	a	$\tau(S, \nu^{-1/2}\sigma)$	2	1	0	0	$\mathbf{1}$ $+$	2 $+-$	$\left[\frac{n^2}{4}\right]$
	b	$\tau(T, \nu^{-1/2}\sigma)$	2	1	0	0	0	0	0
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	1	$-\sigma(\varpi)$	0	$\mathbf{1}$ $-$	1 $-$	2 $--$	$\left[\frac{n+1}{2}\right]$
	d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$	0	1	$\mathbf{1}$ $+$	1 $+$	2 $++$	2 $++$	$\left[\frac{n+2}{2}\right]$

A.9 Hecke Eigenvalues

Table A.14 lists the Hecke eigenvalues of the Hecke operators $T_{0,1}$ and $T_{1,0}$ defined in Sect. 6.1 on the newform of an irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. The "N" column in Table A.14 gives the minimal paramodular level. Representations with no paramodular vectors have been marked by a "—" in the N, λ and μ columns. Otherwise λ denotes the eigenvalue of $T_{0,1}$ on the local newform, and μ denotes the eigenvalue of $T_{1,0}$. See Theorem 7.5.2 for how these eigenvalues are computed.

For typesetting reasons, some of the eigenvalues are given as (A), (B), (C) below.

(A) $q^2\big(\chi_1(\varpi)+\chi_2(\varpi)+\chi_1(\varpi)^{-1}+\chi_2(\varpi)^{-1}+1-q^{-2}\big)$
(B) $q^{3/2}(q+1)(\chi(\varpi)+\chi^{-1}(\varpi))+q^2-1$
(C) $q^2(\chi(\varpi)+\chi^{-1}(\varpi)+q+1)+q-1$

Table A.14. Hecke eigenvalues

	inducing data	N	λ	μ
I	σ,χ_1,χ_2 unr.	0	$q^{3/2}\sigma(\varpi)\big(1+\chi_1(\varpi)$ $+\chi_2(\varpi)+\chi_1(\varpi)\chi_2(\varpi)\big)$	(A)
	σ unr., χ_1,χ_2 ram.	$a(\chi_1)+a(\chi_2)$	$q^{3/2}(\sigma(\varpi)+\sigma(\varpi)^{-1})$	0
	σ ram., $\sigma\chi_i$ unr.	$2a(\sigma)$	$q^{3/2}((\chi_1\sigma)(\varpi)+(\chi_2\sigma)(\varpi))$	0
	σ ram., $\sigma\chi_i$ ram.	$2a(\chi_1\sigma)+2a(\sigma)$	0	$-q^2$
IIa	σ,χ unr.	1	$q^{3/2}(\sigma(\varpi)+\sigma(\varpi)^{-1})$ $+(q+1)(\sigma\chi)(\varpi)$	$q^{3/2}(\chi(\varpi)$ $+\chi(\varpi)^{-1})$
	σ,χ ram., $\chi\sigma$ unr.	$2a(\sigma)+1$	$q(\chi\sigma)(\varpi)$	$-q^2$
	σ unr., $\chi\sigma$ ram.	$2a(\chi)$	$q^{3/2}(\sigma(\varpi)+\sigma(\varpi)^{-1})$	0
	σ ram., $\chi\sigma$ ram.	$2a(\sigma)+2a(\chi\sigma)$	0	$-q^2$
IIb	σ,χ unr.	0	$q^{3/2}(\sigma(\varpi)+\sigma(\varpi)^{-1})$ $+q(q+1)(\sigma\chi)(\varpi)$	(B)
	σ,χ ram., $\chi\sigma$ unr.	$2a(\sigma)$	$q(q+1)(\sigma\chi)(\varpi)$	0
	$\chi\sigma$ ram.	—	—	—

	inducing data	N	λ	μ
IIIa	σ unr.	2	$q(\sigma(\varpi)+\sigma(\varpi)^{-1})$	$-q(q-1)$
	σ ram.	$4a(\sigma)$	0	$-q^2$
IIIb	σ unr.	0	$q(q+1)\sigma(\varpi)(1+\chi(\varpi))$	(C)
	σ ram.	—	—	—
IVa	σ unr.	3	$\sigma(\varpi)$	$-q^2$
	σ ram.	$4a(\sigma)$	0	$-q^2$
IVb	σ unr.	2	$\sigma(\varpi)(1+q^2)$	$-q(q-1)$
	σ ram.	—	—	—
IVc	σ unr.	1	$\sigma(\varpi)(q^3+q+2)$	q^3+1
	σ ram.	—	—	—
IVd	σ unr.	0	$\sigma(\varpi)(q^3+q^2+q+1)$	$q(q^3+q^2+q+1)$
	σ ram.	—	—	—
Va	ξ, σ unr.	2	0	$-q^2-q$
	σ unr., ξ ram.	$2a(\xi)+1$	$\sigma(\varpi)q$	$-q^2$
	σ ram., $\sigma\xi$ unr.	$2a(\sigma)+1$	$-\sigma(\varpi)q$	$-q^2$
	$\sigma, \sigma\xi$ ram.	$2a(\xi\sigma)+2a(\sigma)$	0	$-q^2$
Vb	ξ, σ unr.	1	$\sigma(\varpi)(q^2-1)$	$-q^2-q$
	σ unr., ξ ram.	$2a(\xi)$	$\sigma(\varpi)q(q+1)$	0
	σ ram., $\sigma\xi$ unr.	—	—	—
	$\sigma, \sigma\xi$ ram.	—	—	—
Vc	ξ, σ unr.	1	$-\sigma(\varpi)(q^2-1)$	$-q^2-q$
	σ unr., ξ ram.	—	—	—
	σ ram., $\sigma\xi$ unr.	$2a(\sigma)$	$-\sigma(\varpi)q(q+1)$	0
	$\sigma, \sigma\xi$ ram.	—	—	—
Vd	ξ, σ unr.	0	0	$-(q^3+q^2+q+1)$
	ξ or σ ram.	—	—	—

	inducing data	N	λ	μ
VIa	σ unr.	2	$2q\sigma(\varpi)$	$-q(q-1)$
	σ ram.	$4a(\sigma)$	0	$-q^2$
VIb	σ unr.	—	—	—
	σ ram.	—	—	—
VIc	σ unr.	1	$\sigma(\varpi)(q+1)^2$	$q(q+1)$
	σ ram.	—	—	—
VId	σ unr.	0	$2q(q+1)\sigma(\varpi)$	$(q+1)(q^2+2q-1)$
	σ ram.	—	—	—
VII		$2a(\pi)$	0	$-q^2$
VIIIa		$2a(\pi)$	0	$-q^2$
VIIIb		—	—	—
IXa		$2a(\pi)+1$	0	$-q^2$
IXb		—	—	—
X	σ unr.	$a(\sigma\pi)$	$q^{3/2}(\sigma(\varpi)+\sigma(\varpi)^{-1})$	0
	σ ram.	$a(\sigma\pi)+2a(\sigma)$	0	$-q^2$
XIa	σ unr.	$a(\sigma\pi)+1$	$q\sigma(\varpi)$	$-q^2$
	σ ram.	$a(\sigma\pi)+2a(\sigma)$	0	$-q^2$
XIb	σ unr.	$a(\sigma\pi)$	$q(q+1)\sigma(\varpi)$	0
	σ ram.	—	—	—
super-cuspidal	generic	≥ 2	0	$-q^2$
	non-generic	—	—	—

A.10 Parahori-invariant Vectors

For the convenience of the reader, in this final section we list the dimensions of the subspaces of vectors fixed by $\mathrm{GSp}(4,\mathfrak{o})$, $\mathrm{K}(\mathfrak{p})$, $\mathrm{Kl}(\mathfrak{p})$, $\mathrm{Si}(\mathfrak{p})$ and I in Iwahori-spherical representations. This table appears on p. 269 of [Sch2]. The Atkin–Lehner eigenvalues for $\pi(u_1)$ listed in this table are correct if one assumes that: in group II, where the central character is $\chi^2\sigma^2$, the character $\chi\sigma$ is trivial; in groups IV, V and VI, where the central character is σ^2, the character σ itself is trivial. If these assumptions are not met, then one has to interchange all plus and minus signs.

Table A.15. Iwahori-spherical representations: Dimensions of spaces of parahori-invariant vectors

		representation	a	$\varepsilon(1/2,\varphi)$	$\mathrm{GSp}(4,\mathfrak{o})$	$\mathrm{K}(\mathfrak{p})$	$\mathrm{Kl}(\mathfrak{p})$	$\mathrm{Si}(\mathfrak{p})$	I
I		$\chi_1\times\chi_2\rtimes\sigma$ (irreducible)	0	1	1	$\underset{+-}{2}$	4	$\underset{\substack{++\\--}}{4}$	$\underset{\substack{++++\\----}}{8}$
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$	1	$-(\sigma\chi)(\varpi)$	0	$\underset{-}{1}$	2	$\underset{-}{1}$	$\underset{+---}{4}$
II	b	$\chi\mathbf{1}_{\mathrm{GL}(2)}\rtimes\sigma$	0	1	1	$\underset{+}{1}$	2	$\underset{++-}{3}$	$\underset{+++-}{4}$
III	a	$\chi\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	2	1	0	0	1	$\underset{+-}{2}$	$\underset{++--}{4}$
III	b	$\chi\rtimes\sigma\mathbf{1}_{\mathrm{GSp}(2)}$	0	1	1	$\underset{+-}{2}$	3	$\underset{+-}{2}$	$\underset{++--}{4}$
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	3	$-\sigma(\varpi)$	0	0	0	0	$\underset{-}{1}$
IV	b	$L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	2	1	0	0	1	$\underset{+-}{2}$	$\underset{++-}{3}$
IV	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma)$	1	$-\sigma(\varpi)$	0	$\underset{-}{1}$	2	$\underset{-}{1}$	$\underset{+--}{3}$
IV	d	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$	0	1	1	$\underset{+}{1}$	1	$\underset{+}{1}$	$\underset{+}{1}$
V	a	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	2	-1	0	0	1	0	$\underset{+-}{2}$
V	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	1	$\sigma(\varpi)$	0	$\underset{+}{1}$	1	$\underset{+}{1}$	$\underset{++}{2}$
V	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\xi\nu^{-1/2}\sigma)$	1	$-\sigma(\varpi)$	0	$\underset{-}{1}$	1	$\underset{-}{1}$	$\underset{--}{2}$
V	d	$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$	0	1	1	0	1	$\underset{+-}{2}$	$\underset{+-}{2}$
VI	a	$\tau(S,\nu^{-1/2}\sigma)$	2	1	0	0	1	$\underset{-}{1}$	$\underset{+--}{3}$
VI	b	$\tau(T,\nu^{-1/2}\sigma)$	2	1	0	0	0	$\underset{+}{1}$	$\underset{+}{1}$
VI	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	1	$-\sigma(\varpi)$	0	$\underset{-}{1}$	1	0	$\underset{-}{1}$
VI	d	$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$	0	1	1	$\underset{+}{1}$	2	$\underset{+-}{2}$	$\underset{++-}{3}$

Frequently Used Notations

F	non-archimedean local field of characteristic zero	27
$\mathfrak{o}$	ring of integers of F	27
$\mathfrak{p}$	maximal ideal of $\mathfrak{o}$	27
ϖ	generator of $\mathfrak{p}$	27
v	normalized valuation	27
ν	normalized absolute value (same as $\lvert\,\rvert$)	27
q	number of elements of $\mathfrak{o}/\mathfrak{p}$	27
ψ	character of F, trivial on $\mathfrak{o}$, non-trivial on $\mathfrak{p}^{-1}$	27
$a(\chi)$	conductor of the character χ	27
J	standard symplectic form	27
GSp(4)	group of similitudes of a 4-dimensional symplectic space	27
λ	multiplier homomorphism on GSp(4)	27
Sp(4)	kernel of λ	27
Z	center of GSp(4)	28
B	Borel subgroup of GSp(4) (upper triangular matrices)	28
U	unipotent radical of B	28
P	Siegel parabolic subgroup of GSp(4)	29
Q	Klingen parabolic subgroup of GSp(4)	29
A'	conjugate-inverse-transpose of the 2×2 matrix A	29
G^J	Jacobi subgroup of GSp(4)	30
Z^J	center of G^J	30
W	the eight-element Weyl group of GSp(4)	30
s_1, s_2	Weyl group elements	30
$\mathrm{K}(\mathfrak{p}^n)$	paramodular group of level $\mathfrak{p}^n$	31
u_n	Atkin–Lehner element	31
t_n	special element in $\mathrm{K}(\mathfrak{p}^n)$	31
$\mathrm{Kl}(\mathfrak{p}^n)$	Klingen congruence subgroup of level $\mathfrak{p}^n$	32
$\mathrm{Si}(\mathfrak{p}^n)$	Siegel congruence subgroup of level $\mathfrak{p}^n$	32
$\pi^\vee$	contragredient of the representation π	33
ω_π	central character of the representation π	33
δ	modulus character	33
Ind_P^G	normalized induction	33
R_U	normalized Jacquet module	33
ψ_{c_1,c_2}	character of $U(F)$	34
$\mathcal{W}(\pi, \psi_{c_1,c_2})$	Whittaker model of π with respect to ψ_{c_1,c_2}	34
$\chi_1 \times \chi_2 \rtimes \sigma$, $\pi \rtimes \sigma$, $\chi \rtimes \pi$	parabolic induction	35
$\tau\pi$	twist of π by the character τ	36
$\mathrm{St}_{\mathrm{GL}(2)}$	Steinberg representation of $\mathrm{GL}(2, F)$	37
$\mathbf{1}_{\mathrm{GL}(2)}$	trivial representation of $\mathrm{GL}(2, F)$	37
$e(\chi)$	exponent of the character χ	38
$X^*(T)$	algebraic homomorphisms $T \to \mathbb{G}_m$	41
$X_*(T)$	algebraic homomorphisms $\mathbb{G}_m \to T$	41

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