Arithmetic Geometry
and Automorphic Forms
ALM 19, pp. 505–530

On the Number of Local Newforms in a Metaplectic Representation*

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Abstract

The nonarchimedean local analogues of modular forms of half-integral weight with level and character are certain vectors in irreducible, admissible, genuine representations of the metaplectic group over a nonarchimedean local field of characteristic zero. Two natural level raising operators act on such vectors, leading to the concepts of oldforms and newforms. We prove that the number of newforms for a given representation and character is finite and equal to the number of square classes with respect to which the representation admits a Whittaker model.

2000 Mathematics Subject Classification: 11F37, 11F70.
Keywords and Phrases: Modular forms of half-integral weight, metaplectic group, oldforms, newforms.

Let F be a nonarchimedean local field of characteristic zero with Hilbert symbol $(\cdot\,,\cdot)$ and ring of integers $\mathfrak{o}$, let $\mathfrak{p} \subset \mathfrak{o}$ be the maximal ideal of $\mathfrak{o}$, let ϖ be a generator for $\mathfrak{p}$, and fix a character ψ of F with conductor $\mathfrak{o}$. Let $\widetilde{\mathrm{SL}}(2,F)$ be the two-fold cover of $\mathrm{SL}(2,F)$, as defined below. For m and a in $F^\times$ let $\gamma_m(a)$ be the Weil index of ax^2 with respect to ψ^m, and define $\delta_m(a) = (-1,a)\gamma_m(a)\gamma_m(1)^{-1}$. Let (τ, V) be an irreducible, admissible, genuine representation of $\widetilde{\mathrm{SL}}(2,F)$. The center of $\widetilde{\mathrm{SL}}(2,F)$ consists of the four elements

$$\left(\begin{bmatrix}\varepsilon & \\ & \varepsilon\end{bmatrix}, \varepsilon'\right)$$

where $\varepsilon, \varepsilon' = \pm 1$. Consider the operator

$$\tau\left(\begin{bmatrix}-1 & \\ & -1\end{bmatrix}, 1\right).$$

By Schur's Lemma, this operator acts by a scalar, and the square of this scalar is the Hilbert symbol $(-1,-1)$. Also, $\delta_1(-1)^2 = (-1,-1)$. It follows that there

*This work is a contribution in honor of Stephens. Kudla. Kudla's work spans the theory of modular forms, representation theory, and arithmetic geometry. Though it is difficult to emulate, we have been inspired by his dedication to creating theories and making arithmetic applications.

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exists $\varepsilon(\tau,\psi)=\pm 1$ such that

$$\tau\left(\begin{bmatrix}-1 & \\ & -1\end{bmatrix},1\right)=\varepsilon(\tau,\psi)\delta_1(-1).$$

We let $F_\psi(\tau)$ be the set of a in $F^\times$ such that τ admits a Whittaker model with respect to ψ^a. The group $F^{\times 2}$ acts on $F_\psi(\tau)$. Let χ be a character of $\mathfrak{o}^\times$. For n an integer, we let $V_\psi(\tau,n,\chi)$ be the subspace of vectors v in V such that

$$\tau\left(\begin{bmatrix}1 & b\\ & 1\end{bmatrix},1\right)v=v \qquad \text{for all } b \text{ in } \mathfrak{o}, \tag{1}$$

$$\tau\left(\begin{bmatrix}a & \\ & a^{-1}\end{bmatrix},1\right)v=\delta_1(a)\chi(a)v \qquad \text{for all } a \text{ in } \mathfrak{o}^\times, \tag{2}$$

$$\tau\left(\begin{bmatrix}1 & \\ c & 1\end{bmatrix},1\right)v=v \qquad \text{for all } c \text{ in } \mathfrak{p}^n. \tag{3}$$

We refer to the vectors in the spaces $V_\psi(\tau,n,\chi)$ as *metaplectic vectors*, and say that the vectors in $V_\psi(\tau,n,\chi)$ have level $\mathfrak{p}^n$. Any metaplectic vector of level $\mathfrak{p}^n$ is a metaplectic vector of level $\mathfrak{p}^{n+1}$. That is, the inclusion of $V_\psi(\tau,n,\chi)$ in $V_\psi(\tau,n+1,\chi)$ is a level raising operator. There is another natural level raising operator that takes metaplectic vectors of level $\mathfrak{p}^n$ to metaplectic vectors of level $\mathfrak{p}^{n+2}$. Define

$$\alpha_2 : V_\psi(\tau,n,\chi)\longrightarrow V_\psi(\tau,n+2,\chi)$$

by

$$\alpha_2 v=\tau\left(\begin{bmatrix}\varpi^{-1} & \\ & \varpi\end{bmatrix},1\right)v. \tag{4}$$

We note that the definition of α_2 does not depend on n. We define the subspace $V_\psi(\tau,n,\chi)_{\mathrm{old}}$ of *oldforms* in $V_\psi(\tau,n,\chi)$ as the subspace spanned by the images of vectors of lower level, i.e., as the subspace generated by $V_\psi(\tau,n-1,\chi)$ and $\alpha_2 V_\psi(\tau,n-2,\chi)$. We define

$$V_\psi(\tau,n,\chi)_{\mathrm{new}}=V_\psi(\tau,n,\chi)/V_\psi(\tau,n,\chi)_{\mathrm{old}}.$$

In this paper we study the dimensions of the spaces $V_\psi(\tau,n,\chi)_{\mathrm{new}}$ and prove the following theorem.

Main Theorem. *Let (τ,V) be an irreducible, admissible, genuine representation of $\widetilde{\mathrm{SL}}(2,F)$, and let χ be a character of $\mathfrak{o}^\times$. If $\chi(-1)\neq\varepsilon(\tau,\psi)$, then $V_\psi(\tau,n,\chi)$ is zero for all n. Assume that $\chi(-1)=\varepsilon(\tau,\psi)$. The sum $\sum_n \dim V_\psi(\tau,n,\chi)_{\mathrm{new}}$ is finite and*

$$\sum_n \dim V_\psi(\tau,n,\chi)_{\mathrm{new}}=\#F_\psi(\tau)/F^{\times 2}. \tag{5}$$

This result has a GL(2) analogue. Let (π,V) be a generic, irreducible, admissible representation of $\mathrm{GL}(2,F)$. For n a non-negative integer, let $V(\pi,n)$ be the subspace of vectors v in V that are stabilized by the subgroup of elements

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

of $\mathrm{GL}(2,\mathfrak{o})$ such that $c \equiv 0 \bmod \mathfrak{p}^n$ and $d \equiv 1 \bmod \mathfrak{p}^n$. In this setting, the inclusion of $V(\pi,n)$ in $V(\pi,n+1)$ is again a level raising operator, and there is another level raising operator from $V(\pi,n,\chi)$ to $V(\pi,n+1,\chi)$ that sends v to

$$\pi(\begin{bmatrix} 1 & \\ & \varpi \end{bmatrix})v.$$

In this GL(2) case, the sum analogous to the sum in the main theorem has value 1, so that there is an essentially unique newform. This GL(2) result is directly analogous to the result of the main theorem because π admits a Whittaker model with respect to ψ^a for all a in $F^\times$.

The result presented here builds on the works of Waldspurger, but also introduces some new ideas. As far as we know, the spaces $V_\psi(\tau,n,\chi)$ for $F=\mathbb{Q}_p$ were first considered in [W2]; some subsequent works that also used these spaces are [BM] and [M]. For the case $F=\mathbb{Q}_p$ it should be possible to deduce the main theorem from results in Waldspurger's work. However, our approach is more abstract than the approach in [W2]. To prove the main theorem we introduce the concept of *primitive vectors*. Primitive vectors comprise the kernel of a certain projection μ on the union $V_\psi(\tau,\infty,\chi)$ of the spaces $V_\psi(\tau,n,\chi)$, and the dimension of the subspace of primitive vectors is equal to the sum in the main theorem. Proving the main theorem is thus reduced to computing the dimension of the space of primitive vectors. This is achieved by using the Kirillov-type model for τ. This method can be deployed in other settings. For example, an analogous argument proves the above mentioned analogue for GL(2), as we explain at the end of this paper.

Our interest in the spaces $V_\psi(\tau,n,\chi)$ stems from our project to understand the subspaces $W_0(n)$ of vectors in irreducible, admissible representations (π,W) of $\mathrm{GSp}(4,F)$ with trivial central character that are stabilized by the groups $\Gamma_0(\mathfrak{p}^n)$ of elements

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

of $\mathrm{GSp}(4,\mathfrak{o})$ with $C \equiv 0 \bmod \mathfrak{p}^n$ (we use the notation from [RS] for GSp(4)). We refer to the elements of $W_0(n)$ as *Siegel vectors*. If (π,W) is a Saito-Kurokawa representation of $\mathrm{GSp}(4,F)$, then the quotient $W_{Z^J,\psi^{-1}}$ of W by the subspace spanned by the vectors $\pi(g)w-\psi(-x)w$ for w in W and g of the form

$$g=\begin{bmatrix} 1 & & & x \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

for x in F is isomorphic to $\pi_{SW}^{-1}\otimes\tau$, as a representation of the Jacobi group G^J of $\mathrm{GSp}(4,F)$, for some irreducible, admissible, genuine representation (τ,V) of $\widetilde{\mathrm{SL}}(2,F)$. Here, G^J consists of the elements of $\mathrm{GSp}(4,F)$ of the form

$$\begin{bmatrix} 1 & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & 1 \end{bmatrix},$$

and π_{SW}^{-1} is the Schrödinger-Weil representation of G^J (see [BS] for the definition of π_{SW}^{-1}). Note that the subgroup of g as above is the center of G^J. It turns out that there is a natural connection between Siegel vectors and metaplectic vectors in τ. If the residual characteristic of F is even, then one must additionally consider certain other subspaces $V_{\psi,j}(\tau,n,1)$ of V, where j varies between 0 and val(2); the space $V_\psi(\tau,n,1)$ from above is $V_{\psi,\mathrm{val}(2)}(\tau,n,1)$. In particular, in the case of even residual characteristic the consideration of unramified Saito-Kurokawa representations leads to the definition of the Kohnen plus space in $V_\psi(\tau,2\mathrm{val}(2),1)$. We plan to return to these topics in subsequent publications.

Acknowledgements. We would like to thank the referee for some useful comments.

1 Background

In this section we gather some necessary basic definitions and results about the underlying field, the metaplectic group $\widetilde{\mathrm{SL}}(2,F)$, and representations of $\widetilde{\mathrm{SL}}(2,F)$. Throughout this paper, F is a nonarchimedean local field of characteristic zero with ring of integers $\mathfrak{o}$, maximal ideal $\mathfrak{p}$ in $\mathfrak{o}$, and Hilbert symbol $(\,\cdot\,,\,\cdot\,)$. Let ϖ be a generator of $\mathfrak{p}$, and let q be the order of $\mathfrak{o}/\mathfrak{p}$. We will use the absolute value $|\cdot|$ on F such that $|\varpi|=1/q$. Fix a character ψ of F with conductor $\mathfrak{o}$, i.e., $\psi(\mathfrak{o})=1$ but $\psi(\mathfrak{p}^{-1})\neq 1$. We will always use the Haar measure on F that assigns $\mathfrak{o}$ volume 1. If $n=0$ we take $1+\mathfrak{p}^n$ to be $\mathfrak{o}^\times$.

Number theory

Lemma 1.1. *Assume that F has even residual characteristic.*

(i) *The map $\mathfrak{o}/\mathfrak{p}\to\mathfrak{o}/\mathfrak{p}$ sending x to x^2+x is a group homomorphism and is 2-to-1.*
(ii) *Let a be in $\mathfrak{o}$. The congruence $a\equiv x^2+x$ mod $\mathfrak{p}$ has a solution if and only if the equation $a=x^2+x$ has a solution in $\mathfrak{o}$.*
(iii) *The group $(1+4\mathfrak{o})/(1+2\mathfrak{o})^2$ has two elements. By* (i)*, there exist a in $\mathfrak{o}$ such that the congruence $a\equiv x^2+x$ mod $\mathfrak{p}$ has no solution, and for any such a the element $1+4a$ is a representative for the non-trivial coset of $(1+4\mathfrak{o})/(1+2\mathfrak{o})^2$.*
(iv) *If a in $\mathfrak{o}$ is such that $a\equiv x^2+x$ mod $\mathfrak{p}$ has no solution, then $(\varpi,1+4a)=-1$.*
(v) *The Hilbert symbol satisfies $(\mathfrak{o}^\times,1+4\mathfrak{o})=1$.*

Proof. (i) It is easy to check that the map is a group homomorphism. Also, it is easy to see that x and $x+1$ have the same image. Assume that $x^2+x=y^2+y$. Then $x^2-y^2+x-y=0$, i.e., $(x-y)(x+y+1)=0$. It follows that $x=y$ or $x+y+1=0$. The latter is equivalent to $y=x+1$.

(ii) Assume that $a\equiv c^2+c$ mod $\mathfrak{p}$ for some c in $\mathfrak{o}$. Let $f(X)=X^2+X-a$. Then $|f(c)|<|f'(c)|^2$. By Hensel's Lemma, there exists y in $\mathfrak{o}$ such that $f(y)=0$.

(iii) Let a be any element of $\mathfrak{o}$ such that $a\equiv x^2+x$ mod $\mathfrak{p}$ has no solution; by (i), such an a exists. We need to prove that 1 and $1+4a$ represent all the distinct cosets in $(1+4\mathfrak{o})/(1+2\mathfrak{o})^2$. It is easy to see that they represent distinct cosets. Let b in $\mathfrak{o}$ be such that $1+4b$ is not in $(1+2\mathfrak{o})^2$. Then the identity

$(1+2x)^2 = 1+4(x^2+x)$ implies that the equation $b = x^2+x$ has no solution in $\mathfrak{o}$. By (ii), the congruence $b \equiv x^2+x \mod \mathfrak{p}$ has no solution. By (i), the congruence $b-a \equiv x^2+x \mod \mathfrak{p}$ has a solution. By (ii), there exists x in $\mathfrak{o}$ such that $b-a = x^2+x$. Hence

$$1+4b = (1+4a)\left(1+4\frac{x^2+x}{1+4a}\right).$$

We have

$$\frac{x^2+x}{1+4a} \equiv x^2+x \mod \mathfrak{p}.$$

Therefore, by (ii), there exists y in $\mathfrak{o}$ such that

$$\frac{x^2+x}{1+4a} = y^2+y.$$

Hence

$$1+4b = (1+4a)\big(1+4(y^2+y)\big) = (1+4a)(1+2y)^2.$$

This proves (iii).

(iv) Let a in $\mathfrak{o}$ be such that $a \equiv x^2+x$ has no solution mod $\mathfrak{p}$; clearly, a is in $\mathfrak{o}^\times$. Assume that $(1+4a,\varpi)=1$; we will obtain a contradiction. By the definition of the Hilbert symbol, there exist x and y in F such that

$$x^2-(1+4a)y^2 = \varpi.$$

Since the valuation on the right side is odd, x and y must have the same valuation. Write $x = \varpi^k x'$ and $y = \varpi^k y'$ with k in $\mathbb{Z}$ and x' and y' in $\mathfrak{o}^\times$. Then

$$x'^2-(1+4a)y'^2 = \varpi^{1-2k}.$$

Assume that $1-2k < 2\mathrm{val}(2)$. Then it follows from $(x'-y')(x'+y') = u\varpi^{1-2k} + 4ay'^2$ that $\mathrm{val}(x'-y')+\mathrm{val}(x'+y') = 1-2k$. Now $x'+y' = x'-y'+2y'$. Therefore, if $\mathrm{val}(x'-y') \geq \mathrm{val}(2)$, then $\mathrm{val}(x'+y') \geq \mathrm{val}(2)$, and consequently $1-2k \geq 2\mathrm{val}(2)$, a contradiction. Hence, $\mathrm{val}(x'-y') < \mathrm{val}(2)$. But then $\mathrm{val}(x'+y') = \mathrm{val}(x'-y')$, so that $\mathrm{val}(x'-y')+\mathrm{val}(x'+y')$ is an even number; this is also a contradiction. Thus, $1-2k \geq 2\mathrm{val}(2)$, and then in fact $1-2k > 2\mathrm{val}(2)$. Again, $(x'-y')(x'+y') = u\varpi^{1-2k}+4ay'^2$; this now implies that $\mathrm{val}(x'-y')+\mathrm{val}(x'+y') = 2\mathrm{val}(2)$. Using again $x'+y' = x'-y'+2y'$, we see that necessarily $\mathrm{val}(x'+y') \geq \mathrm{val}(2)$ and $\mathrm{val}(x'-y') \geq \mathrm{val}(2)$, and indeed $\mathrm{val}(x'+y') = \mathrm{val}(x'-y') = \mathrm{val}(2)$. Write $x'-y' = 2w$ with $w \in \mathfrak{o}^\times$. Then $2w(2w+2y') = \varpi^{1-2k}+4ay'^2$, which implies $w(w+y') \equiv ay'^2 \mod \mathfrak{p}$. Hence $a \equiv (\frac{w}{y'})^2 + \frac{w}{y'} \mod \mathfrak{p}$, contradicting the choice of a.

(v) Let v be in $\mathfrak{o}^\times$. Let a in $\mathfrak{o}$ be such that $a \equiv x^2+x \mod \mathfrak{p}$ has no solution. Such an a exists by (i). By (iii), to prove that $(v, 1+4\mathfrak{o}) = 1$ it suffices to prove that $(v, 1+4a) = 1$. Now by iv) we have $(\varpi, 1+4a) = (v\varpi, 1+4a) = -1$. Therefore, $(v,1+4a) = (v\varpi^2, 1+4a) = (-1)(-1) = 1$. □

Lemma 1.2. *The following statements hold about the Hilbert symbol of F.*

(i) *Every element of $1+4\varpi\mathfrak{o}$ is a square, so that $(F^\times, 1+4\varpi\mathfrak{o}) = 1$.*
(ii) $(\mathfrak{o}^\times, (1+4\mathfrak{o}) \cap \mathfrak{o}^\times) = 1$.
(iii) $(\varpi, (1+4\mathfrak{o}) \cap \mathfrak{o}^\times) \neq 1$.

Proof. (i) Let a be in $\mathfrak{o}$ and define $f(X) = X^2 - (1+4\varpi a)$. Then $|f(1)| = |4\varpi a| < |2|^2 = |f'(1)|^2$. By Hensel's Lemma, the equation $f(X) = 0$ has a solution in $\mathfrak{o}$.

(ii) If the residual characteristic of F is odd, then the assertion is $(\mathfrak{o}^\times, \mathfrak{o}^\times) = 1$, which is well-known. If the residual characteristic of F is even, this is v) of Lemma 1.1.

(iii) If the residual characteristic of F is odd, then the assertion is $(\varpi, \mathfrak{o}^\times) \neq 1$, which is well-known. If the residual characteristic of F is even, then this follows from (iv) of Lemma 1.1. □

The cocycle

In this paper we define $\widetilde{\mathrm{SL}}(2,F)$ using the same cocycle c as is commonly used in [G], [W1], [W2] and [W3] (though c is denoted by β in these works). The cocycle c is a Borel measurable function

$$c : \mathrm{SL}(2,F) \times \mathrm{SL}(2,F) \to \{\pm 1\}$$

such that

$$c(g_1 g_2, g_3) c(g_1, g_2) = c(g_1, g_2 g_3) c(g_2, g_3) \tag{6}$$

for g_1, g_2 and g_3 in $\mathrm{SL}(2,F)$, and $c(g,1) = c(1,g) = 1$ for g in $\mathrm{SL}(2,F)$. As a set $\widetilde{\mathrm{SL}}(2,F) = \mathrm{SL}(2,F) \times \{\pm 1\}$, and the group law for $\widetilde{\mathrm{SL}}(2,F)$ is

$$(g,\varepsilon)(g',\varepsilon') = (gg', \varepsilon\varepsilon' c(g,g'))$$

for g and g' in $\mathrm{SL}(2,F)$ and ε and ε' equal to ± 1. Explicitly, c is given by the formula

$$c(g,g') = (x(g), x(g'))(-x(g)x(g'), x(gg'))s(g)s(g')s(gg'),$$

where

$$x\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c = 0 \end{cases}$$

and

$$s\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{cases} (c,d) & \text{if } cd \neq 0 \text{ and } \mathrm{val}(c) \text{ is odd}, \\ 1 & \text{otherwise}. \end{cases}$$

It is known that $c(K^4, K^4) = 1$, where

$$K^4 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathfrak{o}) : a \equiv 1 \ (4\mathfrak{o}),\ c \equiv 0 \ (4\mathfrak{o}) \right\}.$$

Thus, if the residual characteristic of F is odd, then $K^4 = \mathrm{SL}(2,\mathfrak{o})$. The subset $K^4 \times \{1\}$ is a subgroup of $\widetilde{\mathrm{SL}}(2,F)$. Calculations show that the center of $\widetilde{\mathrm{SL}}(2,F)$ consists of the elements

$$\left(\begin{bmatrix} \varepsilon & \\ & \varepsilon \end{bmatrix}, \varepsilon'\right)$$

for ε and ε' equal to ± 1. For n is a positive integer such that $n \geq 2v(2)$, let Γ_n be the subgroup of $\widetilde{\mathrm{SL}}(2,F)$ consisting of all the elements

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, 1\right)$$

with a, b, c and d in $\mathfrak{o}$, $a \equiv d \equiv 1$ $(\mathfrak{p}^n)$, and $b \equiv c \equiv 0$ $(\mathfrak{p}^n)$. The topology for $\widetilde{\mathrm{SL}}(2,F)$ is the topology obtained by taking the subgroups Γ_n for $n \geq 2v(2)$ as a fundamental system of open neighborhoods of the identity. With this topology, $\widetilde{\mathrm{SL}}(2,F)$ is an l-group as defined in [BZ].

The factor $\delta_m(a)$

For m in $F^\times$ define the character $\psi^m : F \to \mathbb{C}^\times$ by $\psi^m(x) = \psi(mx)$, where ψ is our fixed character of F. If m and a are in $F^\times$ then we let $\gamma_m(a)$ denote the *Weil index* of the quadratic form ax^2 on F with respect to ψ^m, as defined in paragraph 24, page 172 of [Weil]. By paragraph 27, page 175 of [Weil], one has

$$\gamma_m(a) = \frac{\lim\limits_{n\to\infty} \int\limits_{\mathfrak{p}^{-n}} \psi^m(ax^2)\,dx}{|\lim\limits_{n\to\infty} \int\limits_{\mathfrak{p}^{-n}} \psi^m(ax^2)\,dx|}. \tag{7}$$

From this formula it follows that $\gamma_m(a) = \gamma_{ma}(1) = \gamma_1(ma)$, $\gamma_{mb^2}(a) = \gamma_m(a)$ and $\gamma_m(ab^2) = \gamma_m(a)$ for a, b and m in $F^\times$. We define

$$\delta_m(a) = (a, -1)\gamma_m(a)\gamma_m(1)^{-1}.$$

The number $\delta_m(a)$ is written as $\chi_{\psi^m}(a)$ in [W3], page 223 and in [W1], page 4, and is denoted by $(a, -1)\gamma_F(a, \psi^m)$ in [Rao], page 367. It is proven in paragraph 28, page 176 of [Weil] (this is the formula on the bottom of this page if one uses Proposition 3 of [Weil], page 172) that

$$\delta_m(ab) = (a, b)\delta_m(a)\delta_m(b) \tag{8}$$

for a, b and m in $F^\times$. From this, and other properties of the Weil index, one can prove that the following hold for a, c and m in $F^\times$:

$$\begin{aligned}
\delta_m(c^2a) &= \delta_m(a), \\
\delta_{mc}(a) &= (a, c)\delta_m(a), \\
\delta_m(-1) &= (-1, -1)\gamma_m(1)^{-2}, \\
\delta_m(a)^{-1} &= (a, -1)\delta_m(a) = \delta_{-m}(a), \\
\delta_m(a)^4 &= 1, \\
\gamma_m(1)^8 &= 1.
\end{aligned}$$

Lemma 1.3. *We have* $\delta_1((1 + 4\mathfrak{o}) \cap \mathfrak{o}^\times) = 1$.

Proof. We will first prove that

$$\gamma_1(a) = \frac{\sum\limits_{z\in\mathfrak{o}/\mathfrak{p}^{\mathrm{val}(2a)}} \psi(az^2\varpi^{-2\mathrm{val}(2a)})}{|\sum\limits_{z\in\mathfrak{o}/\mathfrak{p}^{\mathrm{val}(2a)}} \psi(az^2\varpi^{-2\mathrm{val}(2a)})|} \tag{9}$$

for all non-zero a in $\mathfrak{o}$. Fix a non-zero element a of $\mathfrak{o}$. Let n be a positive integer. Using that ψ has conductor $\mathfrak{o}$ we have

$$\begin{aligned}
\int\limits_{\mathfrak{p}^{-n}} \psi(ax^2)\,dx &= \sum_{z\in\mathfrak{p}^{-n}/\mathfrak{o}} \int\limits_{\mathfrak{o}} \psi(a(x+z)^2)\,dx \\
&= \sum_{z\in\mathfrak{o}/\mathfrak{p}^n} \int\limits_{\mathfrak{o}} \psi(a(x+z\varpi^{-n})^2)\,dx \\
&= \sum_{z\in\mathfrak{o}/\mathfrak{p}^n} \int\limits_{\mathfrak{o}} \psi(a(2xz\varpi^{-n}+z^2\varpi^{-2n}))\,dx \\
&= \sum_{z\in\mathfrak{o}/\mathfrak{p}^n} \psi(az^2\varpi^{-2n}) \int\limits_{\mathfrak{o}} \psi(2axz\varpi^{-n})\,dx \\
&= \sum_{\substack{z\in\mathfrak{o}/\mathfrak{p}^n,\\ \mathrm{val}(2az\varpi^{-n})\geq 0}} \psi(az^2\varpi^{-2n}) \\
&= \sum_{z\in\mathfrak{p}^{n-\mathrm{val}(2a)}/\mathfrak{p}^n} \psi(az^2\varpi^{-2n}) \\
&= \sum_{z\in\mathfrak{o}/\mathfrak{p}^{\mathrm{val}(2a)}} \psi(az^2\varpi^{-2\mathrm{val}(2a)}).
\end{aligned}$$

The statement (9) now follows from (7). Now let a be in $(1+4\mathfrak{o})\cap\mathfrak{o}^\times$. The formula (9) shows that $\gamma_1(a)=\gamma_1(1)$. We now have $\delta_1(a)=(a,-1)\gamma_1(a)\gamma_1(1)^{-1}=(a,-1)=1$ by ii) of Lemma 1.2. □

Representation theory

Let (τ, V) be a representation of $\widetilde{\mathrm{SL}}(2,F)$. We say that τ is *genuine* if $\tau(1,\varepsilon)v=\varepsilon v$ for $\varepsilon=\pm1$ and v in V. We say that τ is *smooth* if for every v in V there exists a positive integer $n\geq 2v(2)$ such that $\tau(k)v=v$ for k in Γ_n, where Γ_n is as defined above. We say that τ is *admissible* if τ is smooth and for any positive integer $n\geq 2v(2)$ the subspace of v in V such that $\tau(k)v=v$ for k in Γ_n is finite-dimensional.

The Kirillov-type model of Waldspurger

The proof of the main theorem will make essential use of a certain type of model for irreducible, admissible, genuine representations τ of $\widetilde{\mathrm{SL}}(2,F)$. This model is analogous to the Kirillov model for infinite-dimensional, irreducible, admissible representations of $\mathrm{GL}(2,F)$. Waldspurger proved the existence of this model for those τ such that $\tau \cong \theta(\pi,\psi)$, where π is an infinite-dimensional, unitary, irreducible, admissible representation of $\mathrm{GL}(2,F)$ with trivial central character and $\theta(\pi,\psi)$ is the representation of $\widetilde{\mathrm{SL}}(2,F)$ defined in [W3], pages 228–231. For such τ, the existence of the model is proved in [W1], and is discussed in Assertion 7, page 396, of [W2] and on pages 228–229 of [W3]. At the suggestion of the referee, we give a complete proof of the existence of the model for all τ because this is missing from the literature. The assertion about the model is as follows.

Theorem 1.4. *Let (τ, V) be an irreducible, admissible, genuine representation of $\widetilde{\mathrm{SL}}(2,F)$. Let χ be a character of $F^\times$ such that $\chi(-1)=\varepsilon(\tau,\psi)$. There exists a space $\mathcal{M}(\tau,\chi)$ of functions $f: F^\times \to \mathbb{C}$ and an action of $\widetilde{\mathrm{SL}}(2,F)$ on $\mathcal{M}(\tau,\chi)$ such that, with this action, $\mathcal{M}(\tau,\chi)$ is isomorphic to τ. Moreover, $\mathcal{M}(\tau,\chi)$ and the action have the following properties:*

(i) *The functions in $\mathcal{M}(\tau,\chi)$ are locally constant, have relatively compact support in F, and are supported in $F_\psi(\tau)$. The space $\mathcal{S}(F_\psi(\tau))$ of locally constant, compactly supported functions on $F_\psi(\tau)$ is contained in $\mathcal{M}(\tau,\chi)$.*

(ii) *For f in $\mathcal{M}(\tau,\chi)$, n in F and x in $F^\times$ we have*

$$\tau\left(\begin{bmatrix}1 & n\\ & 1\end{bmatrix},1\right)f(x)=\psi(nx)f(x).$$

(iii) *For f in $\mathcal{M}(\tau,\chi)$, a in $F^\times$ and x in $F^\times$ we have*

$$\tau\left(\begin{bmatrix}a & \\ & a^{-1}\end{bmatrix},1\right)f(x)=\delta_1(a)\chi(a)f(a^2x).$$

The proof that we will present in the remainder of this section is based on the abstract proof for $\mathrm{GL}(n)$ in [BZ]; in particular, we will not use theta lifts. We require some notation and definitions. Let $\tilde{B}$ be the subgroup of $\widetilde{\mathrm{SL}}(2,F)$ of elements of the form

$$\left(\begin{bmatrix}a & b\\ & a^{-1}\end{bmatrix},\varepsilon\right)$$

for a in $F^\times$, b in F and $\varepsilon=\pm1$, let U be the subgroup of $\tilde{B}$ consisting of the elements

$$\left(\begin{bmatrix}1 & b\\ & 1\end{bmatrix},1\right)$$

with b in F, and let $\tilde{Z}$ be the center of $\widetilde{\mathrm{SL}}(2,F)$. The subgroup $\tilde{Z}$ is contained in $\tilde{B}$, and $\tilde{B}$ normalizes U. Let $\mathcal{C}_0^\infty(F^\times)$ be the complex vector space of locally constant complex valued functions on $F^\times$ that have relatively compact support in F, i.e., vanish outside of a compact subset of F. Let χ be a character of $F^\times$. Define an action of $\tilde{B}$ on $\mathcal{C}_0^\infty(F^\times)$ by

$$\left(\begin{bmatrix}a & b\\ & a^{-1}\end{bmatrix},\varepsilon\right)f(x)=\varepsilon\delta_1(a)\chi(a)\psi(abx)f(a^2x)$$

for f in $\mathcal{C}_0^\infty(F^\times)$, a and x in $F^\times$, b in F and $\varepsilon = \pm 1$. Computations show that this defines a smooth, genuine representation of $\tilde{B}$ on $\mathcal{C}_0^\infty(F^\times)$. Let (τ, V) be an irreducible, admissible, genuine representation of $\widetilde{\mathrm{SL}}(2, F)$, and assume that $\chi(-1) = \varepsilon(\tau, \psi)$. Using Whittaker functionals, we can define a $\tilde{B}$ map from V to $\mathcal{C}_0^\infty(F^\times)$ as follows. By Lemme 3, page 6, of [W1], τ is infinite-dimensional. As in the introductory remarks, let $F_\psi(\tau)$ be the set of a in $F^\times$ such that τ admits a Whittaker model with respect to ψ^a. By Lemme 2, page 226, of [W3] the set $F_\psi(\tau)$ is non-empty. The set $F_\psi(\tau)$ consists of a disjoint union of $F^{\times 2}$ cosets; let $F_\psi(\tau) = a_1 F^{\times 2} \sqcup \cdots \sqcup a_l F^{\times 2}$. For each i in $\{1, \dots, l\}$ let $\lambda_i : V \to \mathbb{C}$ be a non-zero ψ^{a_i} Whittaker functional; by Lemme 2, page 226, of [W3], λ_i is unique up to multiplication by non-zero complex numbers. Let v be in V. We define $f_v : F^\times \to \mathbb{C}$ by

$$f_v(x) = \delta_1(a)^{-1}\chi(a)^{-1}\lambda_i(\tau(\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}, 1)v) \tag{10}$$

if x is in $a_i F^{\times 2}$ for some i in $\{1, \dots, l\}$ and $x = a_i a^2$ for some a in $F^\times$, and by $f_v(x) = 0$ if x is not contained in $F_\psi(\tau)$. A computation using that $\chi(-1) = \varepsilon(\tau, \psi)$ shows that the right-hand side of (10) is the same if a is replaced by $-a$, so that $f_v(x)$ is well-defined for x in $a_i F^{\times 2}$ and i in $\{1, \dots, l\}$. For v in V, the function f_v is locally constant.

Lemma 1.5. *Let (τ, V) be an irreducible, admissible, genuine representation of $\widetilde{\mathrm{SL}}(2, F)$, let χ be a character of $F^\times$ such that $\chi(-1) = \varepsilon(\tau, \psi)$, and let the notation be as above. If v is in V, then f_v is in $\mathcal{C}_0^\infty(F^\times)$. The map*

$$T : V \to \mathcal{C}_0^\infty(F^\times)$$

defined by $T(v) = f_v$ is a $\tilde{B}$ map.

Proof. Let v be in V. Since τ is a smooth representation, there exists a positive integer n such that

$$\tau\left(\begin{bmatrix} 1 & \mathfrak{p}^n \\ & 1 \end{bmatrix}, 1\right) v = v.$$

Let i be in $\{1, \dots, l\}$, let a be in $F^\times$, and set $x = a_i a^2$. If b is in $\mathfrak{p}^n$, we have $f_v(x) = \psi(xb) f_v(x)$. It follows that if x is not in $\mathfrak{p}^{-n}$, then $f_v(x) = 0$. Hence, f_v has relatively compact support in F. A computation shows that T is a $\tilde{B}$ map. ☐

We will use the map T to prove Theorem 1.4. We will show that T is injective and has certain other properties; then we will define $\mathcal{M}(\tau, \chi)$ to be the image of T and prove that it satisfies the claims of the theorem. To do this we need two lemmas. Let (τ, V) be a smooth, genuine representation of U. If a is in F, then we let $V(U, \psi^a)$ be the subspace of V spanned by all the vectors of the form

$$\psi^a(x)v - \tau\left(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}, 1\right) v$$

for v in V and x in F. We will also write $V(U) = V(U, \psi^0)$, $V_{U,\psi^a} = V/V(U, \psi^a)$, and $V_U = V_{U,\psi^0}$ for a in F.

Lemma 1.6. *Let (τ, V) be a smooth representation of U.*

(i) *If v is in V and v is in $V(U, \psi^a)$ for all a in F, then $v = 0$.*
(ii) *If v is in V and A is a compact, open subset of F, then there exists a w in V such that $v - w$ is in $V(U, \psi^a)$ for a in A and w is in $V(U, \psi^a)$ for a not in A.*

Proof. We will use the Fourier transform $\mathcal{S}(F) \to \mathcal{S}(F)$ that sends f to $\hat{f}$, with $\hat{f}$ defined by

$$\hat{f}(x) = \int\limits_F f(y)\psi(-xy)\, dy$$

for x in F. Here, as is our convention, we use the Haar measure that assigns $\mathfrak{o}$ measure 1. As usual, for f in $\mathcal{S}(F)$ and v in V, we define

$$\tau(f)v = \int\limits_F f(x)\tau\left(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}, 1\right) v\, dx.$$

We have $\tau(f_1 * f_2) = \tau(f_1)\tau(f_2)$ for f_1 and f_2 in $\mathcal{S}(F)$, where the convolution $f_1 * f_2$ is defined by

$$(f_1 * f_2)(x) = \int\limits_F f_1(y) f_2(x-y)\, dy.$$

If f_1 and f_2 are in $\mathcal{S}(F)$, then $\widehat{f_1 * f_2} = \hat{f}_1\hat{f}_2$, where $\hat{f}_1\hat{f}_2$ is the pointwise product of $\hat{f}_1$ and $\hat{f}_2$. We define $\tau'(f) = \tau(\hat{f})$ for f in $\mathcal{S}(F)$. We have $\tau'(f_1)\tau'(f_2) = \tau'(f_1 f_2)$ for f_1 and f_2 in $\mathcal{S}(F)$. Also, for t an integer and a in F we define $f_{t,a}$ to be the element of $\mathcal{S}(F)$ defined by $f_{t,a}(x) = \mathrm{char}_{\mathfrak{p}^t}(x-a)$. For t an integer and a in F we have $\hat{f}_{t,a}(x) = q^{-t}\mathrm{char}_{\mathfrak{p}^{-t}}(x)\psi(-ax)$ and the equalities

$$q^{-t}\int\limits_{\mathfrak{p}^{-t}} \psi(-ax)\tau\left(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}, 1\right) v\, dx = \tau(\hat{f}_{t,a})v = \tau'(f_{t,a})v. \tag{11}$$

Let v be in V and let a be in F. We claim that v is in $V(U, \psi^a)$ if and only if there exists f in $\mathcal{S}(F)$ such that $f(a) \neq 0$ and $\tau'(f)v = 0$. Assume that v is in $V(U, \psi^a)$. By 2.33 of [BZ], there exists a positive integer t such that the integral in (11) is zero, so that $\tau'(f_{t,a})v = 0$. Since $f_{t,a}(a) \neq 0$, this proves the claim in one direction. Assume that f in $\mathcal{S}(F)$ is such that $f(a) \neq 0$ and $\tau'(f)v = 0$. Since $\tau'(f_1)\tau'(f) = \tau'(f_1 f)$ for f_1 in $\mathcal{S}(F)$, we may assume that $f = f_{t,a}$ for some positive integer t. By (11), the integral in (11) is zero, so that by 2.33 of [BZ] the vector v is in $V(U, \psi^a)$.

(i) Assume that v is in $V(U, \psi^a)$ for all a. Since τ is a smooth representation, there exists h in $\mathcal{S}(F)$ such that $\tau'(h)v = v$. By the proof of the claim from the previous paragraph, for every a in the support of h there exists a positive integer t_a such that $\tau'(f_{t_a,a})v = 0$ and $a + p^{t_a}$ is in the support of h. Since the support of h is compact, and since for any a and a' in F and positive integers t and t'

either $a+\mathfrak{p}^t$ and $a'+\mathfrak{p}^{t'}$ are disjoint or one coset is contained in the other, it follows that we can write the characteristic function of the support of h as a linear combination of the functions f_{a,t_a} for an appropriate finite subset X of elements a in the support of h. Therefore, $h=\sum_{a\in X} hf_{t_a,a}$. Hence,

$$v=\tau'(h)v=\sum_{a\in X}\tau'(h)\tau'(f_{t_a,a})v=0.$$

(ii) Let v be in V and let A be a compact, open subset of F. Set $w=\tau'(\mathrm{char}_A)v$. Let a be in A. We have $\mathrm{char}_A(a)\neq 0$, and $\tau'(\mathrm{char}_A)(v-w)=\tau'(\mathrm{char}_A)v-\tau'(\mathrm{char}_A^2)v=0$, so that $v-w$ is contained in $V(U,\psi^a)$. Let a be in F but not in A. Let f in $\mathcal{S}(F)$ be such that $f(a)\neq 0$ and $f\mathrm{char}_A=0$. Then $\tau'(f)w=\tau'(f\mathrm{char}_A)v=0$, so that w is in $V(U,\psi^a)$. □

Let (τ,V) be an irreducible, admissible, genuine representation of $\widetilde{\mathrm{SL}}(2,F)$. Evidently, the groups U and $\tilde{Z}$ act naturally on $V(U,\psi^a)$ and V_{U,ψ^a} for a in F, and we have the formula

$$\begin{aligned} z(v+V(U,\psi^a)) &= \omega_\tau(z)v+V(U,\psi^a),\\ \left(\begin{bmatrix}1&b\\&1\end{bmatrix},1\right)(v+V(U,\psi^a)) &= \psi^a(b)v+V(U,\psi^a)\end{aligned}$$

for a and b in F and v in V. Here, ω_τ is the central character of τ. If $a=0$, then the entire group $\tilde{B}$ acts on $V(U)$ and V_U. In the proof of the next lemma we will use the following rule: for any vector u in V, b in F, and a in $F^\times$, we have

$$u\in V(U,\psi^b)\iff \tau\left(\begin{bmatrix}a&\\&a^{-1}\end{bmatrix},1\right)u\in V(U,\psi^{ba^{-2}}). \tag{12}$$

Lemma 1.7. *Let (τ,V) be an irreducible, admissible, genuine representation of $\widetilde{\mathrm{SL}}(2,F)$, let χ be a character of $F^\times$ such that $\chi(-1)=\varepsilon(\tau,\psi)$, and let the notation be as in the paragraph preceding Lemma* 1.5*. For v in V and i in $\{1,\ldots,l\}$, define $h_{v,i}:\tilde{B}\to V_{U,\psi^{a_i}}$ by*

$$h_{v,i}\left(\begin{bmatrix}a&b\\&a^{-1}\end{bmatrix},\varepsilon\right)=\varepsilon\psi^{a_i}(ab)\tau\left(\begin{bmatrix}a&\\&a^{-1}\end{bmatrix},1\right)v+V(U,\psi^{a_i}).$$

(i) *For v in $V(U)=V(U,\psi^0)$ and i in $\{1,\ldots,l\}$ the function $h_{v,i}$ is contained in* $\text{c-Ind}_{\tilde{Z}U}^{\tilde{B}}V_{U,\psi^{a_i}}$*, and the map sending v in $V(U)$ to $\oplus_{i=1}^l h_{v,i}$ defines an isomorphism of $\tilde{B}$ representations*

$$V(U)\xrightarrow{\sim}\oplus_{i=1}^l\text{c-Ind}_{\tilde{Z}U}^{\tilde{B}}V_{U,\psi^{a_i}}. \tag{13}$$

(ii) *The restriction of the function T from Lemma* 1.5 *to $V(U)$ is injective, and for i in $\{1,\ldots,l\}$ the image of* $\text{c-Ind}_{\tilde{Z}U}^{\tilde{B}}V_{U,\psi^{a_i}}$*, regarded as a subspace of $V(U)$ via* (13)*, is $\mathcal{S}(a_iF^{\times 2})$.*

Proof. (i) Let v be in $V(U)$ and let i be in $\{1,\dots,l\}$; the first task is to prove that $h_{v,i}$ is contained $\text{c-Ind}_{\tilde{Z}U}^{\tilde{B}} V_{U,\psi^{a_i}}$. First, it is straightforward to verify that

$$h_{v,i}\left(z\left(\begin{bmatrix}1 & b\\ & 1\end{bmatrix},1\right)p\right)=\omega_\tau(z)\psi^{a_i}(b)h_{v,i}(p)$$

for z in the center $\tilde{Z}$ of $\widetilde{\mathrm{SL}}(2,F)$, b in F, and p in $\tilde{B}$. Second, we claim that

$$h_{v,i}\left(\begin{bmatrix}a & \\ & a^{-1}\end{bmatrix},1\right)=0 \tag{14}$$

for sufficiently small and sufficiently large elements a of $F^\times$. We may assume that

$$v=w-\tau\left(\begin{bmatrix}1 & b\\ & 1\end{bmatrix},1\right)w$$

for some b in F. Let M be a positive integer such that $\psi^{a_i}(a^2b)=1$ for a in $\mathfrak{p}^M$ and

$$\tau\left(\begin{bmatrix}1 & \mathfrak{p}^M\\ & 1\end{bmatrix},1\right)v=v.$$

Let a be in $F^\times$. If a is in $\mathfrak{p}^M$, then

$$h_{v,i}\left(\begin{bmatrix}a & \\ & a^{-1}\end{bmatrix},1\right)=(1-\psi^{a_i}(a^2b))\tau\left(\begin{bmatrix}a & \\ & a^{-1}\end{bmatrix},1\right)w+V(U,\psi^{a_i})=0.$$

On the other hand, assume that a^2 is not in $a_i^{-1}\mathfrak{p}^{-M}$ so that there exists y in $\mathfrak{p}^M$ such that $\psi^{a_i}(a^2y)\neq 1$. We have

$$\begin{aligned}h_{v,i}\left(\begin{bmatrix}a & \\ & a^{-1}\end{bmatrix},1\right)&=\tau\left(\begin{bmatrix}a & \\ & a^{-1}\end{bmatrix},1\right)\tau\left(\begin{bmatrix}1 & y\\ & 1\end{bmatrix},1\right)v+V(U,\psi^{a_i})\\&=\psi^{a_i}(a^2y)h_{v,i}\left(\begin{bmatrix}a & \\ & a^{-1}\end{bmatrix},1\right);\end{aligned}$$

since $\psi^{a_i}(a^2y)\neq 1$, this vector is zero. This proves our claim. Finally, a computation shows that for a and a' in $F^\times$, b and b' in F, and $\varepsilon=\pm 1$,

$$h_{v,i}\left(p\left(\begin{bmatrix}a' & b'\\ & a'^{-1}\end{bmatrix},1\right)\right)=\psi^{a_i}(a^2a'b')h_{w,i}(p)$$

where

$$p=\left(\begin{bmatrix}a & b\\ & a^{-1}\end{bmatrix},\varepsilon\right)\quad\text{and}\quad w=\tau\left(\begin{bmatrix}a' & \\ & a'^{-1}\end{bmatrix},1\right)v.$$

This formula, along with the the smoothness of v and the fact that (14) holds for sufficiently large elements a of $F^\times$, implies that $h_{v,i}$ is right invariant under a compact, open subgroup of $\tilde{B}$.

Next, a computation proves that the map in (i) is $\tilde{B}$ equivariant. To see that it is injective, assume that v is in $V(U)$ and $h_{i,v}=0$ for all i in $\{1,\dots,l\}$. Then

$$\tau\left(\begin{bmatrix}a & \\ & a^{-1}\end{bmatrix},1\right)v\in V(U,\psi^{a_i})$$

for all i in $\{1,\dots,l\}$ and a in $F^\times$. Using the rule (12), and recalling that v is contained in $V(U)$ and that $V=V(U,\psi^a)$ for a not in $F_\psi(\tau)$, it follows that v is in $V(U,\psi^a)$ for all a in F. Therefore, by (i) of Lemma 1.6, we have $v=0$. To prove that the map from (i) is surjective, it will suffice to prove that each direct summand is in the image; for simplicity of notation, we will prove that

$$\text{c-Ind}_{\tilde{Z}U}^{\tilde{B}} V_{U,\psi^{a_1}} \oplus 0 \oplus \cdots \oplus 0$$

is in the image. Let f be in $\text{c-Ind}_{\tilde{Z}U}^{\tilde{B}} V_{U,\psi^{a_1}}$. We have

$$f\left(\left[\begin{matrix} a_1a_2 & \\ & a_1^{-1}a_2^{-1}\end{matrix}\right],1\right) = (a_1,a_2) f\left(\left(\left[\begin{matrix} a_1 & \\ & a^{-1}\end{matrix}\right],1\right)\left(\left[\begin{matrix} a_2 & \\ & a_2^{-1}\end{matrix}\right],1\right)\right)$$

for a_1 and a_2 in $F^\times$. Define $f' : F^\times \to V_{U,\psi^{a_1}}$ by

$$f'(a) = f\left(\left[\begin{matrix} a & \\ & a^{-1}\end{matrix}\right],1\right).$$

The function f' is locally constant and compactly supported. Also,

$$f'(-a) = \varepsilon(\tau,\psi)\delta_1(-1)(-1,a)f'(a)$$

for a in $F^\times$. In particular, $f'(-a)$ is non-zero if and only if $f'(a)$ is non-zero for a in $F^\times$. There exists a finite subset S of V such that for every a in the support of f', there exists v in S such that $f'(a) = v + V(U,\psi^{a_1})$. Let M be a positive integer such that $f'(ak) = f'(a)$ for a in $F^\times$ and k in $1+\mathfrak{p}^M$, such that

$$\tau\left(\left[\begin{matrix} k & \\ & k^{-1}\end{matrix}\right],1\right) v = v \tag{15}$$

for k in $1+\mathfrak{p}^M$ and v in S, such that $1+\mathfrak{p}^M$ is contained in $\mathfrak{o}^{\times 2}$, and such that -1 is not contained in $1+\mathfrak{p}^M$. An argument now shows that there exist $d_1,\dots,d_m$ in $F^\times$ such that $d_1(1+\mathfrak{p}^M), -d_1(1+\mathfrak{p}^M),\dots,d_m(1+\mathfrak{p}^M), -d_m(1+\mathfrak{p}^M)$ are disjoint, and f' is supported in the disjoint union

$$d_1(1+\mathfrak{p}^M) \sqcup -d_1(1+\mathfrak{p}^M) \sqcup \cdots \sqcup d_m(1+\mathfrak{p}^M) \sqcup -d_m(1+\mathfrak{p}^M).$$

Let $v_1,\dots,v_m$ in S be such that $f'(d_1) = v_1 + V(U,\psi^{a_1}),\dots,f'(d_m) = v_m + V(U,\psi^{a_1})$. Then $f'(-d_1) = \varepsilon(\tau,\psi)\delta_1(-1)(-1,d_1)v_1 + V(U,\psi^{a_1}),\dots,f'(-d_m) = \varepsilon(\tau,\psi)\delta_1(-1)(-1,d_m)v_m + V(U,\psi^{a_1})$. Let j be in $\{1,\dots,m\}$. By (ii) of Lemma 1.6, there exists w_j in V such that

$$w_j - \tau\left(\left[\begin{matrix} d_j & \\ & d_j^{-1}\end{matrix}\right],1\right)^{-1} v_j \in V(U,\psi^b) \tag{16}$$

for b in $a_1d_j^2(1+\mathfrak{p}^M)^2$, and such that w_j is in $V(U,\psi^b)$ for b in F but not in $a_1d_j^2(1+\mathfrak{p}^M)^2$. Let k be in $1+\mathfrak{p}^M$ and set $a=d_jk$. By (16) for $b=a_1a^2$, and the rule (12), we have

$$\tau\left(\left[\begin{matrix} a & \\ & a^{-1}\end{matrix}\right],1\right) w_j - \tau\left(\left(\left[\begin{matrix} a & \\ & a^{-1}\end{matrix}\right],1\right)\left(\left[\begin{matrix} d_j & \\ & d_j^{-1}\end{matrix}\right],1\right)^{-1}\right) v_j \in V(U,\psi^{a_1a^2a^{-2}});$$

that is

$$\tau\left(\begin{bmatrix} a & \\ & a^{-1}\end{bmatrix}, 1\right) w_j - v_j \in V(U, \psi^{a_1}) \tag{17}$$

for a in $d_j(1+\mathfrak{p}^M)$. Note that the simplification uses $(F^\times, 1+\mathfrak{p}^M) = 1$, so that the relevant cocycle is trivial, and (15) for $v = v_j$. Since (17) holds for a in $d_j(1+\mathfrak{p}^M)$, we also have

$$\tau\left(\begin{bmatrix} a & \\ & a^{-1}\end{bmatrix}, 1\right) w_j - \varepsilon(\tau,\psi)\delta_1(-1)(-1,d_j)v_j \in V(U, \psi^{a_1})$$

for a in $-d_j(1+\mathfrak{p}^M)$. Next, let a be in $F^\times$ but not in $d_j(1+\mathfrak{p}^M) \sqcup -d_j(1+\mathfrak{p}^M)$. Set $b = a_1 a^2$. Then b is not in $a_1 d_j^2 (1+\mathfrak{p}^M)^2$. Hence, w_j is in $V(U, \psi^b)$, and by the rule (12), we thus have

$$\tau\left(\begin{bmatrix} a & \\ & a^{-1}\end{bmatrix}, 1\right) w_j \in V(U, \psi^{a_1})$$

for a in $F^\times$ and a not in $d_j(1+\mathfrak{p}^M) \sqcup -d_j(1+\mathfrak{p}^M)$. Similarly,

$$\tau\left(\begin{bmatrix} a & \\ & a^{-1}\end{bmatrix}, 1\right) w_j \in V(U, \psi^{a_2}), \dots, \tau\left(\begin{bmatrix} a & \\ & a^{-1}\end{bmatrix}, 1\right) w_j \in V(U, \psi^{a_m})$$

for a in $F^\times$. Finally, as 0 is not in $a_1 d_j^2(1+\mathfrak{p}^M)^2$, the vector w_j is in $V(U)$. Now set $v = w_1 + \cdots + w_m$. Then v is in $V(U)$, $h_{v,1} = f$, and $h_{v,2} = 0, \dots, h_{v,l} = 0$.

(ii) To prove the assertions of (ii) it will suffice to prove that, for every i in $\{1, \dots, l\}$, the restriction of T to the subspace V_i of $V(U)$ corresponding to the subspace $0 \oplus \cdots \oplus 0 \oplus \text{c-Ind}_{\tilde{Z}U}^{\tilde{B}} V_{U,\psi^{a_i}} \oplus 0 \oplus \cdots \oplus 0$ of the direct sum is injective and has image $\mathcal{S}(a_i F^{\times 2})$. For simplicity of notation, we take $i = 1$. Let v be in V_1. We have $h_{v,2} = 0, \dots, h_{v,m} = 0$, so that

$$\tau\left(\begin{bmatrix} a & \\ & a^{-1}\end{bmatrix}, 1\right) v \in V(U, \psi^{a_2}), \dots, \tau\left(\begin{bmatrix} a & \\ & a^{-1}\end{bmatrix}, 1\right) v \in V(U, \psi^{a_l})$$

for a in $F^\times$, and hence

$$\lambda_2\left(\tau\left(\begin{bmatrix} a & \\ & a^{-1}\end{bmatrix}, 1\right) v\right) = \cdots = \lambda_m\left(\tau\left(\begin{bmatrix} a & \\ & a^{-1}\end{bmatrix}, 1\right) v\right) = 0$$

for a in $F^\times$. This implies that $T(v) = f_v$ is supported on $a_1 F^{\times 2}$. Moreover, if a is in $F^\times$, and $x = a_1 a^2$, then

$$f_v(x) = \delta_1(a)^{-1}\chi(a)^{-1}\lambda_1\left(h_{v,1}\left(\begin{bmatrix} a & \\ & a^{-1}\end{bmatrix}, 1\right)\right).$$

Assume that $T(v) = f_v = 0$. Then, since the map sending $v + V(U, \psi^{a_1})$ to $\lambda_1(v)$ is an isomorphism $V_{U,\psi^{a_1}} \cong \mathbb{C}$ of complex vector spaces, we must have $h_{v,1} = 0$; this implies $v = 0$. Finally, let f_1 be in $\mathcal{S}(a_1 F^{\times 2})$. We define $f : \tilde{B} \to V_{U,\psi^{a_1}}$ by the formula

$$\lambda_1\left(f\left(\begin{bmatrix} a & b \\ & a^{-1}\end{bmatrix}, \varepsilon\right)\right) = \varepsilon \psi^{a_1}(ab)\delta_1(a)\chi(a) f_1(a_1 a^2)$$

for a in $F^\times$, b in F and $\varepsilon = \pm 1$. It can be verified that f is in $\text{c-Ind}_{\tilde{Z}U}^{\tilde{B}} V_{U,\psi^{a_1}}$. If v is the corresponding element of V_1, then $T(v) = f_1$. □

We can now give the proof of the existence of Kirillov-type models for metaplectic representations.

Proof of Theorem 1.4. We use the map T from Lemma 1.5. We first prove that T is injective. There is a natural exact sequence

$$0 \to V(U) \to V \to V_U = V/V(U) \to 0$$

of $\tilde{B}$ spaces. If τ is supercuspidal, then $V_U = 0$ by definition. If τ is not supercuspidal, then by Section II of [W1] (see also the summary in [BS], pages 112–115) τ is isomorphic to an irreducible principal series representation, an even Weil representation, or a special representation. In the first case, it can be shown that V_U is two-dimensional; in the second two cases, it can be shown that V_U is one-dimensional. Therefore, V_U is finite-dimensional. There is an inclusion

$$\ker(T)/\ker(T) \cap V(U) \hookrightarrow V_U = V/V(U)$$

of $\tilde{B}$ spaces. By (ii) of Lemma 1.7 we have $\ker(T) \cap V(U) = 0$, so that $\ker(T)$ is finite-dimensional. Since $\ker(T)$ is finite-dimensional, there exists a positive integer n such that each vector in $\ker(T)$ is fixed by the subgroup

$$\left(\begin{bmatrix} 1 & \\ \mathfrak{p}^n & 1 \end{bmatrix}, 1\right)$$

of $\widetilde{\text{SL}}(2,F)$. In particular, this subgroup of $\widetilde{\text{SL}}(2,F)$ preserves $\ker(T)$. Since $\ker(T)$ is also a $\tilde{B}$ subspace, the identity

$$\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, 1\right) = \left(\begin{bmatrix} -\varpi^n & \\ & -\varpi^{-n} \end{bmatrix}, 1\right)\left(\begin{bmatrix} 1 & -\varpi^{-n} \\ & 1 \end{bmatrix}, 1\right)$$
$$\left(\begin{bmatrix} 1 & \\ \varpi^n & 1 \end{bmatrix}, 1\right)\left(\begin{bmatrix} 1 & -\varpi^{-n} \\ & 1 \end{bmatrix}, 1\right)$$

implies that this element preserves $\ker(T)$. Since this element and $\tilde{B}$ together generate $\widetilde{\text{SL}}(2,F)$, $\ker(T)$ is an $\widetilde{\text{SL}}(2,F)$ subspace. As τ is irreducible, we have $\ker(T) = 0$ or $\ker(T) = V$. The map T is non-zero by (ii) of Lemma 1.7; therefore, $\ker(T) = 0$. We now define $\mathcal{M}(\tau,\chi)$ to be the image of T, and transfer the action of $\widetilde{\text{SL}}(2,F)$ to $\mathcal{M}(\tau,\chi)$ via T. The assertions (i), (ii) and (iii) follow from Lemma 1.5 and (ii) of Lemma 1.7. □

2 Basic observations

Let (τ, V) be a smooth, genuine representation of $\widetilde{\text{SL}}(2,F)$ and let χ be a character of $\mathfrak{o}^\times$. In this section we answer two basic questions about the spaces $V_\psi(\tau,n,\chi)$. The first three lemmas determine the general conditions on χ and n that must be satisfied for $V_\psi(\tau,n,\chi)$ to be non-zero. We will prove that if $V_\psi(\tau,n,\chi)$ is non-zero then $n \geq 2\text{val}(2)$ and χ is trivial on $1+\mathfrak{p}^n$.

Lemma 2.1. *Let (τ, V) be a smooth, genuine representation of $\widetilde{\mathrm{SL}}(2,F)$ and let χ be a character of $\mathfrak{o}^\times$. Assume that the space $V_\psi(\tau, 2\mathrm{val}(2), \chi)$ is non-zero. Then χ is trivial on $(1+4\mathfrak{o}) \cap \mathfrak{o}^\times$.*

Proof. Let v be a non-zero vector in $V_\psi(\tau, 2\mathrm{val}(2), \chi)$. Let x be in F, let y be in $F^\times$ and assume that $1+xy$ is in $F^\times$. A computation shows that

$$\begin{aligned}&\left(\begin{bmatrix}1 & \\ y & 1\end{bmatrix}, 1\right)\left(\begin{bmatrix}1 & x\\ & 1\end{bmatrix}, 1r\right)\\ &= \left(\begin{bmatrix}(1+xy\ )^{-1} & \\ & 1+xy\end{bmatrix}, 1\right)\left(\begin{bmatrix}1 & x(1+xy)\\ & 1\end{bmatrix}, 1\right)\\ &\left(\begin{bmatrix}1 & \\ (1+xy)^{-1}y & 1\end{bmatrix}, (-y, 1+xy)\right).\end{aligned} \tag{18}$$

Now set $y=4$ and assume that x is in $\mathfrak{o}$ and $1+4x$ is in $\mathfrak{o}^\times$. Applying both sides of (18) to v, we find that $1 = (-4, 1+4x)\chi(1+4x)^{-1}\delta_1((1+4x)^{-1}) = (-1, 1+4x)\chi(1+4x)^{-1}\delta_1(1+4x)$. By Lemma 1.3 we have $\delta_1(1+4x)=1$ and by (ii) of Lemma 1.2 we have $(-1, 1+4x)=1$, so that $\chi(1+4x)=1$ for all $x \in \mathfrak{o}$ such that $1+4x$ is in $\mathfrak{o}^\times$. □

Lemma 2.2. *Let (τ, V) be a smooth, genuine representation of $\widetilde{\mathrm{SL}}(2,F)$ and let χ be a character of $\mathfrak{o}^\times$. The space $V_\psi(\tau, 2\mathrm{val}(2)-1, \chi)$ is zero.*

Proof. Assume that $V_\psi(\tau, 2\mathrm{val}(2)-1, \chi)$ contains a non-zero vector v; we will obtain a contradiction. Let x in $\mathfrak{p}$ and y in $4\varpi^{-1}\mathfrak{o}$ with y non-zero be such that $1+xy$ is in $\mathfrak{o}^\times$, so that $1+xy$ is in $(1+4\mathfrak{o}) \cap \mathfrak{o}^\times$. Applying both sides of (18) to v, we get $1 = (-y, 1+xy)\chi(1+xy)^{-1}\delta_1(1+xy)$. By Lemma 2.1 we have $\chi(1+xy)=1$; by Lemma 1.3 we have $\delta_1(1+xy)=1$. Therefore, $(-y, 1+xy)=1$ for all x in $\mathfrak{p}$ and non-zero y in $4\varpi^{-1}\mathfrak{o}$ such that $1+xy$ is in $\mathfrak{o}^\times$. Letting y be $-4\varpi^{-1}$ and x be $-\varpi b$ where b is in $\mathfrak{o}$, we find that $(\varpi, 1+4b)=1$ for all b in $\mathfrak{o}$ such that $1+4b$ is in $\mathfrak{o}^\times$. In other words, $(\varpi, (1+4\mathfrak{o}) \cap \mathfrak{o}^\times)=1$. This contradicts (iii) of Lemma 1.2. □

Lemma 2.3. *Let (τ, V) be a smooth, genuine representation of $\widetilde{\mathrm{SL}}(2,F)$, let χ be a character of $\mathfrak{o}^\times$ and let n be an integer. Assume that $V_\psi(\tau, n, \chi)$ is non-zero. Then $n \geq 2\mathrm{val}(2)$ and χ is trivial on $1+\mathfrak{p}^n$.*

Proof. Let v be a non-zero vector in $V_\psi(\tau, n, \chi)$. By Lemma 2.2 we have $n \geq 2\mathrm{val}(2)$. We may assume $n > 2\mathrm{val}(2)$, since the case $n = 2\mathrm{val}(2)$ is Lemma 2.1. Let x be in $\mathfrak{o}$ and y in $\mathfrak{p}^n$ with y non-zero. Applying both sides of (18) to v we obtain $1 = (-y, 1+xy)\chi(1+xy)^{-1}\delta_1((1+xy)^{-1})$. By (i) of Lemma 1.2 we have $(-y, 1+xy)=1$, and by Lemma 1.3 we have $\delta_1(1+xy)=1$. Hence, $1=\chi(1+xy)$. The lemma follows. □

The second question that we deal with in this section concerns an alternative characterization of the spaces $V_\psi(\tau, n, \chi)$. To formulate the question, assume that $V_\psi(\tau, n, \chi)$ is non-zero. By Lemma 2.3 we know that $n \geq 2\mathrm{val}(2)$ and χ is trivial on $1+\mathfrak{p}^n$. Define

$$\tilde{\Gamma}_0(\mathfrak{p}^n) = \Gamma_0(\mathfrak{p}^n) \times \{\pm 1\} \tag{19}$$

where $\Gamma_0(\mathfrak{p}^n)$ is the subgroup of $\mathrm{SL}(2,\mathfrak{o})$ of elements with lower left entries in $\mathfrak{p}^n$. The set $\tilde{\Gamma}_0(\mathfrak{p}^n)$ is a subgroup of $\widetilde{\mathrm{SL}}(2,F)$. Moreover, the group $\tilde{\Gamma}_0(\mathfrak{p}^n)$ is generated by $(1,\pm 1)$ and the elements of $\widetilde{\mathrm{SL}}(2,F)$ that appear in (1), (2) and (3). It follows that for every element (k,ε) of $\tilde{\Gamma}_0(\mathfrak{p}^n)$ there exists an element $\tilde{\chi}(k,\varepsilon)$ of $\mathbb{C}^\times$ such that

$$\tau(k,\varepsilon)v = \tilde{\chi}(k,\varepsilon)v \tag{20}$$

for all v in $V_\psi(\tau,n,\chi)$. Evidently, the function that sends (k,ε) to $\tilde{\chi}(k,\varepsilon)$ is a character of $\tilde{\Gamma}_0(\mathfrak{p}^n)$. The next two results determine the formula for the character $\tilde{\chi}$ on an arbitrary element of $\tilde{\Gamma}_0(\mathfrak{p}^n)$. Though we will not need this formula to prove the main theorem, we include it because it may be of some use in other investigations. For example, this formula is essential for determining explicit information about metaplectic vectors in principal series representations if the residual characteristic of F is even.

Lemma 2.4. *Let χ be a character of $\mathfrak{o}^\times$ and let n be an integer such that $n \geq 2\mathrm{val}(2)$ and χ is trivial on $1+\mathfrak{p}^n$. Define a function $f : \tilde{\Gamma}_0(\mathfrak{p}^n) \to \mathbb{C}^\times$ in the following way. If $n=0$, then define $f(k,\varepsilon)=\varepsilon$. If n is positive, then define*

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \varepsilon\right) = \varepsilon y\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\chi(d)^{-1}\delta_1(d)$$

where $y : \Gamma_0(\mathfrak{p}) \to \{\pm 1\}$ is given by

$$y\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{cases} 1 & \text{if } c = 0, \\ (d,-1) & \text{if } c \neq 0 \text{ and } \mathrm{val}(c) \text{ is odd}, \\ (d,-c) & \text{if } c \neq 0 \text{ and } \mathrm{val}(c) \text{ is even.} \end{cases} \tag{21}$$

The function f is a character of $\tilde{\Gamma}_0(\mathfrak{p}^n)$.

Proof. If F has odd residual characteristic, then it is straightforward to verify that f is a character: note that in this case the cocycle c is trivial on $\Gamma_0(\mathfrak{p}^n)$, the function y is constantly 1, and δ_1 is 1 on $\mathfrak{o}^\times$ by Lemma 1.3. Assume that F has even residual characteristic, and let

$$k = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad k' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$$

be in $\Gamma_0(\mathfrak{p}^n)$. Since we are assuming that F has even residual characteristic, the integer n is positive and a, d, a' and d' are in $\mathfrak{o}^\times$. We have to prove that

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, 1\right) f\left(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}, 1\right) = f\left(\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, 1\right)\left(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}, 1\right)\right).$$

Using the definition of f and (8) this is equivalent to

$$y(k)y(k') = y(kk')c(k,k')(d,d'). \tag{22}$$

Using the definitions and the formula for the cocycle, some computations show that (22) is true if $c = 0$ or $c' = 0$. Assume that $c \neq 0$ and $c' \neq 0$. The formulas for y and the cocycle imply that, in general,

$$y\left(\begin{bmatrix} 1 & * \\ & 1 \end{bmatrix} g\right) = y(g) \qquad \text{and} \qquad c\left(\begin{bmatrix} 1 & * \\ & 1 \end{bmatrix} g, g'\right) = c(g, g').$$

We may therefore assume that $b = 0$. In other words, we are reduced to proving that

$$\begin{aligned} & y\left(\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}\begin{bmatrix} 1 & \\ c & 1 \end{bmatrix}\right) y(k') \\ &= y\left(\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}\begin{bmatrix} 1 & \\ c & 1 \end{bmatrix} k'\right) c\left(\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}\begin{bmatrix} 1 & \\ c & 1 \end{bmatrix}, k'\right)(a, d'). \end{aligned} \tag{23}$$

Now (22) has already been verified in general for upper triangular k. Applying this observation to the first term on the left hand side and the first term on the right hand side, using the cocycle property (6), and using the $(\mathfrak{o}^\times, 1+4\mathfrak{o}) = 1$ rule of (ii) of Lemma 1.2, we find that (23) reduces to

$$y(k') = y\left(\begin{bmatrix} 1 & \\ c & 1 \end{bmatrix} k'\right) c\left(\begin{bmatrix} 1 & \\ c & 1 \end{bmatrix}, k'\right). \tag{24}$$

Writing $k' = \begin{bmatrix} a'd' & b'd'^{-1} \\ c'd' & 1 \end{bmatrix}\begin{bmatrix} d'^{\,-1} & \\ & d' \end{bmatrix}$ and using a similar argument, (24) reduces to

$$\begin{aligned} 1 &= y\left(\begin{bmatrix} 1 & \\ c & 1 \end{bmatrix}\begin{bmatrix} a' & b' \\ c' & 1 \end{bmatrix}\right) c\left(\begin{bmatrix} 1 & \\ c & 1 \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & 1 \end{bmatrix}\right) \\ &= y\left(\begin{bmatrix} a' & b' \\ c' + ca' & 1 + cb' \end{bmatrix}\right) c\left(\begin{bmatrix} 1 & \\ c & 1 \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & 1 \end{bmatrix}\right). \end{aligned} \tag{25}$$

Assume that $c' + ca' = 0$. Then (25) is equivalent to

$$1 = (c, c')(-cc', 1 + cb'). \tag{26}$$

Consider the second Hilbert symbol. Since $c' = -ca'$, $\mathrm{val}(-cc')$ is even. Hence, the second Hilbert symbol is 1 because of the $(\mathfrak{o}^\times, 1 + 4\mathfrak{o}) = 1$ rule. Using the determinant condition $a' - b'c' = 1$ and $c' + ca' = 0$, we get $c' = -(1 + b'c)^{-1}c$. Therefore, $(c, c') = (c, -(1 + b'c)^{-1}c) = (c, 1 + b'c)$. If $\mathrm{val}(c) = 2\mathrm{val}(2)$, this is of the form $(\mathfrak{o}^\times, 1 + 4\mathfrak{o}) = 1$. If $\mathrm{val}(c) > 2\mathrm{val}(2)$, then $(c, 1 + b'c) = 1$ by the $(F^\times, 1 + 4\varpi\mathfrak{o}) = 1$ rule of (i) of Lemma 1.2, so that $(c, c') = 1$. Hence $(c, c') = 1$ in both cases, and (26) is verified.

Assume that $\mathrm{val}(c' + ca')$ is non-zero. Applying the definitions of y and c shows that (25) is equivalent to

$$1 = (1 + cb', -1)(c, c')(-cc', c' + ca')(c' + ca', 1 + cb'). \tag{27}$$

The first Hilbert symbol is 1 by the $(\mathfrak{o}^\times, 1 + 4\mathfrak{o}) = 1$ rule. Using the determinant condition $a' - b'c' = 1$ to eliminate a', we get

$$1 = (c, c')(-cc', c + c' + cc'b')(1 + cb', c + c' + cc'b'). \tag{28}$$

Assume that $\mathrm{val}(c') > \mathrm{val}(c)$. Then $c + c' + cc'b' = c(1 + c^{-1}c')\left(1 + \frac{c'b'}{1+c^{-1}c'}\right)$. Since $\mathrm{val}(c') > \mathrm{val}(c) \geq 2\mathrm{val}(2)$, we have $\left(1 + cb', 1 + \frac{c'b'}{1+c^{-1}c'}\right) = 1$ by the $(F^\times, 1+4\varpi\mathfrak{o}) = 1$ rule; also, $1 + c^{-1}c'$ is in $\mathfrak{o}^\times$. Hence, we have to show

$$1 = (c, c')(-cc', c(1 + c^{-1}c'))(1 + cb', c(1 + c^{-1}c')), \tag{29}$$

which is

$$1 = (-cc', 1 + c^{-1}c')(1 + cb', c(1 + c^{-1}c')). \tag{30}$$

The first Hilbert symbol is 1 since $(x, 1 - x) = 1$ for all x in F such that x and $1 - x$ are in $F^\times$. Hence we are reduced to

$$1 = (1 + cb', 1 + c^{-1}c')(1 + cb', c). \tag{31}$$

The first Hilbert symbol is 1 by the $(\mathfrak{o}^\times, 1 + 4\mathfrak{o}) = 1$ rule. If $\mathrm{val}(c) > 2\mathrm{val}(2)$, then the second Hilbert symbol is also 1 by the $(F^\times, 1 + 4\varpi\mathfrak{o}) = 1$ rule. If $\mathrm{val}(c) = 2\mathrm{val}(2)$, which is even, then the second Hilbert symbol is 1 by the $(\mathfrak{o}^\times, 1 + 4\mathfrak{o}) = 1$ rule. Hence (31) is verified.

Now assume that $\mathrm{val}(c) > \mathrm{val}(c')$. Then $c + c' + cc'b' = c'(1 + c'^{-1}c)\left(1 + \frac{cb'}{1+c'^{-1}c}\right)$. Again, $1 + c'^{-1}c$ is in $\mathfrak{o}^\times$ and $1 + \frac{cb'}{1+c'^{-1}c}$ is in $1 + 4\varpi\mathfrak{o}$. Hence we have to show

$$1 = (c, c')(-cc', c'(1 + c'^{-1}c))(1 + cb', c'(1 + c'^{-1}c)), \tag{32}$$

which is

$$1 = (-cc', 1 + c'^{-1}c)(1 + cb', c'(1 + c'^{-1}c)). \tag{33}$$

The first Hilbert symbol is 1 by the $(x, 1 - x) = 1$ rule. Hence we are reduced to

$$1 = (1 + cb', 1 + c'^{-1}c)(1 + cb', c'). \tag{34}$$

The first Hilbert symbol is 1 by the $(\mathfrak{o}^\times, 1 + 4\mathfrak{o}) = 1$ rule. Since $\mathrm{val}(c) > 2\mathrm{val}(2)$, the element $1 + cb'$ is in $1 + 4\varpi\mathfrak{o}$, and again the second Hilbert symbol is also 1 by the $(F^\times, 1 + 4\varpi\mathfrak{o}) = 1$ rule. Hence (34) is verified.

Finally, assume that $\mathrm{val}(c) = \mathrm{val}(c')$. Write $c = u\varpi^k$ and $c' = v\varpi^k$ with u and v in $\mathfrak{o}^\times$ and $k \geq n \geq 2\mathrm{val}(2)$. Then (28) is equivalent to

$$1 = (u, v)(-uv, u + v + uvb'\varpi^k)(1 + u\varpi^k b', \varpi^k)(1 + u\varpi^k b', u + v + uvb'\varpi^k). \tag{35}$$

If $k > 2\mathrm{val}(2)$, this simplifies to

$$1 = (-uv, 1 + u^{-1}v + vb'\varpi^k). \tag{36}$$

But

$$\begin{aligned}(-uv, 1 + u^{-1}v + vb'\varpi^k) =& (-u^{-1}v, 1 + u^{-1}v + vb'\varpi^k)\\ &\cdot (1 + ub'\varpi^k, 1 + u^{-1}v + vb'\varpi^k)\\ =& (-u^{-1}v - vb'\varpi^k, 1 + u^{-1}v + vb'\varpi^k)\\ =& 1\end{aligned}$$

by the $(1-x,x)=1$ rule. Hence (35) is verified if $k>2\mathrm{val}(2)$. Assume now that $k=2\mathrm{val}(2)$, so that in particular k is even. Then (35) is equivalent to

$$1=(-uv,1+u^{-1}v+vb'\varpi^k)(1+u\varpi^k b',1+u^{-1}v+vb'\varpi^k). \tag{37}$$

If $u+v$ is in $\mathfrak{o}^\times$, then this is equivalent to

$$1=(-uv,1+u^{-1}v). \tag{38}$$

This is true by the $(1-x,x)=1$ rule. Assume that $u+v$ is in $\mathfrak{p}$. Write $v=u(-1+w\varpi^t)$ with w in $\mathfrak{o}^\times$ and $t\geq 1$. Substituting $u^{-1}v=-1+w\varpi^t$ and $-uv=u^2(1-w\varpi^t)$ into (37), we get

$$1=(1-w\varpi^t,w\varpi^t+vb'\varpi^k)(1+u\varpi^k b',w\varpi^t+vb'\varpi^k). \tag{39}$$

Since u is in $-v+\mathfrak{p}$, the second Hilbert symbol equals $(1-v\varpi^k b',w\varpi^t+vb'\varpi^k)$ by the $(F^\times,1+4\varpi\mathfrak{o})=1$ rule. Hence (39) is equivalent to

$$1=((1-w\varpi^t)(1-v\varpi^k b'),w\varpi^t+vb'\varpi^k). \tag{40}$$

Multiplying out, we get

$$1=(1-w\varpi^t-vb'\varpi^k+vw\varpi^{t+k}b',w\varpi^t+vb'\varpi^k). \tag{41}$$

The term $vw\varpi^{t+k}b'$ can be omitted by the $(F^\times,1+4\varpi\mathfrak{o})=1$ rule because $t+k>2\mathrm{val}(2)$. Then (41) holds by the $(x,1-x)=1$ rule. This completes the proof. □

Proposition 2.5. *Let (τ,V) be a smooth, genuine representation of $\widetilde{\mathrm{SL}}(2,F)$, let χ be a character of $\mathfrak{o}^\times$, and let n be an integer. If $V_\psi(\tau,n,\chi)$ is non-zero, then $n\geq 2\mathrm{val}(2)$, χ is trivial on $1+\mathfrak{p}^n$, and the character $\tilde{\chi}$ of $\tilde{\Gamma}_0(\mathfrak{p}^n)$ defined in* (20) *is the character f from Lemma* 2.4.

Proof. Assume that $V_\psi(\tau,n,\chi)$ is non-zero. Then $n\geq 2\mathrm{val}(2)$ and χ is trivial on $1+\mathfrak{p}^n$ by Lemma 2.3. To prove that $\tilde{\chi}$ is f it suffices to prove that these two characters agree on the elements in (1), (2) and (3). This follows from the involved formulas. □

3 Proof of the main theorem

In this section we prove the main theorem. To do so, we will first make two algebraic reductions, and then use the Kirillov-type model from Theorem 1.4. Making the reductions requires some definitions and facts. Let (τ,V) be a smooth, genuine representation of $\widetilde{\mathrm{SL}}(2,F)$ and let χ be a character of $\mathfrak{o}^\times$. We define $V_\psi(\tau,\infty,\chi)$ to be the union of all the spaces $V_\psi(\tau,n,\chi)$ as n runs over the integers. The set $V_\psi(\tau,\infty,\chi)$ is a subspace of V because the $V_\psi(\tau,n,\chi)$ are an ascending sequence of vector spaces. Because τ is a smooth representation, a vector v in V is contained in $V_\psi(\tau,\infty,\chi)$ if and only if (1) and (2) hold. We define

$$\alpha_2: V_\psi(\tau,\infty,\chi)\to V_\psi(\tau,\infty,\chi)$$

by the formula (4). For all n, this operator extends the level raising operator α_2 from $V_\psi(\tau,n,\chi)$ to $V_\psi(\tau,n+2,\chi)$. The following lemma characterizes vectors in the image of α_2.

Lemma 3.1. *Let (τ,V) be a smooth, genuine representation of $\widetilde{\mathrm{SL}}(2,F)$ and let χ be a character of $\mathfrak{o}^\times$. Let n be an integer, and let v be in $V_\psi(\tau,n,\chi)$. The following statements are equivalent:*

(i) *There exists w in $V_\psi(\tau,n-2,\chi)$ such that $v=\alpha_2 w$.*
(ii) *There exists w in $V_\psi(\tau,\infty,\chi)$ such that $v=\alpha_2 w$.*
(iii) *The vector v is invariant under the subgroup*

$$\left(\begin{bmatrix}1&\mathfrak{p}^{-2}\\&1\end{bmatrix},1\right). \tag{42}$$

Proof. (i) $\Longrightarrow$ (ii). This is clear.

(ii) $\Longrightarrow$ (iii). Suppose that $v=\alpha_2 w$ for some w in $V_\psi(\tau,\infty,\chi)$. We have

$$\tau\left(\begin{bmatrix}1&\mathfrak{p}^{-2}\\&1\end{bmatrix},1\right)v=\alpha_2\tau\left(\begin{bmatrix}1&\mathfrak{o}\\&1\end{bmatrix},1\right)w=\alpha_2 w=v,$$

so that v is invariant under the group in (42).

(iii) $\Longrightarrow$ (ii). Assume that v is invariant under the group in (42). Define

$$w=\tau\left(\begin{bmatrix}\varpi^{-1}&\\&\varpi\end{bmatrix},1\right)^{-1}v=\tau\left(\begin{bmatrix}\varpi&\\&\varpi^{-1}\end{bmatrix},(\varpi,\varpi)\right)v,$$

so that $v=\alpha_2 w$. The condition (2) for w is satisfied because the subgroup of elements of $\widetilde{\mathrm{SL}}(2,F)$ whose first components are diagonal matrices is commutative. Also,

$$\begin{aligned}\tau\left(\begin{bmatrix}1&\mathfrak{o}\\&1\end{bmatrix},1\right)w&=\tau\left(\begin{bmatrix}\varpi&\\&\varpi^{-1}\end{bmatrix},(\varpi,\varpi)\right)\tau\left(\begin{bmatrix}1&\mathfrak{p}^{-2}\\&1\end{bmatrix},1\right)v\\&=\tau\left(\begin{bmatrix}\varpi&\\&\varpi^{-1}\end{bmatrix},(\varpi,\varpi)\right)v\\&=w,\end{aligned}$$

and

$$\begin{aligned}\tau\left(\begin{bmatrix}1&\\\mathfrak{p}^{n-2}&1\end{bmatrix},1\right)w&=\tau\left(\begin{bmatrix}\varpi&\\&\varpi^{-1}\end{bmatrix},(\varpi,\varpi)\right)\tau\left(\begin{bmatrix}1&\\\mathfrak{p}^{n}&1\end{bmatrix},1\right)v\\&=\tau\left(\begin{bmatrix}\varpi&\\&\varpi^{-1}\end{bmatrix},(\varpi,\varpi)\right)v\\&=w.\end{aligned}$$

It follows that w is in $V_\psi(\tau,n-2,\chi)$. □

The first reduction proves that the sum from the main theorem can be written in terms of $V_\psi(\tau,\infty,\chi)$.

Lemma 3.2. *Let* (τ, V) *be a smooth, genuine representation of* $\widetilde{\mathrm{SL}}(2,F)$ *and let* χ *be a character of* $\mathfrak{o}^\times$. *The vector spaces* $V_\psi(\tau,\infty,\chi)/\alpha_2 V_\psi(\tau,\infty,\chi)$ *and* $\oplus_n V_\psi(\tau,n,\chi)_{\mathrm{new}}$ *have the same dimension.*

Proof. Let n be an integer, and consider the natural map

$$V_\psi(\tau,n,\chi) \to V_\psi(\tau,n+1,\chi)/\alpha_2 V_\psi(\tau,n-1,\chi)$$

induced by the inclusion of $V_\psi(\tau,n,\chi)$ into $V_\psi(\tau,n+1,\chi)$. The kernel of this map is $V_\psi(\tau,n,\chi) \cap \alpha_2 V_\psi(\tau,n-1,\chi)$. The (ii) $\Longrightarrow$ (i) assertion of Lemma 3.1 implies that this subspace is $\alpha_2 V_\psi(\tau,n-2,\chi)$. It follows that there is a sequence of inclusions

$$\cdots \hookrightarrow V_\psi(\tau,n,\chi)/\alpha_2 V_\psi(\tau,n-2,\chi) \hookrightarrow V_\psi(\tau,n+1,\chi)/\alpha_2 V_\psi(\tau,n-1,\chi) \hookrightarrow \cdots.$$

If $n \leq 2\mathrm{val}(2)-1$, then the n-th term of the sequence is zero by Lemma 2.2. For each integer n, we also have a natural map

$$V_\psi(\tau,n,\chi) \to V_\psi(\tau,\infty,\chi)/\alpha_2 V_\psi(\tau,\infty,\chi).$$

The (ii) $\Longrightarrow$ (i) assertion of Lemma 3.1 also implies that the kernel of this map is $\alpha_2 V_\psi(\tau,n-2,\chi)$, so that there is an inclusion

$$V_\psi(\tau,n,\chi)/\alpha_2 V_\psi(\tau,n-2,\chi) \hookrightarrow V_\psi(\tau,\infty,\chi)/\alpha_2 V_\psi(\tau,\infty,\chi).$$

Let W_n be the image of $V_\psi(\tau,n,\chi)/\alpha_2 V_\psi(\tau,n-2,\chi)$. We have a sequence of subspaces

$$\cdots \subset W_n \subset W_{n+1} \subset \cdots \subset V_\psi(\tau,\infty,\chi)/\alpha_2 V_\psi(\tau,\infty,\chi)$$

with $W_n = 0$ for $n \leq 2\mathrm{val}(2)-1$, and the subspace generated by all the W_n is $V_\psi(\tau,\infty,\chi)/\alpha_2 V_\psi(\tau,\infty,\chi)$. Therefore, the vector space $V_\psi(\tau,\infty,\chi)/\alpha_2 V_\psi(\tau,\infty,\chi)$ has the same dimension as the direct sum $\oplus_n W_{n+1}/W_n$. But

$$W_{n+1}/W_n \cong V_\psi(\tau,n+1,\chi)/\big(V_\psi(\tau,n,\chi)+\alpha_2 V_\psi(\tau,n-1,\chi)\big),$$

and this last space is just $V_\psi(\tau,n+1,\chi)_{\mathrm{new}}$, by definition. Therefore, the dimension of $V_\psi(\tau,\infty,\chi)/\alpha_2 V_\psi(\tau,\infty,\chi)$ is the same as the dimension of the direct sum $\oplus_n V_\psi(\tau,n+1,\chi)_{\mathrm{new}}$. □

The previous lemma reduces proving the main theorem to computing the dimension of $V_\psi(\tau,\infty,\chi)/\alpha_2 V_\psi(\tau,\infty,\chi)$. The next lemma reduces the computation of the dimension of this quotient to the computation of the dimension of a concrete space of metaplectic vectors. In the lemma we use the Haar measure on F that assigns $\mathfrak{o}$ volume 1.

Lemma 3.3. *Let* (τ, V) *be a smooth, genuine representation of* $\widetilde{\mathrm{SL}}(2,F)$ *and let* χ *be a character of* $\mathfrak{o}^\times$. *If* v *is in* $V_\psi(\tau,\infty,\chi)$, *then*

$$\mu v = \frac{1}{q^2}\int\limits_{\mathfrak{p}^{-2}} \tau\left(\begin{bmatrix}1 & x\\ & 1\end{bmatrix}, 1\right) v\,dx$$

is contained in $V_\psi(\tau,\infty,\chi)$*, so that this formula defines a linear map*

$$\mu : V_\psi(\tau,\infty,\chi) \to V_\psi(\tau,\infty,\chi).$$

The operator μ *is a projection, i.e.,* $\mu^2=\mu$*, and* $V_\psi(\tau,\infty,\chi) = \ker\mu \oplus \mathrm{im}\,\mu$*. Moreover, the image of* μ *is* $\alpha_2 V_\psi(\tau,\infty,\chi)$*, and hence there is a natural isomorphism*

$$\ker\mu \xrightarrow{\sim} V_\psi(\tau,\infty,\chi)/\alpha_2 V_\psi(\tau,\infty,\chi).$$

Proof. It is straightforward to verify that μv is contained in $V_\psi(\tau,\infty,\chi)$ for v in $V_\psi(\tau,\infty,\chi)$, and a direct computation shows that $\mu^2=\mu$. It is clear from the definition of μ that the vectors in the image of μ are invariant under the group (42), so that such vectors are contained in $\alpha_2 V_\psi(\tau,\infty,\chi)$ by the implication (iii) $\Longrightarrow$ (ii) of Lemma 3.1. Conversely, if v is in $V_\psi(\tau,\infty,\chi)$, then a computation shows that $\mu\alpha_2 v = \alpha_2 v$, so that $\alpha_2 v$ is contained in the image of μ. □

If (τ,V) is a smooth, genuine representation of $\widetilde{\mathrm{SL}}(2,F)$ and χ is a character of $\mathfrak{o}^\times$, then we denote the kernel of μ by $V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)$, and refer to the elements of $V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)$ as *primitive vectors*. By the previous two lemmas, proving the main theorem is reduced to computing the dimension of the space of primitive vectors. In the following proof of the main theorem we compute the dimension of the space of primitive vectors using the Kirillov-type model from Theorem 1.4. It is at this point that we use the assumption from the main theorem that τ is irreducible.

Proof of the Main Theorem. Assume $\chi(-1)\neq\varepsilon(\tau,\psi)$. Let n be an integer and assume that v is in $V_\psi(\tau,n,\chi)$. Then by the definition of $V_\psi(\tau,n,\chi)$ we have

$$\tau\left(\begin{bmatrix}-1&\\&-1\end{bmatrix},1\right)v = \chi(-1)\delta_1(-1)v.$$

On the other hand, by the definition of $\varepsilon(\tau,\psi)$,

$$\tau\left(\begin{bmatrix}-1&\\&-1\end{bmatrix},1\right)v = \varepsilon(\tau,\psi)\delta_1(-1)v.$$

Since $\chi(-1)\neq\varepsilon(\tau,\psi)$ we must have $v=0$.

Assume that $\chi(-1)=\varepsilon(\tau,\psi)$. By Lemma 3.2 and Lemma 3.3, it suffices to prove that

$$\dim V_{\psi,\mathrm{prim}}(\tau,\infty,\chi) = \#F_\psi(\tau)/F^{\times 2}.$$

The character χ of $\mathfrak{o}^\times$ admits an extension to a character of $F^\times$ that we will also refer to as χ. We will use the Kirillov-type model $\mathcal{M}(\tau,\chi)$ for τ from Theorem 1.4. We recall that:

(i) The vectors in $\mathcal{M}(\tau,\chi)$ are certain functions $f: F^\times\to\mathbb{C}$ that are locally constant, have relatively compact support in F, and are supported in $F_\psi(\tau)$; moreover, the space $\mathcal{S}(F_\psi(\tau))$ of locally constant, compactly supported functions on $F_\psi(\tau)$ is contained in $\mathcal{M}(\tau,\chi)$.

(ii) For f in $\mathcal{M}(\tau,\chi)$, n in F and x in $F^\times$ we have

$$\tau\left(\begin{bmatrix}1 & n\\ & 1\end{bmatrix},1\right)f(x)=\psi(nx)f(x).$$

(iii) For f in $\mathcal{M}(\tau,\chi)$, a in $F^\times$ and x in $F^\times$ we have

$$\tau\left(\begin{bmatrix}a & \\ & a^{-1}\end{bmatrix},1\right)f(x)=\delta_1(a)\chi(a)f(a^2x).$$

From (ii), (1), and the fact that ψ has conductor $\mathfrak{o}$, we see that if f is in $V_\psi(\tau,\infty,\chi)$, then the support of f is contained in $\mathfrak{o}$; from (iii) and (2), we see that $f(v^2x)=f(x)$ for all x in $F^\times$ and v in $\mathfrak{o}^\times$. Now let f be in $V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)$. Then

$$0=\mu f=\frac{1}{q^2}\int\limits_{\mathfrak{p}^{-2}}\tau\left(\begin{bmatrix}1 & x\\ & 1\end{bmatrix}\right)f\,dx.$$

Hence, for all y in $\mathfrak{o}$,

$$0=\left(\int\limits_{\mathfrak{p}^{-2}}\psi(xy)\,dx\right)f(y).$$

Since the conductor of ψ is $\mathfrak{o}$, f is supported on $\mathfrak{o}^\times\sqcup\varpi\mathfrak{o}^\times$. Using this and (i), it follows that f is determined by its values on the set

$$\big(F_\psi(\tau)\cap\mathfrak{o}^\times\big)/\mathfrak{o}^{\times 2}\sqcup\big(F_\psi(\tau)\cap\varpi\mathfrak{o}^\times\big)/\mathfrak{o}^{\times 2}. \tag{43}$$

The natural map from this set to $F_\psi(\tau)/F^{\times 2}$ is a bijection. Therefore, the dimension of the vector space $V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)$ is at most $\#F_\psi(\tau)/F^{\times 2}$. Conversely, suppose that $t\mathfrak{o}^{\times 2}$ is in the set in (43) with t in $F_\psi(\tau)\cap\mathfrak{o}^\times$ or in $F_\psi(\tau)\cap\varpi\mathfrak{o}^\times$. Let $f_{t\mathfrak{o}^{\times 2}}$ be the characteristic function of $t\mathfrak{o}^{\times 2}$. This function lies in the model $\mathcal{M}(\tau,\chi)$ by i). Moreover, a calculation shows that $f_{t\mathfrak{o}^{\times 2}}$ is in $V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)$. The functions $f_{t\mathfrak{o}^{\times 2}}$ as $t\mathfrak{o}^{\times 2}$ varies over the set (43) are linearly independent elements of $V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)$. Therefore, the dimension of $V_{\psi,\mathrm{prim}}(\tau,\infty,\chi)$ is at least $\#F_\psi(\tau)/F^{\times 2}$. This completes the proof. □

To end this paper we briefly describe how similar reasoning proves the analogous theorem in the GL(2) setting. Let (π,V) be a generic, irreducible, admissible representation of $\mathrm{GL}(2,F)$. For n a non-negative integer, let $V(\pi,n)$ be the subspace of vectors v in V that are stabilized by the subgroup of elements

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

of $\mathrm{GL}(2,\mathfrak{o})$ such that $c\equiv 0 \bmod \mathfrak{p}^n$ and $d\equiv 1 \bmod \mathfrak{p}^n$. Define $\alpha: V(\pi,n)\longrightarrow V(\pi,n+1)$ by

$$\alpha v=\pi\left(\begin{bmatrix}1 & \\ & \varpi\end{bmatrix}\right)v.$$

Define the subspace $V(\pi,n)_{\rm old}$ of *oldforms* in $V(\pi,n)$ as the subspace spanned by $V(\pi,n-1)$ and $\alpha V(\pi,n-1)$. Our goal is to prove that $\sum_n \dim V(\pi,n)/V(\pi,n)_{\rm old}$ is one. Define $V(\pi,\infty)$ to be the subspace that is the union of all the spaces $V(\pi,n)$. We have $\sum_n \dim V(\pi,n)/V(\pi,n)_{\rm old} = \dim V(\pi,\infty)/\alpha V(\pi,\infty)$, as in the $\widetilde{\rm SL}(2)$ case. Define $\mu_{{\rm GL}(2)} : V(\pi,\infty) \to V(\pi,\infty)$ by

$$\mu_{{\rm GL}(2)} v = \frac{1}{q} \int\limits_{\mathfrak{p}^{-1}} \pi\left(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\right) v\,dx.$$

The operator $\mu_{{\rm GL}(2)}$ is a well-defined projection, and $\ker \mu_{{\rm GL}(2)}$ is isomorphic to $V(\pi,\infty)/\alpha V(\pi,\infty)$, so that we are reduced to proving that the space $\ker \mu_{{\rm GL}(2)}$ of primitive vectors is one-dimensional. A computation now shows that if the space of π is taken to be the Kirillov model of π with respect to ψ, then the space of primitive vectors is spanned by the characteristic function of $\mathfrak{o}^\times$, which completes the proof. In closing, we note that if π is supercuspidal, then the characteristic function of $\mathfrak{o}^\times$ in the Kirillov model with respect to ψ is the newform of π; the above development shows that this vector is also significant in the non-supercuspidal case.

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