Dirichlet characters and Dirichlet's theorem

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Abstract

In this expository talk we will recall the L-series associated to a Dirichlet character, and we will describe how these L-series can be used to prove Dirichlet's famous theorem on primes in an arithmetic progression.

We will also indicate some further directions.

Who was Dirichlet?

- Dirichlet was born in a small town close to Cologne, lived from 1805 to 1859, and died at age 54.
- Dirichlet's full name was Johann Peter Gustav Lejeune Dirichlet. The surname "Lejeune Dirichlet" is a version of the French "le jeune de Richelette" ("the youth from Richelette").
- Dirichlet failed to get an Abitur (academic high school degree) in Cologne because of a poor knowledge of Latin.
- Dirichlet studied university level mathematics in Paris.
- Dirichlet's thesis was on the $n = 5$ and $n = 14$ cases of Fermat's Last Theorem (this was a significant accomplishment). However, the University of Bonn couldn't awarded him a PhD because he couldn't defend in Latin; instead, he was awarded an honorary PhD.
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- Dirichlet made fundamental contributions in number theory and analysis.

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Some timelines

Some other mathematicians overlapping with Dirichlet are:

Fourier	1768	1830	(62)
Gauss	1777	1855	(77)
Poisson	1781	1840	(58)
Cauchy	1789	1857	(67)
Dirichlet	1805	1859	(54)
Galois	1811	1832	(20)
Weiestrass	1815	1897	(81)
Kronecker	1823	1891	(68)
Riemann	1826	1866	(39)
Dedekind	1831	1916	(84)

Dirichlet's theorem on primes in arithmetic progressions

In this talk we will describe how one can prove the following theorem:

Theorem (Dirichlet)

Let N be a positive integer, and let a be an integer that is relatively prime to N, i.e., $(a, N) = 1$. Then the arithmetic progression

$$(a + kN)_{k=1}^{\infty}$$

contains infinitely many primes. In fact,

$$\sum_{\substack{p \text{ prime} \\ p \equiv a \pmod{N}}} \frac{1}{p} = \infty.$$

Example: $N = 15$, $a = 7$

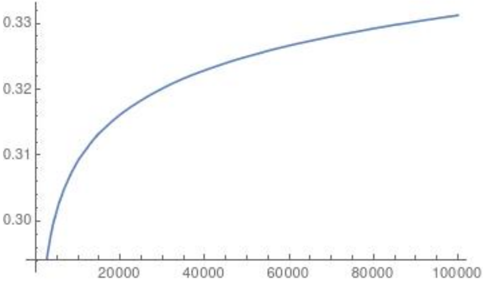
Let $N = 15$ and $a = 7$. We list (k, $a + kN$, prime true or false):

(0, 7, True), (1, 22, False), (2, 37, True), (3, 52, False), (4, 67, True), (5, 82, False), (6, 97, True), (7, 112, False), (8, 127, True), (9, 142, False), (10, 157, True), (11, 172, False), (12, 187, False), (13, 202, False), (14, 217, False), (15, 232, False), (16, 247, False), (17, 262, False), (18, 277, True), (19, 292, False), (20, 307, True), (21, 322, False), (22, 337, True), (23, 352, False), (24, 367, True), (25, 382, False), (26, 397, True), (27, 412, False), (28, 427, False), (29, 442, False), (30, 457, True), (31, 472, False), (32, 487, True), (33, 502, False), (34, 517, False), (35, 532, False), (36, 547, True), (37, 562, False), (38, 577, True), (39, 592, False), (40, 607, True), (41, 622, False), (42, 637, False), (43, 652, False), (44, 667, False), (45, 682, False), (46, 697, False), (47, 712, False), (48, 727, True), (49, 742, False), (50, 757, True), ...

Example: $N = 15$, $a = 7$, continued

Here is a graph of

$$f(x) = \sum_{\substack{0 \le k \le x, \\ 7+15k \text{ is prime}}} \frac{1}{7+15k}:$$



Actually, this graph doesn't convince one that $\lim_{x \to \infty} f(x) = \infty$ ...

Characters

To prove Dirichlet's theorem we will use a number of ingredients.

First of all, we need a bit of background from group theory. Let G be a finite abelian group. A **character** of G is a homomorphism

$$\chi : G \longrightarrow \mathbb{C}^\times.$$

The set

$$\mathrm{Hom}(G, \mathbb{C}^\times)$$

of all characters of G is also an abelian group under pointwise multiplication of characters.

We can determine the structure of $\mathrm{Hom}(G, \mathbb{C}^\times)$. Write

$$G \cong Z_{a_1} \times \cdots \times Z_{a_n}$$

where $a_1, \ldots, a_n$ are integers greater than 1 such that $a_1 \mid \cdots \mid a_n$ and Z_{a_i} is the cyclic group of order a_i.

Characters, continued

We observe that if x_j is a generator for Z_{a_j}, and χ is a character of G, then

$$\chi(x_j)^{a_j} = \chi(x_j^{a_j}) = \chi(1) = 1.$$

Hence,

$$\chi(x_j) \in \mu_{a_j} = \left\{ \begin{array}{l} \text{the group of } a_j\text{-th roots of unity} \\ e^{2\pi i k/a_j},\ 0 \leq k \leq a_j - 1, \text{ in } \mathbb{C}^\times \end{array} \right\}.$$

It turns out that the map

$$\mathrm{Hom}(G, \mathbb{C}^\times) \longrightarrow \mu_{a_1} \times \cdots \times \mu_{a_n}, \qquad \chi \mapsto (\chi(x_1), \ldots, \chi(x_n))$$

is actually an isomorphism of groups. Thus,

$$\mathrm{Hom}(G, \mathbb{C}^\times) \cong \mu_{a_1} \times \cdots \times \mu_{a_n} \cong Z_{a_1} \times \cdots \times Z_{a_n} \cong G,$$

and in particular, $\#\mathrm{Hom}(G, \mathbb{C}) = \#G$.

Dirichlet characters

Now let N be a positive integer. A **Dirichlet character** mod N is a character of $(\mathbb{Z}/N\mathbb{Z})^\times$, and is thus a homomorphism

$$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times.$$

Here, $(\mathbb{Z}/N\mathbb{Z})^\times$ is the group of units in the ring $\mathbb{Z}/N\mathbb{Z}$.

Thus, the elements of $(\mathbb{Z}/N\mathbb{Z})^\times$ are the $a + N\mathbb{Z}$ such that a is relatively prime to N and this is a group under multiplication.

From above, we have

$$\#\mathrm{Hom}((\mathbb{Z}/N\mathbb{Z})^\times, \mathbb{C}^\times) = \#\,(\mathbb{Z}/N\mathbb{Z})^\times.$$

This is given by the Euler function

$$\phi(N) = N \prod_{p|N} \left(1 - \frac{1}{p}\right)$$

where p runs over the primes dividing N.

An example: $N = 15$

Assume that $N = 15$. Then

$$\begin{aligned}\phi(15) &= 15 \prod_{p|15} \left(1 - \frac{1}{p}\right) \\ &= 15 \cdot \left(1 - \frac{1}{3}\right) \cdot \left(1 - \frac{1}{5}\right) \\ &= 15 \cdot \frac{2}{3} \cdot \frac{4}{5} \\ &= 8.\end{aligned}$$

Thus, there are 8 Dirichlet characters mod 15:

$$\chi_0, \quad \chi_1, \quad \chi_2, \quad \dots, \quad \chi_6, \quad \chi_7$$

Here, and in the rest of the talk, χ_0 will be the **principal character** mod N that sends every element of $(\mathbb{Z}/N\mathbb{Z})^\times$ to 1. We will see this example again.

Dirichlet characters, continued

One might now say that there isn't there isn't much more to Dirichlet characters ....

But this turns out to be very, very wrong ....

- Since the domain of χ is $(\mathbb{Z}/N\mathbb{Z})^\times$, there are potential connections to number theory.
- Since the codomain of χ is $\mathbb{C}^\times$, there are potential connections to complex analysis.

In fact ... when these connections are made, then one of the deepest open problem in mathematics concerns the principal character χ_0! (The famous **Riemann hypothesis**). In addition, Dirichlet characters are the prototypes for modular forms.

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Dirichlet series

To make the suggested connections we need to introduce another concept.

Let $(a(n))_{n=1}^{\infty}$ be a sequence of complex numbers. The **Dirichlet series** associated to $(a(n))_{n=1}^{\infty}$ is

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s}.$$

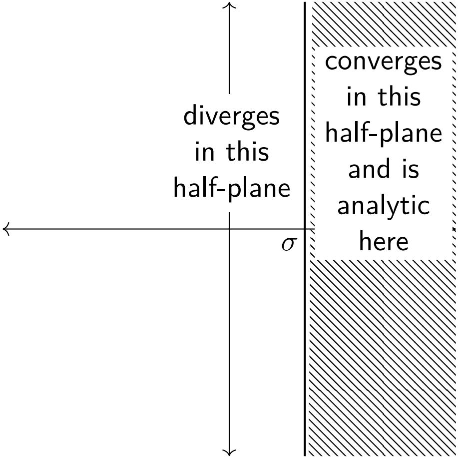
Here, $s \in \mathbb{C}$ and $n^s = e^{s \log(n)}$.

The convergence properties of this series are as follows:

If the series converges at s_0, then the series converges uniformly on compact sets for all s with $\mathrm{Re}(s) > \mathrm{Re}(s_0)$. Hence, there exists a real number σ (we also allow $\sigma = \pm\infty$) such that the series converges for $\mathrm{Re}(s) > \sigma$ and diverges for $\mathrm{Re}(s) < \sigma$.

Moreover, in the half-plane $\mathrm{Re}(s) > \sigma$ the series defines a holomorphic function of s.

The number σ is called **abscissa of convergence**; it is reminiscent of the radius of convergence of a power series (but differs in some important aspects). We have the following picture.



An important extra property: Euler products

Further assume that $(a(n))_{n=1}^{\infty}$ is **completely multiplicative**:

$$a(mn) = a(m)a(n), \qquad m, n \in \mathbb{Z}_{>0}.$$

Assume that $s \in \mathbb{C}$ is such that $\mathrm{Re}(s) > \sigma$ and in fact

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

converges absolutely. Then

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_{p \in P} \left(1 + \frac{a(p)}{p^s} + \frac{a(p)^2}{p^{2s}} + \cdots \right) = \prod_{p \in P} \frac{1}{1 - \dfrac{a(p)}{p^s}}$$

with each product converging absolutely. In this setting we say that the Dirichlet series admits an **Euler product**. Here, and below,

$$P = \text{the set of all primes} = \{2, 3, 5, \dots\}.$$

$L(s, \chi)$

We now apply this to Dirichlet characters.

Let χ be a Dirichlet character mod N.

To χ we associate a function $\chi : \mathbb{Z} \to \mathbb{C}$ via

$$\chi(n) = \begin{cases} \chi(n + N\mathbb{Z}) & \text{if } (n, N) = 1, \\ 0 & \text{if } (n, N) > 1. \end{cases}$$

It's straightforward to verify that $\chi : \mathbb{Z} \to \mathbb{C}$ is completely multiplicative.

We associate to χ the Dirichlet series

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

This is called the L-**series** or L-**function** (perhaps for "Lejeune") of χ.

The L-function $L(s, \chi)$ has the following analytic properties:

$L(s, \chi)$, continued

- The abscissa of convergence of $L(s, \chi_0)$ is $\sigma = 1$ and $L(s, \chi_0)$ extends to a unique meromorphic function on $\mathbb{C}$ with just a simple pole at $s = 1$.
- If $\chi \neq \chi_0$, then the abscissa of convergence of $L(s, \chi)$ is $\sigma = 0$, and $L(s, \chi)$ extends to a unique holomorphic function on $\mathbb{C}$.
- Define

$$\Lambda(s, \chi) = L(s, \chi) \times \begin{cases} \pi^{-s/2}\Gamma\left(\frac{s}{2}\right) & \text{if } \chi(-1) = 1, \\ \pi^{-(s+1)/2}\Gamma\left(\frac{s+1}{2}\right) & \text{if } \chi(-1) = -1. \end{cases}$$

Then

$$\Lambda(s, \chi) = \varepsilon N^{-s+1/2}\Lambda(1 - s, \bar{\chi}) \qquad \text{(the \textbf{functional equation})}$$

with ε a known constant with $|\varepsilon| = 1$.

An example: $L(s, \chi_0)$

Assume that $N = 1$ and consider χ_0. We have $\chi_0(n) = 1$ for all $n \in \mathbb{Z}_{>0}$. Hence, for $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 1$,

$$L(s, \chi_0) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) \qquad \text{(the } \textbf{Riemann zeta function}\text{)}$$

$$= \prod_{p \in P} \frac{1}{1 - \frac{1}{p^s}}.$$

Now assume N is arbitrary. Then for $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 1$,

$$L(s, \chi_0) = \sum_{n=1}^{\infty} \frac{\chi_0(n)}{n^s}$$

$$= \sum_{\substack{n=1 \\ (n,N)=1}}^{\infty} \frac{\chi_0(n)}{n^s}$$

An example: $L(s, \chi_0)$, continued

$$
\begin{aligned}
L(s,\chi_0) &= \sum_{\substack{n=1 \\ (n,N)=1}}^{\infty} \frac{\chi_0(n)}{n^s} \\
&= \prod_{p \in P} \frac{1}{1 - \frac{\chi_0(p)}{p^s}} \\
&= \prod_{\substack{p \in P \\ (p,N)=1}} \frac{1}{1 - \frac{1}{p^s}} \\
&= \prod_{p \mid N} \left(1 - \frac{1}{p^s}\right) \zeta(s).
\end{aligned}
$$

An example: $L(s, \chi_0)$, continued

From the formula

$$L(s, \chi_0) = \prod_{p|N} \left(1 - \frac{1}{p^s}\right) \zeta(s)$$

we see that $L(s, \chi_0)$ has a pole at $s = 1$ because $\zeta(s)$ has a pole at $s = 1$. Note that since

$$\log(t) = \int_1^t \frac{dx}{x},$$

since $\lim_{t\to\infty} \log(t) = \infty$, and since the formula for $\zeta(s)$ is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

it's not surprising that $\zeta(s)$ has pole at $s = 1$.

Some assumptions

Now we will indicate how to prove Dirichlet's theorem on primes in an arithmetic progression.

Because of time constraints we will need to assume two preliminary results.

To explain the first assumption, recall that if $\chi \neq \chi_0$, then the abscissa of convergence of $L(s, \chi)$ is $\sigma = 0$. Thus, if $\chi \neq \chi_0$, then

$$L(1, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n}$$

is a well-defined complex number.

We will need to assume the following fact:

If $\chi \neq \chi_0$, then $L(1, \chi) \neq 0$.

An example: $L(1, \chi)$ for $N = 15$

Let's look at this assumption for our example $N = 15$. There are 8 Dirichlet characters $\chi_0, \chi_1 \ldots, \chi_7$ mod 15. Calculating

$$\sum_{n=1}^{100{,}000} \frac{\chi_i(n)}{n}$$

for $i = 0, \ldots, 7$ one obtains (approximately; and $i = \sqrt{-1}$, as usual):

$$\begin{aligned}
\chi_0 &\rightsquigarrow \quad 6.95563, \\
\chi_1 &\rightsquigarrow \quad 0.796751 + 0.492424i, \\
\chi_2 &\rightsquigarrow \quad 0.573869, \\
\chi_3 &\rightsquigarrow \quad 0.796751 - 0.492424i, \\
\chi_4 &\rightsquigarrow \quad 0.725523, \\
\chi_5 &\rightsquigarrow \quad 0.738567 - 0.316793i, \\
\chi_6 &\rightsquigarrow \quad 1.62232, \qquad (\longleftarrow \text{we'll see this again!}) \\
\chi_7 &\rightsquigarrow \quad 0.738567 + 0.316793i.
\end{aligned}$$

Another ingredient

To prove Dirichlet's theorem we will need one more ingredient.

As in the statement of Dirichlet's theorem, fix a positive integer a such that $(a, N) = 1$.

For k a positive integer define the following set, which depends on a and k:

$$S_k = \{p \in P : p^k \equiv a \pmod{N}\}.$$

We then have

$$\frac{1}{\phi(N)} \sum_{\chi \pmod{N}} \bar{\chi}(a)\chi(p)^k = \begin{cases} 1 & \text{if } p \in S_k, \\ 0 & \text{if } p \notin S_k. \end{cases}$$

We leave the proof as an exercise.

It's actually this formula that connects Dirichlet characters to our problem.

The proof

We are now ready to prove Dirichlet's theorem.

Recall that N is a fixed positive integer, and a is a fixed positive integer such that $(a, N) = 1$. We want to prove that

$$\sum_{\substack{p \text{ prime} \\ p \equiv a \pmod{N}}} \frac{1}{p} = \infty$$

or equivalently using our new notation

$$\lim_{s \to 1^+} \sum_{p \in S_1} \frac{1}{p^s} = \infty.$$

The proof, continued

Now let $s \in \mathbb{C}$ be such that $\mathrm{Re}(s) > 1$. We will examine

$$\sum_{k=1}^{\infty} \sum_{p \in S_k} \frac{1}{kp^{ks}}$$

in two different ways. First:

$$\sum_{k=1}^{\infty} \sum_{p \in S_k} \frac{1}{kp^{ks}} = \sum_{k=1}^{\infty} \sum_{p \in S_k} \left(\frac{1}{\phi(N)} \sum_{\chi \pmod{N}} \bar{\chi}(a)\chi(p)^k \right) \frac{1}{kp^{ks}}$$

$$= \sum_{k=1}^{\infty} \sum_{p \in P} \left(\frac{1}{\phi(N)} \sum_{\chi \pmod{N}} \bar{\chi}(a)\chi(p)^k \right) \frac{1}{kp^{ks}}$$

$$= \frac{1}{\phi(N)} \sum_{\chi \pmod{N}} \bar{\chi}(a) \sum_{p \in P} \left(\sum_{k=1}^{\infty} \frac{\chi(p)^k}{kp^{ks}} \right)$$

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&= \sum_{k=1}^{\infty} \sum_{p \in P} \left(\frac{1}{\phi(N)} \sum_{\chi \pmod{N}} \bar{\chi}(a)\chi(p)^k \right) \frac{1}{kp^{ks}} \\
&= \frac{1}{\phi(N)} \sum_{\chi \pmod{N}} \bar{\chi}(a) \sum_{p \in P} \left(\sum_{k=1}^{\infty} \frac{\chi(p)^k}{kp^{ks}} \right)
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in two different ways. First:

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\sum_{k=1}^{\infty} \sum_{p \in S_k} \frac{1}{kp^{ks}} &= \sum_{k=1}^{\infty} \sum_{p \in S_k} \left(\frac{1}{\phi(N)} \sum_{\chi \pmod{N}} \bar{\chi}(a)\chi(p)^k \right) \frac{1}{kp^{ks}} \\
&= \sum_{k=1}^{\infty} \sum_{p \in P} \left(\frac{1}{\phi(N)} \sum_{\chi \pmod{N}} \bar{\chi}(a)\chi(p)^k \right) \frac{1}{kp^{ks}} \\
&= \frac{1}{\phi(N)} \sum_{\chi \pmod{N}} \bar{\chi}(a) \sum_{p \in P} \left(\sum_{k=1}^{\infty} \frac{\chi(p)^k}{kp^{ks}} \right)
\end{aligned}$$

The proof, continued

(Continuing the calculation...)

$$\sum_{k=1}^{\infty} \sum_{p \in S_k} \frac{1}{kp^{ks}} = \frac{1}{\phi(N)} \sum_{\chi \pmod{N}} \bar{\chi}(a) \sum_{p \in P} \left(\sum_{k=1}^{\infty} \frac{\chi(p)^k}{kp^{ks}} \right)$$

$$= \frac{1}{\phi(N)} \sum_{\chi \pmod{N}} \bar{\chi}(a) \sum_{p \in P} \log \left(\left(1 - \frac{\chi(p)}{p^s} \right)^{-1} \right)$$

$$= \frac{1}{\phi(N)} \sum_{\chi \pmod{N}} \bar{\chi}(a) \log \left(\prod_{p \in P} \left(1 - \frac{\chi(p)}{p^s} \right)^{-1} \right)$$

$$\sum_{k=1}^{\infty} \sum_{p \in S_k} \frac{1}{kp^{ks}} = \frac{1}{\phi(N)} \sum_{\chi \pmod{N}} \bar{\chi}(a) \log \left(L(s, \chi) \right).$$

The proof, continued

(Continuing the calculation...)

$$\sum_{k=1}^{\infty} \sum_{p \in S_k} \frac{1}{kp^{ks}} = \frac{1}{\phi(N)} \sum_{\chi \pmod N} \bar{\chi}(a) \sum_{p \in P} \left(\sum_{k=1}^{\infty} \frac{\chi(p)^k}{kp^{ks}} \right)$$

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The proof, continued

(Continuing the calculation...)

$$\sum_{k=1}^{\infty}\sum_{p\in S_k}\frac{1}{kp^{ks}} = \frac{1}{\phi(N)}\sum_{\chi \pmod{N}}\bar{\chi}(a)\sum_{p\in P}\left(\sum_{k=1}^{\infty}\frac{\chi(p)^k}{kp^{ks}}\right)$$

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(Continuing the calculation...)

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The proof, continued

So far we've proven that

$$\sum_{k=1}^{\infty}\sum_{p\in S_k}\frac{1}{kp^{ks}} = \frac{1}{\phi(N)}\sum_{\chi \pmod N}\bar{\chi}(a)\log\left(L(s,\chi)\right)$$

From before we have

$$\chi \neq \chi_0 \implies L(1,\chi)\neq 0 \implies \lim_{s\to 1^+}\log(L(s,\chi)) \quad \text{is well-defined,}$$
$$L(s,\chi_0) \text{ has a pole at } s=1 \implies \lim_{s\to 1^+}\log(L(s,\chi_0)) = \infty.$$

Hence,

$$\lim_{s\to 1^+}\sum_{k=1}^{\infty}\sum_{p\in S_k}\frac{1}{kp^{ks}} = \infty.$$

Now let's look at

$$\sum_{k=1}^{\infty} \sum_{p \in S_k} \frac{1}{kp^{ks}}$$

a different way. We have

$$\sum_{k=1}^{\infty} \sum_{p \in S_k} \frac{1}{kp^{ks}} = \sum_{p \in S_1} \frac{1}{p^s} + \sum_{k=2}^{\infty} \sum_{p \in S_k} \frac{1}{kp^{ks}}.$$

Let's consider the last sum when $s = 1$. We have

$$\sum_{k=2}^{\infty} \sum_{p \in S_k} \frac{1}{kp^k} \leq \sum_{k=2}^{\infty} \sum_{p \in P} \frac{1}{kp^k} = \sum_{p \in P} \sum_{k=2}^{\infty} \frac{1}{kp^k} \leq \sum_{p \in P} \sum_{k=2}^{\infty} \frac{1}{2p^k}$$

The proof, continued

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The proof, continued

(Continuing the estimate...)

$$\sum_{k=2}^{\infty} \sum_{p \in S_k} \frac{1}{kp^k} \leq \sum_{p \in P} \sum_{k=2}^{\infty} \frac{1}{2p^k} = \sum_{p \in P} \frac{1}{2p(p-1)} \leq \sum_{n=2}^{\infty} \frac{1}{2n(n-1)} = \frac{1}{2}.$$

We now conclude that

$$\lim_{s \to 1^+} \sum_{k=1}^{\infty} \sum_{p \in S_k} \frac{1}{kp^{ks}} = \lim_{s \to 1^+} \sum_{p \in S_1} \frac{1}{p^s} + 1/2$$

$$\infty = \lim_{s \to 1^+} \sum_{p \in S_1} \frac{1}{p^s} + 1/2$$

$$\infty = \lim_{s \to 1^+} \sum_{p \in S_1} \frac{1}{p^s}.$$

This is the desired result and completes the proof!

The proof, continued

(Continuing the estimate...)

$$\sum_{k=2}^{\infty} \sum_{p \in S_k} \frac{1}{kp^k} \leq \sum_{p \in P} \sum_{k=2}^{\infty} \frac{1}{2p^k} = \sum_{p \in P} \frac{1}{2p(p-1)} \leq \sum_{n=2}^{\infty} \frac{1}{2n(n-1)} = \frac{1}{2}.$$

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The proof, continued

(Continuing the estimate...)

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The proof, continued

(Continuing the estimate...)

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This is the desired result and completes the proof!

Another application of $L(1, \chi)$

Dirichlet also proved that if

- $D < 0$ is a **fundamental discriminant**,
- $h(D) =$ the class number of $\mathbb{Q}(\sqrt{D})$,
- $\chi = \left(\frac{D}{\cdot}\right)$ (Kronecker symbol), a Dirichlet character mod $|D|$,
- and

$$w = \begin{cases} 6 & \text{if } D = -3, \\ 4 & \text{if } D = -4, \\ 2 & \text{if } D < -4, \end{cases}$$

then

$$L(1, \chi) = -\frac{\pi}{|D|^{3/2}} \sum_{n=1}^{|D|-1} \chi(n)n$$

and

$$h(D) = \frac{\sqrt{|D|}(w/2)}{\pi} L(1, \chi).$$

The example $D = -15$

Let $D = -15$. Then D is a fundamental discriminant. Let $\chi = \left(\frac{-15}{\cdot}\right)$. Then χ is a Dirichlet character mod 15 and

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\chi(n)$	0	1	1	0	1	0	0	-1	1	0	0	-1	0	-1	-1

This is the character χ_6 from our previous list. Using the above formulas we find that:

$$L(1, \chi) = 1.62231... \qquad h(-15) = 2.$$

Further directions

- It's natural to consider the generalization of Dirichlet characters to any number field. These are called **Hecke characters**.
- It turns out that there are theorems and conjectures connecting Hecke characters and their values at special points like $s = 1$ to number theoretic entities (e.g., **Dedekind's class number formula**).
- It also happens that there are "higher dimensional" analogues of Hecke characters called modular forms. In important cases these are holomorphic functions on complex manifolds arising from the groups $\mathrm{GL}(2)$, $\mathrm{GSp}(2n)$, and other reductive algebraic groups over number fields. These have had important applications (they are essential to the proof of **Fermat's Last Theorem**).
- This area of mathematics uses lots of concepts from many parts of mathematics: it's pure mathematics applied to pure mathematics.

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