The final exam will cover the material presented in class from chapters 1, 2, 3, 4, 5, 6, 7, 8, and 10 of our text, and will take place on Friday, December 16, 2022 from 10:15 AM to 12:15 AM. For the exam you should:

be able to define:

- integral domain;
- field;
- Euclidean domain;
- principal ideal domain (PID);
- unique factorization domain (UFD);
- Noetherian ring;
- unit in a ring;
- zero divisor in a ring;
- nilpotent element of a ring;
- irreducible element in an integral domain;
- prime element in an integral domain;
- comaximal ideals;
- radical of an ideal;
- nilradical of R;
- ideal quotient (colon ideal);
- maximal ideal;
- prime ideal;
- primary ideal;
- decomposable ideal;
- irreducible ideal;
- minimal prime ideal of I;
- embedded prime ideal of I;
- minimal primary decomposition;
- $\mathrm{Spec}(R)$;
- $\mathrm{Var}(I)$;
- $\mathrm{Min}(I)$;
- $\mathrm{ass}_R(I)$;

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- multiplicatively closed set;
- the ring of fractions $S^{-1}R$;
- R_P if P is a prime ideal of R;
- quasi-local ring;
- quasi-semi-local ring;
- local ring;
- semi-local ring;
- R-module;
- $\mathrm{Ann}(M)$;
- short exact sequence;
- free R-module;
- $\mathrm{rank}(F)$ for F free R-module;
- Noetherian R-module;
- Artinian R-module;
- Artinian ring;
- simple module;
- composition series;
- composition factor;
- the length $\ell(M)$ of a composition series for M;
- Jacobson radical $\mathrm{Jac}(R)$;

be able to state:

- Chinese remainder theorem (see the online lecture notes for the statement);
- First uniqueness theorem for primary decomposition;
- Second uniqueness theorem for primary decomposition;

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- First, second, and third isomorphism theorem for R-modules;
- Jordan-Hölder theorem;
- Hilbert basis theorem;
- Nakayama's lemma;
- Krull's intersection theorem;
- Elementary divisors theorem (see online lecture notes);
- Structure theorem for finitely generated modules over a PID (see online lecture notes);

understand and know:

- The natural map surjective map $R \to R/I$ that has kernel I;
- The isomorphism theorem;
- Given an ideal I of R, the bijection between the ideals J such that $I \subseteq J$ and the ideals of R/I;
- Given a ring homomorphism $f : R \to S$, the contraction and extension maps on ideals determined by f, especially when f is surjective;

- How to characterize an ideal I of R being maximal, prime, or primary in terms of the quotient R/I;
- The equivalent conditions for a non-zero ideal (r) in a PID to be a prime ideal;
- If R is an Euclidean domain, then R is a PID;
- If R is a PID, then R is a UFD;
- If K is a field, $K[X]$ is Euclidean, hence a PID, hence a UFD;
- If R is a UFD, then $R[X]$ is a UFD;
- Every proper ideal is included in a maximal ideal;
- $\sqrt{I} = \bigcap_{P \in \mathrm{Var}(I)} P = \bigcap_{P \in \mathrm{Min}(I)} P$;
- Formulas involving radicals:
  $\sqrt{IJ} = \sqrt{(I \cap J)} = \sqrt{I} \cap \sqrt{J}$;
  $\sqrt{P^n} = P$ if P is prime;
- $\sqrt{Q}$ maximal $\implies$ Q primary;
- M maximal $\implies$ M^n primary;
- What are the primary ideals in a PID;
- If R is Noetherian, then every proper ideal has a primary decomposition;

. . . .

- The universal property of the natural map $R \to S^{-1}R$;
- If P is a prime ideal of R, then $S = R - P$ is a multiplicatively closed set, and $S^{-1}R = R_P$ is quasi-local;
- For the natural map $f : R \to S^{-1}R$,
  $(J^c)^e = J$;
  $(\sqrt{I})^e = \sqrt{I^e}$;
  $I \cap S \neq \emptyset \iff I^e = S^{-1}R$;
- There are bijections:

$$\{P \in \mathrm{Spec}(R) : P \cap S = \emptyset\} \overset{e,c}{\rightleftarrows} \mathrm{Spec}(S^{-1}R),$$

$$\{Q \text{ primary}, Q \cap S = \emptyset\} \overset{e,c}{\rightleftarrows} \{Q' \subseteq S^{-1}R \text{ primary}\};$$

- If R is Noetherian, then $S^{-1}R$ is Noetherian;
- Given an R-module M with submodule N, the bijection between submodules N' of M such that $N \subseteq N'$ and submodules of M/N;
- External and internal direct sums;
- Every R-module is the quotient of free R-module;
- M is Noetherian if and only if every submodule of M is finitely generated;
- For an R-module M with submodule N:

$$M \text{ is Noetherian (Artinian)}$$
$$\Updownarrow$$
$$N \text{ and } M/N \text{ are Noetherian (Artinian)};$$

- For R-modules $M_1, \ldots, M_n$,

$$\bigoplus_{i=1}^{n} M_i \text{ Noetherian (Artinian)}$$
$$\Updownarrow$$
$$M_1, \ldots, M_n \text{ Noetherian (Artinian)};$$

- If R is Noetherian (Artinian), then every finitely generated R-module is Noetherian (Artinian);
- N is a simple R-module if and only if $N \cong R/M$ with M a maximal ideal;
- If R is a PID and not a field, and M is an R-module, then M has finite length if and only if M is finitely generated and there exists $r \in R$, $r \neq 0$, such that $rM = 0$.
- If R is a non-zero Noetherian ring, and N is an R-module, then N has finite length if and only if N is finitely generated and there exists maximal ideals $M_1, \ldots, M_n$ such that $M_1 \cdots M_n \cdot N = 0$;
- If $f : R \to R'$ is a surjective ring homomorphism and R is Noetherian, then R' is Noetherian;
- If I is a finitely generated ideal of R, then there exists $n \in \mathbb{N}$ such that $(\sqrt{I})^n \subseteq I$;
- R is Artinian if and only if R is Noetherian and every prime ideal of R is maximal;
- If R is Artinian then R is semi-local;

and be able to demonstrate knowledge of the above concepts and results in the context of simple problems and questions for the examples $\mathbb{Z}$, $\mathbb{Z}/N\mathbb{Z}$, $K[X]$, and $K[X_1, \ldots, X_n]$.