https://www.webpages.uidaho.edu/∼brooksr/

Assignment number	due date	Problems
1	Friday, Sept. 2	1.7, 1.16, 1.19, 1.20, 1.29, 1.30
2	Friday, Sept. 9	1.43, 2.4, 2.5
3	Friday, Sept. 16	2.16, 2.22, 2.25, 2.30, 2.33, 2.40
4	Friday, Sept. 23	3.29, 3.31, 3.42, 3.47
5	Friday, Sept. 30	3.50, 3.51, 3.53, 4.7, 4.8
6	Friday, Oct. 7	4.21, 4.22, 4.28
7	Friday, Oct. 21	5.11, 5.18, 5.22
8	Friday, Oct. 28	5.26, 5.34, 6.5, 6.11
9	Friday, Nov. 4	6.24, 6.44, 6.48, 6.52
10	Friday, Nov. 11	6.59, 7.2, 7.8, 7.23
11	Friday, Nov. 18	7.45, 7.46, 7.47
12	Friday, Dec. 2	8.5, 8.15, 8.28

Contents

Homework grading scheme

Each problem is worth ten points. Points for a problem are assessed as follows:

points	marking guide
10-9	Correct and complete solution with possibly a small mistake or oversight
8-7	Essentially a correct solution, with a bigger mistake or oversight
6-5	Correct idea for a solution, but substantially incomplete
5-0	Attempted problem, with parts of a solution

Hints

Assignment 2

1.43. If $f = \sum_{i=0}^{\infty} f_i, g = \sum_{i=0}^{\infty} g_i \in R[[X_1, \ldots, X_n]]$ then $fg = \sum_{i=0}^{\infty}(\sum_{j=0}^{i} f_j g_{i-j})$ (see Sharp p. 11). Hence, $fg = 1$ if and only if

$$\begin{aligned} 1 &= f_0 g_0, \\ 0 &= f_0 g_1 + f_1 g_0, \\ 0 &= f_0 g_2 + f_1 g_1 + f_2 g_0, \\ &\ \cdots \end{aligned}$$

2.5 Use the binomial theorem, which is valid in any commutative ring R: If $x, y \in R$, and $n \in \mathbb{N}$, then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}, \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

2.22 It may be useful to use the (total) degree function $\deg : K[X_1, X_2] - 0 \to \mathbb{N}$ (see p. 9 of Sharp). This function satisfies $\deg(pq) = \deg(p) + \deg(q)$ for non-zero elements p and q of $K[X_1, X_2]$.

Assignment 5

For Exercise **3.50** and Exercise **3.51** consider using Corollary **3.49**. For Exercise **4.7** first read and understand Exercise **2.46**. For Exercise **4.8** consider using Exercise **4.7**.

Assignment 6

For Exercise **4.28**, prove that the ideal (X^3, XY, Y^n) is primary by finding a maximal ideal M and $k \in \mathbb{N}$ such that

$$M^k \subseteq (X^3, XY, Y^n) \subseteq M,$$

take radicals, and apply Proposition 4.9.

Assignment 8

For Exercise **5.34**, assume that R does admit a non-zero nilpotent element x and obtain a contradiction via the following idea. Consider $I = \{r \in R : rx = 0\}$. Then $I = (0 : x)$, and I is thus an ideal of R. If $I = R$, then $1 \cdot x = 0$, which is a contradiction. Assume that $I \subsetneqq R$, so that I is a proper ideal. Since I is a proper ideal, I is included inside a maximal ideal M. Since M is a maximal ideal, M is a prime ideal. Consider R_M and the image $x/1$ in R_M of x under the natural map. The element $x/1$ is nilpotent. By the hypothesis of this exercise we must have $x/1 = 0/1$. Now obtain the final contradiction.

Suggested solutions to selected problems

Assignment 1

1.16 *Let R' be a commutative ring, and let $\xi_1, \dots, \xi_n \in R'$ be algebraically independent over the subring R of R'. Let T be a commutative R-algebra with structural ring homomorphism $f : R \to T$ and let $\alpha_1, \dots, \alpha_n \in T$. Show that there is exactly one ring homomorphism*

$$g : R[\xi_1, \dots, x_n] \longrightarrow T$$

which extends f (that is, is such that $g|_R = f$) and is such that $g(\xi_i) = \alpha_i$ for all $i = 1, \dots, n$.

Suggest solution: We begin with some notation. For $\lambda = (i_1, \dots, i_n) \in \mathbb{N}_0$ we will write

$$\xi^\lambda = \xi_1^{i_1} \cdots \xi_n^{i_n}.$$

With this notation every element p of $R[\xi_1, \dots, \xi_n]$ can be written uniquely in the form

$$p = \sum_{\lambda \in \mathbb{N}_0^n} r_\lambda \xi^\lambda$$

where $r_\lambda \in R$ for $\lambda \in \mathbb{N}_0^n$ and $r_\lambda = 0$ for all but finitely many $\lambda \in \mathbb{N}_0^n$ (see 1.14). If

$$q = \sum_{\lambda \in \mathbb{N}_0^n} s_\lambda \xi^\lambda$$

is another element of $R[\xi_1, \dots, \xi_n]$, then we have

$$p + q = \sum_{\lambda \in \mathbb{N}_0^n} (r_\lambda + s_\lambda) \xi^\lambda,$$

$$pg = \sum_{\lambda \in \mathbb{N}_0^n} \left(\sum_{\substack{\lambda_1, \lambda_2 \in \mathbb{N}_0, \\ \lambda_1 + \lambda_2 = \lambda}} r_{\lambda_1} s_{\lambda_2} \right) \xi^\lambda.$$

We now define

$$g : R[\xi_1, \dots, x_n] \longrightarrow T$$

by

$$g(p) = \sum_{\lambda \in \mathbb{N}_0^n} f(r_\lambda)\alpha^\lambda$$

for p as above; here, for $\lambda = (i_1, \dots, i_n) \in \mathbb{N}_0$ we define $\alpha^\lambda = \alpha_1^{\lambda_1} \cdots \alpha_n^{\lambda_n}$. With p and q as above, and using that f is a ring homomorphism, we have:

$$\begin{aligned} g(p+q) &= g\left(\sum_{\lambda \in \mathbb{N}_0^n} (r_\lambda + s_\lambda)\xi^\lambda\right) \\ &= \sum_{\lambda \in \mathbb{N}_0^n} f(r_\lambda + s_\lambda)\alpha^\lambda \\ &= \sum_{\lambda \in \mathbb{N}_0^n} f(r_\lambda)\alpha^\lambda + \sum_{\lambda \in \mathbb{N}_0^n} f(s_\lambda)\alpha^\lambda \\ &= g(p) + g(q). \end{aligned}$$

And:

$$\begin{aligned} g(pq) &= g\left(\sum_{\lambda \in \mathbb{N}_0^n} \left(\sum_{\substack{\lambda_1,\lambda_2 \in \mathbb{N}_0, \\ \lambda_1+\lambda_2=\lambda}} r_{\lambda_1} s_{\lambda_2}\right)\xi^\lambda\right) \\ &= \sum_{\lambda \in \mathbb{N}_0^n} f\left(\sum_{\substack{\lambda_1,\lambda_2 \in \mathbb{N}_0, \\ \lambda_1+\lambda_2=\lambda}} r_{\lambda_1} s_{\lambda_2}\right)\alpha^\lambda \\ &= \left(\sum_{\lambda \in \mathbb{N}_0^n} f(r_\lambda)\alpha^\lambda\right)\left(\sum_{\lambda \in \mathbb{N}_0^n} f(s_\lambda)\alpha^\lambda\right) \\ &= g(p)g(q). \end{aligned}$$

It is clear that $g(1) = 1$. It follows that g is a ring homomorphism. It is also clear that g extends f. Finally, to prove that g has the required uniqueness property, assume that $h : R[\xi_1, \dots, \xi_n] \to T$ is another right homomorphism such that $h|_R = f$ and $h(\xi_i) = \alpha_i$ for all $i = 1, \dots, n$. Let p be as above. We then have

$$\begin{aligned} h(p) &= h\left(\sum_{\lambda \in \mathbb{N}_0^n} r_\lambda \xi^\lambda\right) \\ &= \sum_{\lambda \in \mathbb{N}_0^n} h(r_\lambda)h(\xi^\lambda) \\ &= \sum_{\lambda \in \mathbb{N}_0^n} f(r_\lambda)\alpha^\lambda \\ &= g(p). \end{aligned}$$

It follows that $h = g$.

1.19 *Let K be an infinite field, let Λ be a finite subset of K, and let $f \in K[X_1, \dots, X_n]$, the ring of polynomials over K in the indeterminates $X_1, \dots, X_n$. Suppose that $f \neq 0$. Show that there exist infinitely many choices of*

$$(\alpha_1, \dots, \alpha_n) \in (K - \Lambda)^n$$

for which $f(\alpha_1, \dots, \alpha_n) \neq 0$.

Suggest solution: We prove this by induction on n. The case $n = 1$ is clear because a non-zero polynomial in one variable over K has finitely many distinct roots and $K - \Lambda$ is infinite. Assume that $n > 1$ and that the statement holds for $n - 1$; we will prove that it holds for n. There exists a non-negative integer N such that

$$f(X_1, \dots, X_n) = \sum_{k=0}^{N} f_k(X_1, \dots, X_{n-1}) X_n^k$$

where $f_k(X_1, \dots, X_{n-1}) \in K[X_1, \dots, X_{n-1}]$ for $k = 1, \dots, N$, and $f_N(X_1, \dots, X_{n-1})$ is non-zero. By the induction hypothesis, there exists $(\alpha_1, \dots, \alpha_{n-1}) \in (K - \Lambda)^{n-1}$ such that $f_N(\alpha_1, \dots, \alpha_{n-1}) \neq 0$. Consider the polynomial

$$g(X_n) = f(\alpha_1, \dots, \alpha_{n-1}, X_n) = \sum_{k=0}^{N} f_k(\alpha_1, \dots, \alpha_{n-1}) X_n^k$$

in the variable X_n. This polynomial is non-zero because $f_N(\alpha_1, \dots, \alpha_{n-1}) \neq 0$. By the case $n = 1$, there exist infinitely many $\alpha_n \in K - \Lambda$ such that $g(\alpha_n) \neq 0$, i.e., $f(\alpha_1, \dots, \alpha_n) \neq 0$; moreover, for any such α_n we have $(\alpha_1, \dots, \alpha_n) \in (K - \Lambda)^n$. This proves the statement for n.

Assignment 2

1.43 *Let R be a commutative ring, and consider the ring $R[[X_1, \dots, X_n]]$ of formal power series over R in indeterminates $X_1, \dots X_n$. Let*

$$f = \sum_{i=0}^{\infty} f_i \in R[[X_1, \dots, X_n]],$$

where f_i is either zero or a homogeneous polynomial of degree i in $R[X_1, \dots, X_n]$ (for each $i \in \mathbb{N}_0$). Prove that f is a unit of $R[[X_1, \dots, X_n]]$ if and only if f_0 is a unit of R.

Suggest solution: Assume that f is a unit. Let $g \in R[[X_1, \dots, X_n]]$ be such that $fg = 1$. Let $g = \sum_{i=0}^{\infty} g_i$ be the standard representation of g. Now

$$fg = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} f_j g_{i-j} \right)$$

and this expression is the standard representation of fg in $R[[X_1, \dots, X_n]]$. Since $fg = 1$ we must therefore have

$$1 = f_0 g_0 \quad \text{and} \quad 0 = \sum_{j=0}^{i} f_j g_{i-j} \quad \text{for } i > 0.$$

In particular, we see that $f_0 g_0 = 1$, i.e., f_0 is a unit. Now assume that f_0 is a unit. We inductively define a sequence $(g_i)_{i \in \mathbb{N}_0}$ by setting $g_0 = f_0^{-1}$, and for $i > 0$,

$$g_i = -f_0^{-1} \left(\sum_{j=1}^{i} f_i g_{i-j} \right).$$

Evidently, each g_i is either zero or a homogeneous polynomial of degree i in $R[X_1, \dots, X_n]$. Also, we have $f_0 g_0 = 1$ and for $i > 0$,

$$0 = \sum_{j=0}^{i} f_j g_{i-j}.$$

Now define

$$g = \sum_{i=0}^{\infty} g_i.$$

Then g is in $R[[X_1, \dots, X_n]]$, and this is the standard representation of g. Using the above formula for fg we see that $fg = 1$.

Assignment 3

2.22 *Let K be a field. Show that the ideal (X_1, X_2) of the commutative ring $K[X_1, X_2]$ (of polynomials over K in indeterminates X_1, X_2) is not principal.*

Suggest solution: Assume that $(X_1, X_2) = (f)$ for some $f \in K[X_1, X_2]$; we will obtain a contradiction. Since $X_1, X_2 \in (f)$, there exist $g_1, g_2 \in K[X_1, X_2]$ such that

$$X_1 = g_1 f, \qquad X_2 = g_2 f.$$

Applying the degree function to the first equation we obtain

$$\begin{aligned} \deg(X_1) &= \deg(g_1 f) \\ 1 &= \deg(g_1) + \deg(f). \end{aligned}$$

Similarly,

$$1 = \deg(g_2) + \deg(f).$$

Since $\deg(f), \deg(g_1)$, and $\deg(f)$ are in $\mathbb{N}_0$, we must have $\deg(f) = 0$ or $\deg(f) = 1$. Assume first that $\deg(f) = 0$. Then $f \in K$. Moreover, since $f \neq 0$ (otherwise $X_1 = 0$ and $X_2 = 0$, which is impossible), f is a unit in K and hence a unit in $K[X_1, X_2]$. Now $f \in (X_1, X_2)$. Hence, there exist

$h_1, h_2 \in K[X_1, X_2]$ such that

$$f = h_1 X_1 + h_2 X_2.$$

Evaluating both sides at $X_1 = 0$ and $X_2 = 0$, we obtain $f = 0$, a contradiction (recall that we just showed that f is a non-zero constant). Hence, $\deg(f) = 1$. It follows that $\deg(g_1) = \deg(g_2) = 0$, so that $g_1, g_2 \in K$. Again, we see that g_1 and g_2 are non-zero and are hence units in K and hence units in $K[X_1, X_2]$. Now

$$X_1 = g_1 f = g_1 g_2^{-1} g_2 f = g_1 g_2^{-1} X_2.$$

That is,

$$X_1 = (g_1 g_2^{-1}) X_2.$$

Evaluating both sides at $X_1 = 1$ and $X_2 = 0$, we obtain $1 = 0$, a contradiction.

2.30 *Let I, J be ideals of the commutative ring R. Show that*

$$\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}.$$

Let $r \in \sqrt{IJ}$. Then there exists $n \in \mathbb{N}$ such that $r^n \in IJ$. Since $IJ \subseteqq I \cap J$ we have $r^n \in I \cap J$. Hence, $r \in \sqrt{I \cap J}$. It follows that

$$\sqrt{IJ} \subseteq \sqrt{I \cap J}.$$

Let $r \in \sqrt{I \cap J}$. Then there exists $n \in \mathbb{N}$ such that $r^n \in I \cap J$. Since $I \cap J \subseteq I$ and $I \cap J \subseteq J$ we have $r \in \sqrt{I}$ and $r \in \sqrt{J}$. Thus, $r \in \sqrt{I} \cap \sqrt{J}$. It follows that

$$\sqrt{I \cap J} \subseteq \sqrt{I} \cap \sqrt{J}.$$

Let $r \in \sqrt{I} \cap \sqrt{J}$. Then there exist $m, n \in \mathbb{N}$ such that $r^m \in I$ and $r^n \in J$. Hence, $r^{mn} = r^m r^n \in IJ$ so that $r \in \sqrt{IJ}$. It follows that

$$\sqrt{I} \cap \sqrt{J} \subseteq \sqrt{IJ}.$$

We have proven that

$$\sqrt{IJ} \subseteq \sqrt{I \cap J} \subseteq \sqrt{I} \cap \sqrt{J} \subseteq \sqrt{IJ}.$$

This implies that

$$\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}.$$

Assignment 4

3.29 *Determine the prime ideals of the ring $\mathbb{Z}/60\mathbb{Z}$ of residue classes of integrs modulo* 60.

Suggest solution: By 3.28, every prime ideal of $\mathbb{Z}/60\mathbb{Z}$ is of the form $P/60\mathbb{Z}$ where P is a prime ideal of $\mathbb{Z}$ such that $60\mathbb{Z} \subseteq P$. By 3.34, every prime ideal P of $\mathbb{Z}$ such that $60\mathbb{Z} \subseteq P$ is of the form $P = p\mathbb{Z}$, where p is a prime of $\mathbb{Z}$ such that $60\mathbb{Z} \subset p\mathbb{Z}$, i.e., $p \mid 60$. It follows that the prime ideals of $\mathbb{Z}/60\mathbb{Z}$ are $2\mathbb{Z}/60\mathbb{Z}$, $3\mathbb{Z}/60\mathbb{Z}$, and $5\mathbb{Z}/60\mathbb{Z}$.

3.31 *Let R be an integral domain. Recall that for $a_1, \dots, a_n \in R$, where $n \in \mathbb{N}$, a greatest common divisor (GCD for short) or highest common factor of $a_1, \dots, a_n$ is an element d of R such that*

(i) $d \mid a_i$ for all $i = 1, \dots, n$, and

(ii) whenever $c \in R$ is such that $c \mid a_i$ for all $i = 1, \dots, n$, then $c \mid d$.

Show that every non-empty finite set of elements in a PID has a GCD.

Suggest solution: Assume that R is a PID, and let $a_1, \dots, a_n \in R$. Consider the ideal $(a_1, \dots, a_n)$. Since R is a PID, there exists $d \in R$ such that $(a_1, \dots, a_n) = (d)$. We claim that d is a GCD of $a_1, \dots, a_n$. Since $a_1, \dots, a_n \in (a_1, \dots, a_n) = (d)$, we see that $d \mid a_i$ for $i = 1, \dots, n$. Assume that $c \in R$ is such that $c \mid a_i$ for $i = 1, \dots, n$. Let $r_i \in R$ be such that $a_i = r_i c$ for $i = 1, \dots, n$. Also, let $x_1, \dots, x_n$ be such that $x_1 a_1 + \cdots + x_n a_n = d$; note that $x_1, \dots, x_n$ exist because $d \in (a_1, \dots, a_n)$. Then

$$d = x_1 a_1 + \cdots + x_n a_n = x_1 r_1 c + \cdots x_n r_n c = (x_1 r_1 + \cdots x_n r_n) c.$$

Thus, $c \mid d$.

3.42 *Show that an irreducible element in a unique factorization domain R generates a prime ideal of R.*

Suggest solution: Let $r \in R$ be irreducible. Then by definition r is non-zero and not a unit. Since r is not a unit we have $(r) \subsetneqq R$ (otherwise, $1 \in (r)$ so that r is a unit). Let $a, b \in R$ be such that $ab \in (r)$; to prove that (r) is a prime ideal it will suffice to prove that $a \in (r)$ or $b \in (r)$. If $a = 0$ or $b = 0$, then clearly $a \in (r)$ or $b \in (r)$; we may thus assume that $a \neq 0$ and $b \neq 0$. If a or b is a unit, then also $a \in (r)$ or $b \in (r)$; we may thus also assume that a and b are non-units. Since $ab \in (r)$, there exists $s \in R$ such that $ab = rs$. Since R is an integral domain we have $rs = ab \neq 0$; also, rs is not a unit (otherwise (r) contains a unit, contradicting $(r) \subsetneqq R$). As R is a UFD, there exist irreducible elements $p_1, \dots, p_k$, $q_1, \dots, q_\ell$, $y_1, \dots, y_n$ in R such that

$$a = p_1 \cdots p_k, \qquad b = q_1 \cdots q_\ell, \qquad rs = y_1 \cdots y_n;$$

Since r is irreducible, we may assume that $y_1 = vr$ for some unit v in R. Since $ab = rs$ we have

$$p_1 \cdots p_k q_1 \cdots q_\ell = v r y_2 \cdots y_n.$$

Since r is irreducible and R is a UFD, there exists a unit u in R such that $r = u p_i$ for some $i \in \{1, \dots, k\}$ or $r = u q_j$ for some $j \in \{1, \dots, \ell\}$. Hence, $p_i \in (r)$ for some $i \in \{1, \dots, k\}$ or $q_j \in (r)$ for some $j \in \{1, \dots, \ell\}$ (recall that u is a unit, so that $p_i = u^{-1} r$ or $q_j = u^{-1} r$). This implies that $a \in (r)$ or $b \in (r)$, as desired.

3.47 *Let P be a prime ideal of the commutative ring R. Show that $\sqrt{P^n} = P$ for all $n \in \mathbb{N}$.*

Suggest solution: Let $n \in \mathbb{N}$. Let $x \in \sqrt{P^n}$. Then there exits $m \in \mathbb{N}$ such that $x^m \in P^n$. Now $P^n \subseteq P$. Hence, $x^m \in P$. Since P is prime we have $x \in P$. This proves that $\sqrt{P^n} \subseteq P$. Let $x \in P$. Then $x^n \in P^n$. Therefore, $x \in \sqrt{P^n}$. This proves that $P \subseteq \sqrt{P^n}$. We conclude that $P = \sqrt{P^n}$.

Assignment 5

3.50 *Let R be a commutative ring, and let N be the nilradical of R. Show that the ring R/N has zero nilradical.*

Suggest solution: Let $x \in R/N$ and assume that $n \in \mathbb{N}$ is such that $x^n = 0_{R/N}$; we need to prove that $x = 0_{R/N}$. Let $a \in R$ be such that $x = a + N$. Then $0_{R/N} = x^n = (a+N)^n = a^n + N$. This means that $N = a^n + N$ so that $a^n \in N$. Since $a^n \in N$ there exists $m \in \mathbb{N}$ such that $(a^n)^m = 0$, i.e., $a^{nm} = 0$. Therefore, $a \in N$. We now have $x = a + N = N = 0_{R/N}$, as desired.

3.51 *Let R be a non-trivial commutative ring. Show that R has exactly one prime ideal if and only if each element of R is either a unit or nilpotent.*

Suggest solution: Assume that R has exactly one prime ideal P. Let $x \in R$. Assume x is not a unit; we need to prove that x is nilpotent. Since x is not a unit (x) is a proper ideal, and is hence included in a maximal ideal; since every maximal ideal is prime and P is unique, $(x) \subseteq P$. Now by **3.49** we have

$$\sqrt{0} = \bigcap_{P' \in \operatorname{Spec}(R)} P' = \bigcap_{P' \in \{P\}} P' = P.$$

Hence, $x \in (x) \subseteq P = \sqrt{0}$. This implies that x is nilpotent.
Next, assume that every element of R is either a unit or nilpotent. Since R is non-trivial, $0 \neq 1$. Hence, the ideal $0 = (0)$ is a proper ideal. Since 0 is proper, the ideal 0 is included in a maximal ideal; since every maximal ideal is prime, this proves that R has at least one prime ideal. Let P be a prime ideal of R; we will prove that $P = \sqrt{0}$, which will show that P is unique. Let $r \in P$. Since P is proper the element r is not a unit. Hence, r is nilpotent so that $r \in \sqrt{0}$. This proves that $P \subseteq \sqrt{0}$. Conversely, let $r \in \sqrt{0}$. Let $n \in \mathbb{N}$ be such that $r^n = 0$. Then $r^n = 0 \in P$. Since P is prime we have $r \in P$. It follows that $\sqrt{0} \subseteq P$. We conclude that $P = \sqrt{0}$ so that P is unique.

3.53 *Let P, I be ideals of the commutative ring R with P prime and $I \subseteq P$. Show that the non-empty set*

$$\Theta = \{P' \in \operatorname{Spec}(R) : I \subseteq P' \subseteq P\}$$

has a minimal member with respect to inclusion.

Suggest solution: We partially order Θ by declaring that $P_1 \leq P_2$ if and only if $P_2 \subseteq P_1$. The set Θ is non-empty because $P \in \Theta$. Let Y be a totally ordered non-empty subset of Θ; we need to prove that Y has an upper bound in Θ. Let Q be the intersection of all the elements of Y. We claim that $Q \in \Theta$. Evidently, Q is an ideal because Q is the intersection of ideals. Also, it is clear that $I \subseteq Q \subseteq P$; in particular, Q is proper because P is proper. Let $a, b \in R$ be such that $ab \in Q$. Assume that $a \notin Q$; to prove that Q is prime it will suffice to prove that $b \in Q$. Let $P' \in Y$; to prove that $b \in Q$ it will suffice to prove that $b \in P'$. Now since $a \notin Q$ there exists $P'' \in Y$ such that $a \notin P''$. Consider P' and P''. Since Y is totally ordered we have $P' \subseteq P''$ or $P'' \subseteq P'$. Assume first that $P' \subseteq P''$. Now $ab \in Q \subset P'$. Since P' is prime we have $a \in P'$ or $b \in P'$. We cannot have $a \in P'$ for otherwise $a \in P' \subseteq P''$, contradicting $a \notin P''$. Therefore, $b \in P'$. Assume now that $P'' \subseteq P'$. We have $ab \in Q \subseteq P''$. Since P'' is prime we have $a \in P''$ or $b \in P''$. However, $a' \notin P''$;

hence, $b \in P'' \subseteq P'$. We have proven that $b \in P'$; thus, Q is a prime ideal of R. It follows now that $Q \in \Theta$. Clearly, Q is an upper bound in Θ for Y. By Zorn's Lemma the set Θ has a minimal member with respect to inclusion.

4.7 Let $f : R \to S$ *be a surjective homomorphism of commutative rings. Us the extension and contraction notation of 2.41 and 2.45 in conjunction with* f*. Note that, by 2.46,* $\mathcal{C}_R = \{I \in \mathcal{I}_R : \ker(f) \subseteq I\}$ *and* $\mathcal{E}_S = \mathcal{I}_S$*. Let* $I \in \mathcal{C}_R$*. Show that*

(i) I *is a primary ideal of* R *if and only if* I^e *is a primary ideal of* S*.*

(ii) When this is the case, $\sqrt{I} = (\sqrt{I^e})^c$ *and* $\sqrt{I^e} = (\sqrt{I})^e$*.*

Suggest solution: We first note that by **2.46** we have $J^e = f(J)$ for $J \in \mathcal{C}_R$, and also the maps

$$\mathcal{C}_R \overset{\text{extension}}{\longrightarrow} \mathcal{I}_S \quad \text{and} \quad \mathcal{C}_R \overset{\text{contraction}}{\longleftarrow} \mathcal{I}_S$$

are inverses of each other.

(i) Define $g : R \to S/I^e = S/f(I)$ by $g(r) = f(r) + f(I)$. It is straightforward to verify that g is a ring homomorphism. Since f is surjective, g is also surjective. Also, for $r \in R$ we have

$$\begin{aligned} g(r) = 0 &\iff f(r) + f(I) = f(I) \\ &\iff \text{there exists } x \in I \text{ such that } f(r) = f(x) \\ &\iff \text{there exists } x \in I \text{ such that } f(r - x) = 0 \\ &\iff \text{there exists } x \in I \text{ such that } r - x \in \ker(f) \\ &\iff r \in I \qquad (\text{because } \ker(f) \subseteq I). \end{aligned}$$

Thus, $\ker(g) = I$. By the Isomorphism Theorem, g induces an isomorphism of rings

$$R/I \xrightarrow{\sim} S/f(I).$$

Since R/I and $S/f(I)$ are isomorphic the ideal I is primary if and only if $f(I)$ is primary (see **4.3**).

(ii) We first prove that $\sqrt{I^e} = (\sqrt{I})^e$. Since $I^e = f(I)$ and $(\sqrt{I})^e = f(\sqrt{I})$, we need to prove that $\sqrt{f(I)} = f(\sqrt{I})$. Let $s \in \sqrt{f(I)}$. Let $r \in R$ be such that $f(r) = s$. Since $s \in \sqrt{f(I)}$, there exists $n \in \mathbb{N}$ such that $s^n \in f(I)$. Let $a \in I$ be such that $s^n = f(a)$. We now have $f(r^n - a) = 0$. Since $\ker(f) \subseteq I$, this implies that $r^n \in I$. That is, $r \in \sqrt{I}$. Applying f, we obtain $s = f(r) \in f(\sqrt{I})$. We have proven that $\sqrt{f(I)} \subseteq f(\sqrt{I})$. Next, let $s \in f(\sqrt{I})$. Let $r \in \sqrt{I}$ be such that $f(r) = s$. Since $r \in \sqrt{I}$ there exists $n \in \mathbb{N}$ such that $r^n \in I$. Therefore, $s^n = f(r^n) \in f(I)$. This implies that $s \in \sqrt{f(I)}$, so that $f(\sqrt{I}) \subseteq \sqrt{f(I)}$. Hence, $\sqrt{f(I)} = f(\sqrt{I})$.

Now

$$\begin{aligned} (\sqrt{I^e})^c &= (\sqrt{f(I)})^c \qquad (\text{because } I^e = f(I)) \\ &= (f(\sqrt{I}))^c \qquad (\text{by } \sqrt{f(I)} = f(\sqrt{I})) \\ &= \sqrt{I} \qquad (\text{by } \mathbf{2.46}\text{; see the above summary}). \end{aligned}$$

4.8 *Let I be a proper ideal of the commutative ring R, and let P and Q be ideals of R which contain I. Prove that Q is a P-primary ideal of R if and only if Q/I is a P/I-primary ideal of of R/I.*

Suggest solution: It will suffice to prove that Q is primary if and only if Q/I is primary and that $\sqrt{Q}/I = \sqrt{Q/I}$. Let $f : R \to R/I$ be the natural map. Then f is a surjective ring homomorphism. By **4.7** (i), we have Q is primary if and only if $f(Q) = Q/I$ is primary. It remains to prove that $\sqrt{Q}/I = \sqrt{Q/I}$. Now

$$\begin{aligned}\sqrt{Q/I} &= \sqrt{f(Q)}\\ &= f(\sqrt{Q}) \qquad \text{(by 4.7 (ii))}\\ &= \sqrt{Q}/I.\end{aligned}$$

Assignment 6

4.21 *Let $f : R \to S$ be a homomorphism of commutative rings, and use the contraction notation of 2.41 in conjunction with f. let I be a decomposable ideal of S.*

(i) Let

$$I = Q_1 \cap \cdots \cap Q_n \quad \textit{with} \quad \sqrt{Q_i} = P_i \quad \textit{for} \quad i = 1, \ldots, n$$

be a primary decomposition of I. Show that

$$I^c = Q_1^c \cap \cdots \cap Q_n^c \quad \textit{with} \quad \sqrt{Q_i^c} = P_i^c \quad \textit{for} \quad i = 1, \ldots, n$$

is a primary decomposition of I. Deduced that I^c is a decomposable ideal of R and that

$$\mathrm{ass}_R(I^c) \subseteq \{P^c : P \in \mathrm{ass}_R(I)\}.$$

(ii) Now assume that f is surjective. Show that, if the first primary decomposition in (i) is minimal, then so too is the second, and deduce that in these circumstances,

$$\mathrm{ass}_R(I^c) = \{P^c : P \in \mathrm{ass}_R(I)\}.$$

Suggest solution: (i) We have

$$\begin{aligned}I^c &= f^{-1}(I)\\ &= f^{-1}(Q_1 \cap \cdots \cap Q_n)\\ &= f^{-1}(Q_1) \cap \cdots \cap f^{-1}(Q_n)\\ &= Q_1^c \cap \cdots \cap Q_n^c.\end{aligned}$$

Also, for $i \in \{1, \ldots, n\}$,

$$\sqrt{Q_i^c} = (\sqrt{Q_i})^c \qquad \text{(2.43(iv))}$$

$$= P_i^c.$$

Next we prove that Q_i^c is primary for $i \in \{1, \ldots, n\}$. Let $i \in \{1, \ldots, n\}$. The ideal Q_i^c is proper (otherwise, $1 \in Q_i^c$ so that $1 = f(1) \in Q_i$, a contradiction). Let $a, b \in R$ and assume that $ab \in Q_i^c$ and $a \notin Q_i^c$; we need to prove that $b \in \sqrt{Q_i^c}$. Since $ab \in Q_i^c = f^{-1}(Q_i)$ we have $f(ab) = f(a)f(b) \in Q_i$. Since Q_i is primary, we have $f(a) \in Q_i$ or $f(b) \in \sqrt{Q_i}$. If $f(a) \in Q_i$, then $a \in f^{-1}(Q_i) = Q_i^c$, a contradiction. Hence, $f(b) \in \sqrt{Q_i}$. This means that $b \in f^{-1}(\sqrt{Q_i}) = (\sqrt{Q_i})^c = \sqrt{Q_i^c}$. Hence, Q_i^c is primary. This completes the proof that the above is a primary decomposition of I^c and thus I^c is decomposable. We have $\mathrm{ass}_R(I^c) \subseteq \{P^c : P \in \mathrm{ass}_R(I)\}$ because the above primary decomposition can be refined to a minimal primary decomposition (see 4.16 or the lecture notes).

(ii) Assume that f is surjective. Assume that the first primary decomposition in (i) is minimal; we need to prove that second primary decomposition is also minimal. First we verify that $P_1^c, \ldots, P_n^c$ are pairwise unequal. Assume that $P_i^c = P_j^c$ for some $i, j \in \{1, \ldots, n\}$. Then $f^{-1}(P_i) = f^{-1}(P_j)$. Applying f and using that f is surjective, we find that $P_i = P_j$. As the first primary decomposition is minimal, we must have $i = j$. This implies that $P_1^c, \ldots, P_n^c$ are pairwise unequal. Finally, assume that $i \in \{1, \ldots, n\}$ is such that

$$\bigcap_{\substack{j=1\\ j\neq i}}^{n} Q_j^c \subseteq Q_i^c.$$

Let $y \in \bigcap_{\substack{j=1\\ j\neq i}}^{n} Q_j$. Since f is surjective, there exists $x \in R$ such that $f(x) = y$. Since $y \in Q_j$ for $j \neq i$, we have $x \in f^{-1}(Q_j) = Q_j^c$ for $j \neq i$. Therefore, $x \in \bigcap_{\substack{j=1\\ j\neq i}}^{n} Q_j^c$. By the assumed inclusion, we get $x \in Q_i^c = f^{-1}(Q_i)$. This implies that $y \in Q_i$. We have proven that

$$\bigcap_{\substack{j=1\\ j\neq i}}^{n} Q_j \subseteq Q_i,$$

contradicting the minimality of the first primary decomposition. That $\mathrm{ass}_R(I^c) = \{P^c : P \in \mathrm{ass}_R(I)\}$ follows from definition of $\mathrm{ass}_R(I^c)$.

4.22 *Let $f : R \to S$ be a surjective homomorphism of commutative rings; use the extension notation of 2.41 in conjunction with f. Let $I, Q_1, \ldots, Q_n, P_1, \ldots, P_n$ be ideals of R that contain* $\ker(f)$. *Show that*

$$I = Q_1 \cap \cdots \cap Q_n \quad \textit{with} \quad \sqrt{Q_i} = P_i \quad \textit{for} \quad i = 1, \ldots, n \tag{1}$$

is a primary decomposition of I if and only if

$$I^e = Q_1^e \cap \cdots \cap Q_n^e \quad \textit{with} \quad \sqrt{Q_i^e} = P_i^e \quad \textit{for} \quad i = 1, \ldots, n \tag{2}$$

is a primary decomposition of I^e, and that, when this is the case, the first of these is minimal if and only if the second is. Deduce that I is a decomposable ideal of R if and only if I^e is a decomposable *ideal of S, and when this is the case,*

$$\operatorname{ass}_R(I) = \{P^e : P \in \operatorname{ass}_R(I)\}.$$

Suggest solution: We first note the following fact: if A and B are ideals of R such that $\ker(f) \subseteq A$ and $\ker(f) \subseteq B$, then $f(A \cap B) = f(A) \cap f(B)$. We leave the proof of this as an exercise. Assume that (1) is a primary decomposition. Then

$$\begin{aligned} I &= Q_1 \cap \cdots \cap Q_n \\ f(I) &= f(Q_1 \cap \cdots \cap Q_n) \\ I^e &= f(Q_1) \cap \cdots \cap f(Q_n) \\ I^e &= Q_1^e \cap \cdots \cap Q_n^e. \end{aligned}$$

Also, if $i \in \{1, \ldots, n\}$, then $f(Q_i) = Q_i^e$ is primary and $\sqrt{Q_i^e} = \sqrt{Q_i}^e = P_i^e$ by **4.7**. Thus, (2) is a primary decomposition. Assume that (1) is a minimal primary decomposition; we want to prove that (2) is also a minimal primary decomposition. We first prove that $P_1^e, \ldots, P_n^e$ are pairwise unequal. Assume that $P_i^e = P_j^e$ for some $i, j \in \{1, \ldots, n\}$; we need to prove $i = j$. Now since $P_i^e = P_j^e$ we have $(P_i^e)^c = (P_j^e)^c$. Now $(P_i^e)^c = f^{-1}(f(P_i)) = P_i$ because f is surjective and $\ker(f) \subseteq P_i$; similarly, $(P_j^e)^c = P_j$. We thus get $P_i = P_j$. Since (1) is minimal we must have $i = j$. Finally, assume that $i \in \{1, \ldots, n\}$ is such that

$$\bigcap_{\substack{j=1 \\ j \neq i}}^{n} Q_j^e \subseteq Q_i^e;$$

we will obtain a contradiction. Now

$$\begin{aligned} \bigcap_{\substack{j=1 \\ j \neq i}}^{n} Q_j^e &\subseteq Q_i^e \\ f^{-1}\Big(\bigcap_{\substack{j=1 \\ j \neq i}}^{n} Q_j^e\Big) &\subseteq f^{-1}(Q_i^e) \\ \bigcap_{\substack{j=1 \\ j \neq i}}^{n} f^{-1}(Q_j^e) &\subseteq Q_i \\ \bigcap_{\substack{j=1 \\ j \neq i}}^{n} Q_j &\subseteq Q_i. \end{aligned}$$

This contradicts that (1) is a minimal primary decomposition.

Next, 4.21 implies that if (2) is a primary decomposition, then (1) is a primary decomposition, and also if (2) is a minimal primary decomposition, then (1) is a minimal primary decomposition. The remaining assertion follows immediately from what we have already proven.

4.28 *Let K be a field and let $R = K[X,Y]$ be the ring of polynomials over K in indeterminates X, Y. In R, let $I = (X^3, XY)$.*

(i) Show that, for every $n \in \mathbb{N}$, the ideal (X^3, XY, Y^n) of R is primary.

(ii) Show that $I = (X) \cap (X^3, Y)$ is a minimal primary decomposition of I.

(iii) Construct infinitely many different minimal primary decompositions of I.

Suggest solution: (i) Let $M = (X,Y)$. For $n \in \mathbb{N}$ let $I_n = (X^3, XY, Y^n)$. We have

$$M^3 = (X^3, X^2Y, XY^2, Y^3) \subseteq I_1 = (X^3, XY, Y) \subseteq M = (X,Y),$$
$$M^3 = (X^3, X^2Y, XY^2, Y^3) \subseteq I_2 = (X^3, XY, Y^2) \subseteq M = (X,Y)$$

and if $n \geq 3$,

$$M^n = (X^n, X^{n-1}Y, \dots, XY^{n-1}, Y^n) \subseteq I_n = (X^3, XY, Y^n) \subseteq M = (X,Y).$$

Taking radicals, we obtain

$$\sqrt{M^3} = M \subseteq \sqrt{I_1} \subseteq \sqrt{M} = M,$$
$$\sqrt{M^3} = M \subseteq \sqrt{I_2} \subseteq \sqrt{M},$$

and if $n \geq 3$,

$$\sqrt{M^n} = M \subseteq \sqrt{I_n} \subseteq \sqrt{M} = M.$$

It follows that $\sqrt{I_n} = M$ for all $n \in \mathbb{N}$. By Proposition 4.9 the ideal I_n is primary for all $n \in \mathbb{N}$.

(ii) First we prove that $I = (X) \cap (X^3, Y)$. It is clear that $I \subseteq (X) \cap (X^3, Y)$. Let $g \in (X) \cap (X^3, Y)$. Then there exist $a, b, c \in R$ such that $g = aX$ and $g = bX^3 + cY$. Now $aX = bX^3 + cY$. Substituting $X = 0$ we obtain $0 = c(0,Y)Y^3$. This implies that there exists $d \in R$ such that $c = dX$. We now have $g = bX^3 + dXY$. Hence, $g \in I$ so that $(X) \cap (X^3, Y) \subseteq I$. It follows that $I = (X) \cap (X^3, Y)$. Next, we note that (X) is a prime ideal of R (since $R/(X) \cong K[Y]$, which is an integral domain). Also, we have

$$(X,Y)^3 = (X^3, X^2Y, XY^2, Y^3) \subseteq (X^3, Y) \subseteq (X,Y).$$

Taking radicals, we obtain

$$(X,Y) \subseteq \sqrt{(X^3, Y)} \subseteq (X,Y).$$

Hence, $(X,Y) = \sqrt{(X^3,Y)}$, which implies by Proposition 4.9 that (X^3, Y) is primary (since (X,Y) is maximal). It is clear that the primary decomposition $I = (X) \cap (X^3, Y)$ is minimal.

(iii) Using the method of (ii) we find that

$$I = (X^3, XY) = (X) \cap (X^3, XY, Y^n)$$

for $n \in \mathbb{N}$. The ideal (X) is prime and primary, and (X^3, XY, Y^n) is primary with radical (X, Y) for $n \in \mathbb{N}$ by (i). Hence, this is a primary decomposition of I. It is straightforward to verify that this primary decomposition is minimal. The primary decompositions $I = (X) \cap (X^3, XY, Y^n)$ are all different because $(X^3, XY, Y^n) \neq (X^3, XY, Y^m)$ for $m, n \in \mathbb{N}$ with $m \neq n$.

Assignment 8

5.26. *Let the situation be as in 5.23. Show that if the ring R is Noetherian, then so too is the ring $S^{-1}R$.*

Suggest solution: Assume that R is Noetherian. Let

$$J_1 \subseteq J_2 \subseteq J_3 \subseteq \cdots$$

be a sequence of ideals in $S^{-1}R$. Then

$$(J_1)^c \subseteq (J_2)^c \subseteq (J_3)^c \subseteq \cdots$$

is a sequence of ideals in R. Since R is Noetherian, there exists $n \in \mathbb{N}$ such that for $k \in \mathbb{N}$ with $k \geq n$ we have $(J_{n+k})^c = (J_n)^c$. Therefore, $((J_{n+k})^c)^e = ((J_n)^c)^e$ for $k \geq n$. By 5.24 we have $((J_n)^c)^e$ and $((J_{n+k})^c)^e = J_{n+k}$ for $k \geq n$. Hence, $J_{n+k} = J_n$ for $k \geq n$. It follows that $S^{-1}R$ is Noetherian.

Suggest solution: Alternatively, we can argue as follows. Assume that R is Noetherian. Assume that J is an ideal of $S^{-1}R$; to prove that $S^{-1}R$ is Noetherian, it will suffice to prove that J is finitely generated. Then J^c is an ideal of R. Since R is Noetherian, J^c is finitely generated by, say, $r_1, \ldots, r_t$: $J^c = (r_1, \ldots, r_t)$. We claim that $(J^c)^e$ is generated by $r_1/1, \ldots, r_t/1$. It is clear that $r_1/1, \ldots, r_t/1$ are contained in $(J^c)^e$. Let $x \in (J^c)^e$. By 5.25 there exist $a \in J^c$ and $s \in S$ such that $x = a/s$. Since $a \in J^c$ there exist $c_1, \ldots, c_t \in R$ such that $a = c_1 r_1 + \cdots + c_t r_t$. This implies that

$$\begin{aligned}
x &= a/s \\
&= (c_1 r_1 + \cdots + c_t r_t)/s \\
&= c_1 r_1/s + \cdots + c_t r_t/s \\
&= (c_1/s)(r_1/1) + \cdots + (c_t/s)(r_t/1).
\end{aligned}$$

Thus, $x \in (r_1/1, \ldots, r_t/1)$. We have proven that $(J^c)^e = (r_1/1, \ldots, r_t/1)$, so that $(J^c)^e$ is finitely generated. Since $J = (J^c)^e$ by 5.24, J is finitely generated. This implies that $S^{-1}R$ is Noetherian.

5.34. *Let R be a non-trivial commutative ring, and assume that, for each $P \in \mathrm{Spec}(R)$, the localization R_P has no non-zero nilpotent element. Show that R has no non-zero nilpotent element.*

Suggest solution: Assume that $x \in R$ is such that $x \neq 0$ and x is nilpotent; we will obtain a contradiction. Let $I = \{s \in R : sx = 0\}$. Then $I = (0 : x)$, and I is an ideal of R. Assume that $I = R$. Then $1 \in I$; this implies that $1 \cdot x = 0$, i.e., $x = 0$; this is a contradiction. Hence, $I \subsetneq R$.

Since I is a proper ideal, I is included in a maximal ideal M. Since M is a maximal ideal, M is prime. Consider R_M and the image $x/1$ of x in R_M under the natural map $R \to R_M$. Since x is nilpotent so is $x/1$. By hypothesis, R_M does not contain a non-zero nilpotent element. Therefore, $x/1 = 0_{R_M} = 0/1$. This implies that there exists an element $s \in S = R - M$ such that $sx = 0$. By the definition of I we have $s \in I \subseteq M$. We now have $s \in M \cap (R - M)$; this is a contradiction.

6.11. *Let M be a module over a commutative ring R, and let $J \subseteq M$; let G be the submodule of M generated by J.*

(i) Show that, if $J = \emptyset$, then $G = 0$.

(ii) Show that, if $J \neq \emptyset$, then

$$G = \left\{ \sum_{i=1}^{n} r_i j_i : n \in \mathbb{N}, r_1, \ldots, r_n \in R, j_1, \ldots, j_n \in J \right\}.$$

(iii) Show that, if $\emptyset \neq J = \{l_1, \ldots, l_t\}$, then

$$G = \left\{ \sum_{i=1}^{t} r_i l_i : r_1, \ldots, r_t \in R \right\}.$$

Suggest solution: (i) Assume that $J = \emptyset$. Since G is a submodule of M we have $0 \subseteq G$. Also, 0 is a submodule of M such that $\emptyset \subseteq 0$. This implies that

$$G = \bigcap_{\substack{N \text{ submodule of } M \\ \text{such that } J \subseteq N}} N \subseteq 0.$$

Hence, $G = 0$.
(ii) Define

$$W = \left\{ \sum_{i=1}^{n} r_i j_i : n \in \mathbb{N}, r_1, \ldots, r_n \in R, j_1, \ldots, j_n \in J \right\}.$$

We need to prove that $G = W$. Using the submodule criterion, it is straightforward to verify that W is a submodule of M that contains J. Hence,

$$G = \bigcap_{\substack{N \text{ submodule of } M \\ \text{such that } J \subseteq N}} N \subseteq W.$$

Since G contains J, G also contains all R-linear combinations of elements of J. Thus, $W \subseteq G$. We conclude that $G = W$.
(iii) Let W be as above, and let

$$U = \left\{ \sum_{i=1}^{t} r_i l_i : r_1, \ldots, r_t \in R \right\}.$$

Evidently, $U \subseteq W$. Conversely, let $x = \sum_{i=1}^{n} r_i j_i \in W$. Recalling that $J = \{l_1, \dots, l_t\}$, we have:

$$\begin{aligned}
x &= \sum_{i=1}^{n} r_i j_i \\
&= \left(\sum_{\substack{i=1 \\ j_i = l_1}}^{n} r_i j_i \right) + \cdots + \left(\sum_{\substack{i=1 \\ j_i = l_t}}^{n} r_i j_i \right) \\
&= \left(\sum_{\substack{i=1 \\ j_i = l_1}}^{n} r_i l_1 \right) + \cdots + \left(\sum_{\substack{i=1 \\ j_i = l_t}}^{n} r_i l_t \right) \\
&= \left(\sum_{\substack{i=1 \\ j_i = l_1}}^{n} r_i \right) l_1 + \cdots + \left(\sum_{\substack{i=1 \\ j_i = l_t}}^{n} r_i \right) l_t \\
&\in U.
\end{aligned}$$

Thus, $W \subseteq U$. It follows that $W = U$.

Assignment 11

7.45 *Let G be a module over a non-trivial commutative Noetherian ring R. Show that G has finite length if and only if G is finitely generated and there exist $n \in \mathbb{N}$ and maximal ideals $M_1, \dots, M_n$ of R (not necessarily distinct) such that*

$$M_1 \cdots M_n G = 0.$$

Suggest solution: Assume that G has finite length. By 7.36 the R-module G is Noetherian. By 7.13, G is finitely generated. Let

$$0 = G_0 \subsetneqq G_1 \subsetneqq \cdots \subsetneqq G_{n-1} \subsetneqq G_n = G$$

be a composition series. By definition, G_i/G_{i-1} is simple for $i = 1, \dots, n$. By 7.32, for each $i \in \{1, \dots, n\}$ there exists a maximal ideal M_i of R such that $G_i/G_{i-1} \cong R/M_i$ as R-modules. Now let $g \in G$, and let $m_i \in M_i$ for $i \in \{1, \dots, n\}$. Since $G_n/G_{n-1} \cong R/M_n$, we have $r(x+G_{n-1}) = 0$ for $r \in M_n$ and $x \in G_n$. This implies that $m_n g \in G_{n-1}$. Similarly, $m_{n-1}m_n g \in G_{n-2}$, and continuing, we find that $m_1 \cdots m_n g \in G_0 = 0$. This proves that $M_1 \cdots M_n G = 0$.
Now assume that G is finitely generated and and there exist $n \in \mathbb{N}$ and maximal ideals $M_1, \dots, M_n$ of R (not necessarily distinct) such that

$$M_1 \cdots M_n G = 0.$$

Since R is a Noetherian ring (by assumption), and since G is finitely generated, G is Noetherian by 7.22. By 7.30, G is also Artinian (this uses the hypothesis $M_1 \cdots M_n G = 0$). By 7.36, G has finite length.

7.46 *Let R be a principal ideal domain which is not a field. Let G be an R-module. Show that G has finite length if and only if G is finitely generated and there exists $r \in R$ with $r \neq 0$ such that $rG = 0$.*

Assume that G has finite length. By 7.45, G is finitely generated, and there exist $n \in \mathbb{N}$ and maximal ideals $M_1, \ldots, M_n$ of R (not necessarily distinct) such that

$$M_1 \cdots M_n G = 0.$$

Since R is not a field, 0 is not a maximal ideal of R. This implies that $M_1, \ldots, M_n$ are all non-zero. Since R is a PID, we may write $M_i = (r_i)$ for some $r_i \in R$ for $i \in \{1, \ldots, n\}$. Since $M_1 \cdots M_n G = 0$ we have $rG = 0$ with $r = r_1 \cdots r_n$; note that $r \neq 0$ as $r_1 \neq 0, \ldots, r_n \neq 0$, and R is an integral domain.

Now suppose that G is finitely generated and there exists $r \in R$ with $r \neq 0$ and $rG = 0$. If r is a unit, then $G = 0$, and G has finite length. Assume that r is not a unit. Since R is a PID, R is a UFD by 3.39. Therefore, there exist $n \in \mathbb{N}$ and irreducible elements $p_1, \ldots, p_n \in R$ such that $r = p_1 \cdots p_n$. Let $M_i = (p_i)$ for $i \in \{1, \ldots, n\}$. By 3.34, M_i is a maximal ideal of R for $i \in \{1, \ldots, n\}$. Since $rG = 0$ we have $M_1 \cdots M_n G = 0$. By 7.45 we now conclude that G has finite length.