Student name: ______________________________

1	2	3	4	5	6	7	8	9	10	total

1. Let R be an integral domain and let $r \in R$.

(a) [/ 5] Define what it means for r to irreducible.

The element r is irreducible if

(i) $r \neq 0$ and r is not a unit,

(ii) and if $r = ab$ for some $a, b \in R$, then a is a unit or b is a unit.

(b) [/ 5] Define what it means for r to be prime.

The element r is irreducible if

(i) $r \neq 0$ and r is not a unit,

(ii) and if $r|ab$ for some $a, b \in R$, then $r|a$ or $r|b$.

2. Let R be a commutative ring, and let I and J be ideals of R.

(a) [/ 5] Assume that I is proper. What is a minimal prime ideal of I?

A minimal prime ideal of I is a prime ideal P of R such that

(i) $I \subseteq P$

(ii) and if P' is another prime ideal of R such that $I \subseteq P' \subseteq P$,
then $P = P'$.

(b) [/ 5] Define $(I : J)$.

$(I : J) = \{r \in R : aJ \subseteq I\}$.

3. Let $R = \mathbb{Z}/12\mathbb{Z}$.

(a) $\boxed{\quad / 5}$ What is $\mathrm{Spec}(R)$?

Let $f : \mathbb{Z} \to R = \mathbb{Z}/12\mathbb{Z}$ be the natural map. We have $P \in \mathrm{Spec}(R)$ if and only if $P = f((p))$ where p is a prime element of $\mathbb{Z}$ such that $(12) \subseteq (p)$, i.e., $p \mid 12$. Hence, $P \in \mathrm{Spec}(R)$ if and only if $P = (\bar{2})$ or $P = (\bar{3})$ so that $\mathrm{Spec}(R) = \{(\bar{2}), (\bar{3})\}$.

(b) $\boxed{\quad / 5}$ What is the nilradical of R?

We know that the nilradical of R is $\sqrt{0} = \bigcap_{P \in \mathrm{Spec}(R)} P$. By (a), this is $(\bar{2}) \cap (\bar{3}) = (\bar{6})$.

$\boxed{\quad / 10}$ 4. Let R be an integral domain. Let $a, b \in R$ and assume that a and b are non-zero. Assume that $(a) = (b)$. Prove that there exists a unit u in R such that $b = ua$.

Since $(a) = (b)$, there exist $u, v \in R$ such that $a = vb$ and $b = ua$. Now $a = vb = uva$. Hence, $a(1 - uv) = 0$. Since R is an integral domain, $a = 0$ or $1 - uv = 0$. But $a \neq 0$ by assumption. Hence, $1 - uv = 0$, i.e., $uv = 1$. It follows that u is a unit.

/ 10 5. Let R be a commutative ring, and let I and J be comaximal ideals of R. Prove that $I \cap J = IJ$.

Clearly, $IJ \subseteq I$ and $IJ \subseteq J$. Hence, $IJ \in I \cap J$. For the converse, let $x \in I \cap J$. Since I and J are comaximal, there exist $a \in I$ and $b \in J$ such that $a + b = 1$. We have $x = xa + xb$. Now $xa, xb \in IJ$. Hence, $x \in IJ$, proving that $I \cap J \subseteq IJ$. It follows that $I \cap J = IJ$.

/ 5 6. Let $R = \mathbb{Z}/10000\mathbb{Z}$. Is R a Noetherian ring? Explain your answer.

The ring R is Noetherian because R is finite. Since R is finite, there are only finitely many possible ideals in R. This means that any ascending chain of ideals must eventually be stationary.

/ 5 7. Let $S = \mathbb{Z}[X_1, X_2, X_3, \ldots]$ where $X_1, X_2, X_3, \ldots$ are indeterminates. Is S a Noetherian ring? Explain your answer.

The ring S is not Noetherian. The sequence

$$(X_1) \subsetneqq (X_1, X_2) \subsetneqq (X_1, X_2, X_3) \subsetneqq \cdots$$

has strict containment at each inclusion and thus never becomes stationary.

$\boxed{/\ 10}$ 8. Let K be a field, let X and Y be indeterminates, and let $R = K[X, Y]$. Let $I = (X, Y^2)$. Determine the radical of I.

We have

$$(X, Y)^2 = (X^2, XY, Y^2) \subseteq (X, Y^2) \subseteq (X, Y).$$

Taking radicals, we get

$$\sqrt{(X, Y)^2} = (X, Y) \subseteq \sqrt{(X, Y^2)} \subseteq \sqrt{(X, Y)} = (X, Y)$$

where the first and last radicals are determined by using that (X, Y) is maximal and hence prime. It follows that $\sqrt{(X, Y^2)} = (X, Y)$.

$\boxed{/\ 10}$ 9. State the First Uniqueness Theorem for Primary Decomposition.

See text.

10. Let K be a field, let X be an indeterminate, and let $R = K[X]$. Let I be the principal ideal of R generated by $X^2 + X$.

(a) $\boxed{\ \ / \ 5}$ Find a minimal primary decomposition of I.

We have $I = (X^2+X) = (X)(X+1) = (X)\cap(X+1)$. Here, the last step follows because (X) and $(X+1)$ are distinct maximal ideals of $K[X]$ and hence comaximal. This is a primary decomposition of I because (X) and $(X+1)$ are maximal, hence prime, hence primary. It is minimal because $\sqrt{(X)} = (X)$, $\sqrt{(X+1)} = (X+1)$, and $(X) \neq (X+1)$, and $(X) \not\subseteq (X+1)$ and $(X+1) \not\subseteq (X)$.

(b) $\boxed{\ \ / \ 5}$ What is $\operatorname{ass}_R(I)$?

By the answer to (a) we have $\operatorname{ass}_R(I) = \{(X), (X+1)\}$.