Lecture Notes on Sheaf Theory

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Preface

These notes are for a series of lectures at the University of Idaho during the spring semester of 2025. The main source is the book [10] by Tennison called *Sheaf Theory*. The last update of the notes was on April 2, 2025.

Conventions

In these notes the formal definition of a function is as follows: A **function** is an ordered triple (X, Y, R) of sets such that: $R \subset X \times Y$; if $x \in X$, then there exists $y \in Y$ such that $(x, y) \in R$; if $(x, y), (x, z) \in R$, then $y = z$. Thus, given a set Y, there exists exactly one function from the empty set $X = \emptyset$ to Y, and this function is $(\emptyset, Y, \emptyset)$ (because $\emptyset \times Y = \emptyset$); we refer to this function as the **empty function**.

Chapter 1

Category theory

1.1 Categories

Suppose that we are given

- a collection of objects $\mathrm{Ob}(\mathcal{C})$;
- for any two objects $A, B \in \mathrm{Ob}(\mathcal{C})$, a set $\mathrm{Mor}(A, B) = \mathrm{Mor}_{\mathcal{C}}(A, B)$;
- and for any three objects $A, B, C \in \mathrm{Ob}(\mathcal{C})$ a function

$$\circ : \mathrm{Mor}(B, C) \times \mathrm{Mor}(A, B) \longrightarrow \mathrm{Mor}(A, C).$$

We say that this structure is a **category**, or more briefly, that $\mathcal{C}$ is a category, if the following axioms are satisfied:

(a) Let $A, B, A', B' \in \mathrm{Ob}(\mathcal{C})$. If $A \neq A'$ or $B \neq B'$, then $\mathrm{Mor}(A, B)$ and $\mathrm{Mor}(A', B')$ are disjoint.

(b) For every $A \in \mathrm{Ob}(\mathcal{C})$ there exists $\mathrm{id}_A \in \mathrm{Mor}(A, A)$ such that if $B \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Mor}(A, B)$, then

$$f \circ \mathrm{id}_A = f,$$

and if $B \in \mathrm{Ob}(\mathcal{C})$ and $g \in \mathrm{Mor}(B, A)$, then

$$\mathrm{id}_A \circ g = g.$$

(c) If $A, B, C, D \in \mathrm{Ob}(\mathcal{C})$, and $f \in \mathrm{Mor}(A, B)$, $g \in \mathrm{Mor}(B, C)$, and $h \in \mathrm{Mor}(C, D)$ then

$$(h \circ g) \circ f = h \circ (g \circ f).$$

We note that if $\mathcal{C}$ is a category, and $A, B \in \mathrm{Ob}(\mathcal{C})$, then it may be the case that $\mathrm{Mor}(A, B) = \emptyset$ (the empty set). Given a category $\mathcal{C}$, we will refer to the elements of $\mathrm{Mor}(A, B)$ for $A, B \in \mathrm{Ob}(\mathcal{C})$ as **morphisms**. Let $A, B \in \mathrm{Ob}(\mathcal{C})$, and let $f \in \mathrm{Mor}(A, B)$. We say that f is an **isomorphism** if there exists $g \in \mathrm{Mor}(B, A)$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. Some important categories are:

- $\mathcal{S}et$, the category of sets.

- $\mathcal{Ab}$, the category of abelian groups.
- $\mathcal{Ring}$, the category of commutative rings.
- $\mathcal{Mod}(R)$, the category of R-modules, where R is a commutative ring.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be categories. We say that $\mathcal{C}_1$ is a **subcategory** of $\mathcal{C}_2$, and write $\mathcal{C}_1 \subset \mathcal{C}_2$, if

(a) we have $\mathrm{Ob}(\mathcal{C}_1) \subset \mathrm{Ob}(\mathcal{C}_2)$;

(b) for any two objects $A, B \in \mathrm{Ob}(\mathcal{C}_1)$ we have $\mathrm{Mor}_{\mathcal{C}_1}(A,B) \subset \mathrm{Mor}_{\mathcal{C}_2}(A,B)$;

(c) and for any three objects $A, B, C \in \mathrm{Ob}(\mathcal{C}_1)$, the diagram

$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{C}_1}(A,B) \times \mathrm{Mor}_{\mathcal{C}_1}(B,C) & \xrightarrow{\circ_1} & \mathrm{Mor}_{\mathcal{C}_1}(A,C) \\ \downarrow & & \downarrow \\ \mathrm{Mor}_{\mathcal{C}_2}(A,B) \times \mathrm{Mor}_{\mathcal{C}_2}(B,C) & \xrightarrow{\circ_2} & \mathrm{Mor}_{\mathcal{C}_2}(A,C) \end{array}$$

commutes.

Evidently,

$$\begin{array}{ccccc} \mathcal{Ring} & & & & \\ & \subset & & & \\ & & \mathcal{Ab} & \subset & \mathcal{Set} \\ & \subset & & & \\ \mathcal{Mod}(R) & & & & \end{array}$$

1.2 Functors

Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A **covariant functor** is two functions, which we refer to with the same name,

$$\mathrm{Ob}(\mathcal{A}) \xrightarrow{F} \mathrm{Ob}(\mathcal{B}),$$
$$\{\text{morphisms of } \mathcal{A}\} \xrightarrow{F} \{\text{morphisms of } \mathcal{B}\},$$

such that:

(a) If $A, B \in \mathrm{Ob}(\mathcal{A})$, and $f \in \mathrm{Mor}(A,B)$, then $F(f) \in \mathrm{Mor}(F(A), F(B))$.

(b) For all objects A of $\mathcal{A}$, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

(c) If $A, B, C \in \mathrm{Ob}(\mathcal{A})$, and $f \in \mathrm{Mor}(A,B)$ and $g \in \mathrm{Mor}(B,C)$, then

$$F(g \circ f) = F(g) \circ F(f).$$

We also define an analogous concept which reverses the direction of morphisms. We define a **contravariant functor** to be two functions, which we refer to with the same name,

$$\mathrm{Ob}(\mathcal{A}) \xrightarrow{F} \mathrm{Ob}(\mathcal{B}),$$
$$\{\text{morphisms of } \mathcal{A}\} \xrightarrow{F} \{\text{morphisms of } \mathcal{B}\},$$

such that:

(a) If $A, B \in \mathrm{Ob}(\mathcal{A})$, and $f \in \mathrm{Mor}(A, B)$, then $F(f) \in \mathrm{Mor}(F(B), F(A))$.

(b) For all objects A of $\mathcal{A}$, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

(c) If $A, B, C \in \mathrm{Ob}(\mathcal{A})$, and $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, C)$, then

$$F(g \circ f) = F(f) \circ F(g).$$

1.3 Direct limits

Let $\mathcal{A}$ be a category. Let I be a set with a partial order $\leq$ (this means that $\leq$ is a binary relation on I that is reflexive, antisymmetric, and transitive). Assume further that I is **directed**, i.e., for every $i, j \in I$ there exist $k \in I$ such that $i \leq k$ and $j \leq k$. Assume that we are given $A_i \in \mathrm{Ob}(\mathcal{A})$ for $i \in I$, and for every pair $i, j \in I$ with $i \leq j$, a morphism

$$A_i \xrightarrow{\rho_{ij}} A_j.$$

To avoid excessive notation, we will often not mention the name of the morphism ρ_{ij} and instead indicate such morphisms with an arrow. We say that the A_i and the morphisms ρ_{ij} are a **direct system** in $\mathcal{A}$ if the following hold:

(a) $A_i \to A_i$ is the identity for all $i \in I$;

(b) if $i, j, k \in I$ with $i \leq j \leq k$, then the following diagram commutes:

$$\begin{array}{ccc} A_i & \longrightarrow & A_j \\ & \searrow \quad \swarrow & \\ & A_k & \end{array}$$

Assume that $(A_i)_{i \in I}$ is a direct system in $\mathcal{A}$. A **direct limit** of this direct system is an object $A \in \mathrm{Ob}(\mathcal{A})$ along with a morphism

$$A_i \longrightarrow A$$

for each $i \in I$ such that:

(a) For all $i, j \in I$ with $i \leq j$ the diagram

$$\begin{array}{ccc} A_i & \longrightarrow & A_j \\ & \searrow \quad \swarrow & \\ & A & \end{array}$$

commutes;

(b) if $B \in \mathrm{Ob}(\mathcal{A})$ and

$$A_i \longrightarrow B$$

for $i \in I$ are morphisms such that

$$\begin{array}{ccc} A_i & \longrightarrow & A_j \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

commutes for $i, j \in I$ with $i \le j$, then there exists a **unique** morphism

$$A \longrightarrow B$$

such that

$$\begin{array}{ccc} A_i & \longrightarrow & A \\ & \searrow & \downarrow \\ & & B \end{array}$$

commutes for all $i \in I$.

We refer to (b) of the definition of a direct limit as the **universal property** of $\varinjlim A_i$.

Lemma 1.3.1. *Let $\mathcal{A}$ be a category. Let $(A_i)_{i \in I}$ be a direct system in $\mathcal{A}$, and assume that A and B are direct limits of this direct system. Then there exists a unique isomorphism*

$$A \longrightarrow B$$

such that

$$\begin{array}{ccc} & A_i & \\ \swarrow & & \searrow \\ A & \longrightarrow & B \end{array}$$

commutes for all $i \in I$.

Proof. Since A is a direct limit of $(A_i)_{i \in I}$ there exist a unique morphism $A \to B$ such that

$$\begin{array}{ccc} A_i & \longrightarrow & A \\ & \searrow & \downarrow \\ & & B \end{array}$$

commutes for all $i \in I$. It remains to prove that $A \to B$ is an isomorphism. Since B is also a direct limit of $(A_i)_{i\in I}$ there exists a morphism $B \to A$ such that

$$\begin{array}{ccc} A_i & \longrightarrow & B \\ & \searrow & \downarrow \\ & & A \end{array}$$

commutes for all $i \in I$. It follows that

$$\begin{array}{ccc} A_i & \longrightarrow & A \\ & \searrow & \downarrow \\ & & B \\ & & \downarrow \\ & & A \end{array} \quad \text{and} \quad \begin{array}{ccc} A_i & \longrightarrow & B \\ & \searrow & \downarrow \\ & & A \\ & & \downarrow \\ & & B \end{array}$$

commute for all $i \in I$. Since the same diagrams with

$$A \to B \to A \quad \text{and} \quad B \to A \to B \tag{1.1}$$

replaced with the identity morphisms also commute, the uniqueness property in the definition of a direct limit implies that the compositions in (1.1) are the identity morphisms on A and B, respectively; thus, the morphism $A \to B$ is an isomorphism. □

If $(A_i)_{i\in I}$ is a direct system in $\mathcal{A}$, and $A \in \mathrm{Ob}(\mathcal{A})$ is a direct limit of $(A_i)_{i\in I}$, then we will write

$$\varinjlim A_i = A.$$

Theorem 1.3.2. *Let $\mathcal{A}$ be the category $\mathcal{S}et$, $\mathcal{A}b$, $\mathcal{R}ing$, or $\mathcal{M}od(R)$ where R is a commutative ring. Every direct system in $\mathcal{A}$ has a direct limit.*

Proof. Let $(A_i)_{i\in I}$ be a direct system in $\mathcal{A}$. Then every object A_i for $i \in I$ is a set, and we may assume that these sets are mutually disjoint. Define

$$X = \bigsqcup_{i\in I} A_i,$$

the disjoint union of all the sets A_i for $i \in I$. We define a relation $\sim$ on X in the following way. Let $a, b \in X$, and let $i, j \in I$ be such that $a \in A_i$ and $b \in A_j$. We then define $a \sim b$ if and only if there exists $k \in I$ such that $i \leq k$, $j \leq k$, and $\rho_{ik}(a) = \rho_{jk}(b)$. It is straightfoward to verify that $\sim$ is reflexive, antisymmetric, and transitive, and is thus an equivalence relation. We now define A to be the set of equivalence classes determined by $\sim$:

$$A = X/\sim .$$

If a is in X, then we write the equivalence class determined by a as $[a]$.

Assume first that $\mathcal{A} = \mathcal{S}et$. Then A is an object of $\mathcal{A}$. We define, for $i \in I$,

$$A_i \xrightarrow{\rho_i} A$$

by $\rho_i(a) = [a]$ for $a \in A_i$. Let $i, j \in I$ with $i \leq j$; we need to see that

$$\begin{array}{ccc} A_i & \xrightarrow{\rho_{ij}} & A_j \\ & \searrow^{\rho_i} \quad \swarrow_{\rho_j} & \\ & A & \end{array}$$

commutes. Let $a \in A_i$. Then

$$\rho_j(\rho_{ij}(a)) = [\rho_{ij}(a)] \quad \text{and} \quad \rho_i(a) = [a].$$

Since $j \leq j$ and $i \leq j$, and also $\rho_{ij}(a) = \rho_{jj}(\rho_{ij}(a))$, we have by definition $a \sim \rho_{ij}(a)$. It follows that $\rho_j(\rho_{ij}(a)) = \rho_i(a)$, as desired. Next, assume that $B \in \mathrm{Ob}(\mathcal{S}et)$ and, for $i \in I$,

$$A_i \xrightarrow{\alpha_i} B$$

are such that

$$\begin{array}{ccc} A_i & \xrightarrow{\rho_{ij}} & A_j \\ & \searrow^{\alpha_i} \quad \swarrow_{\alpha_j} & \\ & B & \end{array}$$

commutes for $i, j \in I$ with $i \leq j$. Define

$$A \xrightarrow{\alpha} B$$

by $\alpha([a]) = \alpha_i(a)$ for $[a] \in A$, where i is the unique element of I such that $a \in A_i$. We claim that α is well-defined. To see this, assume that $i, j \in I$, $a \in A_i$, $b \in A_j$, and $a \sim b$, i.e., $[a] = [b]$. Since $a \sim b$, there exists $k \in I$ such that $i \leq k$, $j \leq k$, and $\rho_{ik}(a) = \rho_{jk}(b)$. Then

$$\begin{aligned} \rho_{ik}(a) &= \rho_{jk}(b) \\ \alpha_k(\rho_{ik}(a)) &= \alpha_k(\rho_{jk}(b)) \\ \alpha_i(a) &= \alpha_j(b). \end{aligned}$$

It follows that α is well-defined. It is evident from the definition of α that the diagram

$$\begin{array}{ccc} A_i & \xrightarrow{\rho_i} & A \\ & \searrow^{\alpha_i} & \downarrow \alpha \\ & & B \end{array}$$

commutes for every $i \in I$. Finally, assume that $\beta : A \longrightarrow B$ is another morphism such that

$$\begin{array}{ccc} A_i & \xrightarrow{\rho_i} & A \\ & \searrow^{\alpha_i} & \downarrow \beta \\ & & B \end{array}$$

commutes for all $i \in I$. We need to prove that $\beta = \alpha$. Let $[a] \in A$, and let $i \in I$ be the unique element of I such that $a \in A_i$. Then

$$\begin{aligned}\beta([a]) &= \beta(\rho_i(a))\\ &= \alpha_i(a)\\ &= \alpha([a]).\end{aligned}$$

It follows that $\beta = \alpha$. We conclude that A is a direct limit of $(A_i)_{i\in I}$.

Now assume that $\mathcal{A} = \mathcal{A}b$. We define an addition on A as follows. Let $a, b \in X$, and let $i, j \in I$ be such that $a \in A_i$ and $b \in A_j$. Since the set I is directed, there exists $\ell \in I$ such that $i \leq \ell$ and $j \leq \ell$. We define

$$[a] + [b] = [\rho_{i\ell}(a) + \rho_{j\ell}(b)].$$

We claim that this addition is well-defined. Assume that $[a] = [a']$ and $[b] = [b']$ for some $a', b' \in X$, that $i', j' \in I$ are such that $a' \in A_{i'}$ and $b' \in A_{j'}$, and that $\ell' \in I$ is such that $i' \leq \ell'$ and $j' \leq \ell'$. We need to prove that

$$\rho_{i\ell}(a) + \rho_{j\ell}(b) \sim \rho_{i'\ell'}(a') + \rho_{j'\ell'}(b'), \tag{1.2}$$

or equivalently, there exists $t \in I$ such that $\ell \leq t$, $\ell' \leq t$, and

$$\rho_{\ell t}(\rho_{i\ell}(a) + \rho_{j\ell}(b)) = \rho_{\ell' t}(\rho_{i'\ell'}(a') + \rho_{j'\ell'}(b')). \tag{1.3}$$

Since $a \sim a'$ and $b \sim b'$, there exist $r, s \in I$ such that $i \leq r$, $i' \leq r$, $j \leq s$, $j' \leq s$, and

$$\rho_{ir}(a) = \rho_{i'r}(a') \quad \text{and} \quad \rho_{js}(b) = \rho_{j's}(b'). \tag{1.4}$$

Let $t \in I$ be such that $r \leq t$ and $s \leq t$, and also $\ell \leq t$ and $\ell' \leq t$. Then (1.4) implies that

$$\rho_{it}(a) = \rho_{i't}(a') \quad \text{and} \quad \rho_{jt}(b) = \rho_{j't}(b').$$

Hence,

$$\begin{aligned}\rho_{it}(a) + \rho_{jt}(b) &= \rho_{i't}(a') + \rho_{j't}(b'),\\ \rho_{\ell t}\rho_{i\ell}(a) + \rho_{\ell t}\rho_{j\ell}(b) &= \rho_{\ell' t}\rho_{i'\ell'}(a') + \rho_{\ell' t}\rho_{j'\ell'}(b'),\\ \rho_{\ell t}(\rho_{i\ell}(a) + \rho_{j\ell}(b)) &= \rho_{\ell' t}(\rho_{i'\ell'}(a') + \rho_{j'\ell'}(b')).\end{aligned}$$

This is (1.3). It follows that the addition we have defined on A is well-defined. It is straightforward to verify that this addition is associative and commutative. If $i \in I$, then we denote the additive identity in A_i by 0_i. Let $i, j \in I$. Let $\ell \in I$ be such that $i \leq \ell$ and $j \leq \ell$. Then

$$\rho_{i\ell}(0_i) = 0_\ell = \rho_{j\ell}(0_j)$$

because $\rho_{i\ell}$ and $\rho_{j\ell}$ are homomorphisms. It follows that $0_i \sim 0_j$. We now define $0 \in A$ by $0 = [0_i]$; this definition does not depend on i. It is easy to verify that 0 is an additive identity for A. Thus, A is an abelian group. We define the homomorphisms $\rho_i \colon A_i \to A$ for $i \in I$ exactly as in the case $\mathcal{A} = \mathcal{S}et$. It is straightforward to verify that these homomorphisms are homomorphisms, and, arguing as in the case $\mathcal{A} = \mathcal{S}et$, that A is a direct limit of $(A_i)_{i\in I}$.

The cases $\mathcal{A} = \mathcal{R}ing$ and $\mathcal{A} = \mathcal{M}od(R)$ are similarly treated, and we omit the proofs. □

Lemma 1.3.3. *Let $\mathcal{A}$ be the category $\mathcal{S}et$, $\mathcal{A}b$, $\mathcal{R}ing$, or $\mathcal{M}od(R)$ where R is a commutative ring. Let $(A_i)_{i\in I}$ be a direct system in $\mathcal{A}$. Let $A \in \mathrm{Ob}(\mathcal{A})$, and assume that for each $i \in I$, $\rho_i\colon A_i \longrightarrow A$ is a morphism such that*

$$\begin{array}{ccc} A_i & \xrightarrow{\rho_{ij}} & A_j \\ & {\scriptstyle\rho_i}\searrow \quad \swarrow{\scriptstyle\rho_j} & \\ & A & \end{array}$$

commutes for all $i, j \in I$ with $i \leq j$. Assume additionally the following two conditions:

(a) For every $a \in A$ there exists $i \in I$ and $x \in A_i$ such that $\rho_i(x) = a$.

(b) For all $i, j \in I$, $x \in A_i$, and $y \in A_j$, we have $\rho_i(x) = \rho_j(y)$ if and only if there exists $k \in I$ such that $i \leq k$, $j \leq k$, and $\rho_{ik}(x) = \rho_{jk}(y)$.

Then A is a direct limit of the direct system $(A_i)_{i\in I}$. Moreover, if $\varinjlim A_i$ is as in the construction in the proof of Theorem 1.3.2, then the canonical isomorphism

$$\varinjlim A_i \xrightarrow{\sim} A$$

sends $[a] \in \varinjlim A_i$ to $\rho_i(a)$ if $i \in I$ is such that $a \in A_i$.

Proof. To prove that A is a direct limit of the direct system $(A_i)_{i\in I}$, it will suffice to verify, for A, the universal property from the definition of a direct limit. Assume that $B \in \mathrm{Ob}(\mathcal{A})$, and for all $i \in I$, $\alpha_i\colon A_i \longrightarrow B$ are morphisms such that

$$\begin{array}{ccc} A_i & \xrightarrow{\rho_{ij}} & A_j \\ & {\scriptstyle\alpha_i}\searrow \quad \swarrow{\scriptstyle\alpha_j} & \\ & B & \end{array}$$

commutes for $i, j \in I$ with $i \leq j$. Define $\alpha\colon A \longrightarrow B$ in the following way. Let $a \in A$. By (a), there exist $i \in I$ and $x \in A_i$ such that $\rho_i(x) = a$. Now define $\alpha(a) = \alpha_i(x)$. We claim that α is well-defined. Assume that $i, j \in I$, $x \in A_i$, and $y \in A_j$ are such that $\rho_i(x) = \rho_j(y) = a$. By (b), there exists $k \in I$ such that $i \leq k$ and $j \leq k$. and $\rho_{ik}(x) = \rho_{jk}(y)$. Hence,

$$\alpha_i(x) = \alpha_k(\rho_{ik}(x)) = \alpha_k(\rho_{jk}(y)) = \alpha_j(y).$$

This proves that α is well-defined. We claim that $\alpha \in \mathrm{Mor}(A, B)$. This is clear if $\mathcal{A} = \mathcal{S}et$. Assume that $\mathcal{A} = \mathcal{A}b$. Let $a, b \in A$, and let $i, j \in I$, $x \in A_i$, and $y \in A_j$ be such that $\rho_i(x) = a$ and $\rho_j(y) = b$. Let $k \in I$ be such that $i \leq k$ and $j \leq k$. Then

$$a = \rho_i(x) = \rho_k(\rho_{ik}(x)), \qquad b = \rho_j(y) = \rho_k(\rho_{jk}(y)),$$

so that

$$a + b = \rho_k(\rho_{ik}(x) + \rho_{jk}(y)).$$

Hence,

$$\alpha(a + b) = \alpha_k(\rho_{ik}(x) + \rho_{jk}(y))$$

$$\begin{aligned} &= \alpha_k(\rho_{ik}(x)) + \alpha_k(\rho_{jk}(y)) \\ &= \alpha_i(x) + \alpha_j(y) \\ &= \alpha(a) + \alpha(b). \end{aligned}$$

It follows that $\alpha \in \mathrm{Mor}(A, B)$. The argument that $\alpha \in \mathrm{Mor}(A, B)$ if $\mathcal{A} = \mathcal{Ring}$ or $\mathcal{A} = \mathcal{Mod}(R)$ for a commutative ring R is similar. Next, let $i \in I$. To see that

$$\begin{array}{ccc} A_i & \xrightarrow{\rho_i} & A \\ & \searrow^{\alpha_i} & \downarrow{\scriptstyle \alpha} \\ & & B \end{array}$$

commutes, let $x \in A_i$. Then, by definition, $\alpha(\rho_i(x)) = \alpha_i(x)$. Thus, the diagram commutes. Similarly, we see that if $\alpha' \in \mathrm{Mor}(A, B)$ is such that

$$\begin{array}{ccc} A_i & \xrightarrow{\rho_i} & A \\ & \searrow^{\alpha_i} & \downarrow{\scriptstyle \alpha'} \\ & & B \end{array}$$

commutes for all $i \in I$, then necessarily $\alpha' = \alpha$. This completes the verification that A has the required universal property. That the canonical isomorphism $\varinjlim A_i \xrightarrow{\sim} A$ is defined as described follows from the proof of Theorem 1.3.2. □

Lemma 1.3.4. *Let $\mathcal{A}$ be a category, and let $(A_i)_{i \in I}$ and $(B_i)_{i \in I}$ be direct systems in $\mathcal{A}$. For each $i \in I$, assume that*

$$A_i \xrightarrow{f_i} B_i$$

is a morphism such that if $i, j \in I$ with $i \leq j$, then

$$\begin{array}{ccc} A_i & \longrightarrow & B_i \\ \downarrow & & \downarrow \\ A_j & \longrightarrow & B_j \end{array}$$

commutes. Then there exists a unique morphism

$$\varinjlim A_i \xrightarrow{m} \varinjlim B_i$$

such that

$$\begin{array}{ccc} A_i & \longrightarrow & B_i \\ \downarrow & & \downarrow \\ \varinjlim A_i & \longrightarrow & \varinjlim B_i \end{array}$$

commutes for all $i \in I$.

Proof. Let $i, j \in I$ with $i \le j$. Since the diagrams

$$\begin{array}{ccc} A_i & \longrightarrow & A_j \\ \downarrow & & \downarrow \\ B_i & \longrightarrow & B_j \end{array} \qquad \text{and} \qquad \begin{array}{ccc} B_i & \longrightarrow & B_j \\ & \searrow \quad \swarrow & \\ & \varinjlim B_i & \end{array}$$

commute, the diagram

$$\begin{array}{ccccc} A_i & & \longrightarrow & & A_j \\ & \searrow & & \swarrow & \\ & B_i & & B_j & \\ & & \searrow \quad \swarrow & & \\ & & \varinjlim B_i & & \end{array}$$

commutes. This implies that there exists a unique morphism

$$\varinjlim A_i \longrightarrow \varinjlim B_i$$

such that

$$\begin{array}{ccc} A_i & \longrightarrow & \varinjlim A_i \\ & \searrow & \\ & B_i & \downarrow \\ & & \searrow \\ & & \varinjlim B_i \end{array}$$

commutes for all $i \in I$, proving the desired assertion. □

Let the notation be as in Lemma 1.3.4. Assume further that $\mathcal{A}$ is $\mathcal{Set}$, $\mathcal{Ab}$, $\mathcal{Ring}$, or $\mathcal{Mod}(R)$ where R is a commutative ring, and that $\varinjlim A_i$ and $\varinjlim B_i$ are constructed as in the proof of Theorem 1.3.2. It is then straightforward to verify that the morphism

$$\varinjlim A_i \xrightarrow{m} \varinjlim B_i$$

from Lemma 1.3.4 sends $[a]$ to $[f_i(a)]$ if $i \in I$ is such that $a \in A_i$.

Chapter 2

Presheaves

2.1 The definition

Let X be a topological space. To X we associate a category $\text{Open}(X)$ as follows. The objects in $\text{Open}(X)$ are the open sets in X. If U and V are open sets in X, then we define

$$\text{Mor}(V, U) = \begin{cases} \emptyset & \text{if } V \not\subset U, \\ \{\text{the inclusion map}\} & \text{if } V \subset U. \end{cases}$$

Let $\mathcal{C}$ be a category. A **presheaf** on X with values in $\mathcal{C}$ is contravariant functor

$$\mathcal{F} : \text{Open}(X) \longrightarrow \mathcal{C}.$$

Concretely, a presheaf $\mathcal{F}$ on X provides the following:

(a) for each open subset U in X, an object $\mathcal{F}(U)$ of $\mathcal{C}$;

(b) for open subsets U and V of X such that $V \subset U$, a morphism

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$

such that if $U = V$, then this morphism is the identity, and if W is another open subset of X with $W \subset V \subset U$, then

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{F}(W) \\ & \searrow \qquad \nearrow & \\ & \mathcal{F}(V) & \end{array}$$

commutes.

If $\mathcal{F}$ is a presheaf on X, and U and V are open subsets of X with $V \subset U$, then $\mathcal{F}(U) \to \mathcal{F}(V)$ is referred to as a **restriction morphism**.

2.2 Examples of presheaves

Presheaves of continuous functions. Let X be a topological space, and let Y also be a topological space. We define a presheaf C^Y on X with values in $\mathcal{S}et$ by setting $C^Y(U)$ to be the set of all continuous functions $U \to Y$ for all open subsets U of X, and letting $C^Y(U) \to C^Y(V)$ be restriction of functions for all open subsets U and V of X with $V \subset U$.

Presheaves of differentiable functions. Assume that X is an open subset of $\mathbb{R}^n$ for some positive integer n. Let $r \in \{0, 1, 2, 3, \ldots, \infty\}$. We define a presheaf C^r on X with values in $\mathcal{A}b$ by setting $C^r(U)$ to be the set of all r-times continuous differentiable functions $U \to \mathbb{R}$ for all open subsets U of X, and letting $C^r(U) \to C^r(V)$ be restriction of functions for all open subsets U and V of X with $V \subset U$.

Presheaves of analytic functions. Assume that X is an open subset of $\mathbb{C}^n$ for some positive integer n. We define a presheaf C^ω on X with values in $\mathcal{R}ing$ by setting $C^\omega(U)$ to be the set of all analytic functions $U \to \mathbb{C}$ for all open subsets U of X, and letting $C^\omega(U) \to C^\omega(V)$ be restriction of functions for all open subsets U and V of X with $V \subset U$.

Constant presheaves. Let X be a topological space, and let $\mathcal{C}$ be a category. Let A be an object in $\mathcal{C}$. We define a presheaf A_X on X with values in $\mathcal{C}$ by setting $A_X(U) = A$ for all open subsets U of X, and letting $A_X(U) \to A_X(V)$ be the identity element of $\mathrm{Mor}(A, A)$ for all open subsets U and V of X with $V \subset U$. We refer to A_X as a **constant presheaf**.

Skyscraper presheaves. Let X be a non-empty topological space. Let $A \in \mathrm{Ob}(\mathcal{A}b)$. Fix an element $x_0 \in X$. We define a presheaf $\mathcal{S} = \mathcal{S}_A$ on X with values in $\mathcal{A}b$ by setting

$$\mathcal{S}(U) = \begin{cases} A & \text{if } x_0 \in U, \\ 0 & \text{if } x_0 \notin U \end{cases}$$

for open subsets U of X, and letting

$$\mathcal{S}(U) \longrightarrow \mathcal{S}(V) \quad \text{be} \quad \begin{cases} \mathrm{id}_A & \text{if } x_0 \in V \subset U, \\ 0 & \text{if } x_0 \notin V \subset U \end{cases}$$

for all open subsets U and V of X with $V \subset U$. We refer to $\mathcal{S}$ as a skyscraper presheaf.

A pathological example. Let X be a T_1 topological space with at least two points. (recall that T_1 means that for any two elements $a, b \in X$ with $a \neq b$ there exists open subsets U and V of X such that $a \in U$ and $b \notin U$ and $a \notin V$ and $b \in V$). We define a presheaf $\mathcal{P}$ on X with values in $\mathcal{A}b$ by setting

$$\mathcal{P}(U) = \begin{cases} \mathbb{Z} & \text{if } U = X, \\ 0 & \text{if } U \subsetneq X \end{cases}$$

for open subsets U of X, and letting

$$\mathcal{P}(U) \longrightarrow \mathcal{P}(V) \quad \text{be defined by} \quad \begin{cases} x \mapsto 0 & \text{if } V \subsetneq X, \\ x \mapsto x & \text{if } U = V = X \end{cases}$$

for all open subsets U and V of X with $V \subset U$.

2.3 Stalks of presheaves

Let X be a topological space, and let $\mathcal{C}$ be a category. Let $\mathcal{F}$ be a presheaf on X with values in $\mathcal{C}$. Assume that every direct system in $\mathcal{C}$ has a direct limit. In this situation, the presheaf $\mathcal{F}$ defines a function from X to $\mathrm{Ob}(\mathcal{C})$.

To explain this, let $x \in X$. Consider set $I = I_x$ of all open subsets U of X that contain x. We define a relation $\leq$ on I by letting $U \leq V$ if and only if $V \subset U$ for $U, V \in I$. Evidently, $\leq$ is a partial order on I (i.e., $\leq$ is reflexive, antisymmetric, and transitive). The set is also directed: if $U, V \in I$, then $U \cap V \in I$ and $U \leq U \cap V$ and $V \leq U \cap V$. The presheaf $\mathcal{F}$ associates a direct system to I. Let $U, V \in I$ with $U \leq V$, i.e., $V \subset U$. The presheaf $\mathcal{F}$ provides a morphism

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(V).$$

With these maps, we obtain a direct system $(\mathcal{F}(U))_{U \in I}$. We define

$$\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U).$$

By our assumption on $\mathcal{C}$, this direct limit exists. We refer to $\mathcal{F}_x$ as the **stalk** of $\mathcal{F}$ at x. Assume further that $\mathcal{C}$ is a subcategory of $\mathcal{S}et$. Let U be an open subset of X and let $x \in U$. Evidently, $\mathcal{F}(U) \to \mathcal{F}_x$ is a function between sets. We will sometimes refer to the elements of $\mathcal{F}_x$ as **germs**. If $s \in \mathcal{F}(U)$, then we will denote the image of s under the function $\mathcal{F}(U) \to \mathcal{F}_x$ by s_x. For us, $\mathcal{C}$ will usually be $\mathcal{S}et$, $\mathcal{A}b$, $\mathcal{R}ing$, or $\mathcal{M}od(R)$. We have the following lemma.

Lemma 2.3.1. *Let X be a topological space, and let $\mathcal{C}$ be $\mathcal{S}et$, $\mathcal{A}b$, $\mathcal{R}ing$, or $\mathcal{M}od(R)$. Let $\mathcal{F}$ be a presheaf on X with values in $\mathcal{C}$. Let $x \in X$.*

(a) Let U be an open subset of X such that $x \in U$, and let $e \in \mathcal{F}_x$. There exists an open subset W of X and $s \in \mathcal{F}(W)$ such that $x \in W \subset U$ and $s_x = e$.

(b) Let U and V be open subsets of X such that $x \in U$ and $x \in V$, and let $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$. If $s_x = t_x$, then there exists an open subset W of X such that $x \in W \subset U \cap V$ and

$$s_y = t_y \tag{2.1}$$

for $y \in W$.

Proof. (a) By the construction of $\mathcal{F}_x$ from the proof of Theorem 1.3.2, there exists an open subset V of X such that $x \in V$ and $r \in \mathcal{F}(V)$ such that $r_x = e$. Since the diagram

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\rho_{V,U\cap V}} & \mathcal{F}(U \cap V) \\ & \searrow \quad \swarrow & \\ & \mathcal{F}_x & \end{array}$$

commutes, we have $\rho_{U,U\cap V}(r)_x = e$. Thus, (a) holds with $W = U \cap V$ and $s = \rho_{U,U\cap V}(r)$.

(b) Assume that $s_x = t_x$. By the construction of $\mathcal{F}_x$ from the proof of Theorem 1.3.2, since $s_x = t_x$, there exists an open subset W of X such that $x \in W \subset U \cap V$ and $\rho_{U,W}(t) = \rho_{V,W}(s) \in \mathcal{F}(W)$. Let $y \in W \subset U \cap V$. Then t and $\rho_{U,W}(t)$ define the same germ in $\mathcal{F}_y$, i.e., $t_y = \rho_{U,W}(t)_y$; similarly, $s_y = \rho_{V,W}(s)_y$. Since $\rho_{U,W}(t) = \rho_{V,W}(s)$, we obtain $s_y = t_y$. $\square$

We consider some examples.

The stalks of the presheaf of analytic functions. Let X be an open subset of $\mathbb{C}$, and, as above, let C^ω be the presheaf of analytic functions on X. Let $x \in X$. How can we think of the elements of C^ω_x? Let $G \in C^\omega_x$. By the definition of C^ω_x, the germ G is an equivalence class: let U be an open subset of X such that $x \in U$ and let $f \in C^\omega(U)$ be such that $G = [f]$. The function f is only one representative for G. Assume that h is another representative for G. Then, from the involved definitions, $h \in C^\omega(V)$ where V is another open subset of X such that $x \in V$, and there exists an open subset W of X such that $x \in W \subset U \cap V$ and

$$h|_W = f|_W.$$

Evidently, the germ G encodes the local behavior of f (or any other representative for G) at x. We can make C^ω_x even more concrete. Since f is analytic at x, the function f admits a power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z-x)^n$$

that converges to $f(x)$ in an open disk contained in U and centered at x. We define

$$C^\omega_x \xrightarrow{\sim} \mathbb{C}\{z-x\}$$

by

$$f \mapsto \sum_{n=0}^{\infty} a_n (z-x)^n.$$

Here, $\mathbb{C}\{z-a\}$ is the ring of complex power series in $z-x$ that converge in some open disk centered at x. This map is an isomorphism of rings.

The stalks of a constant presheaf. Let X be a non-empty topological space, let $\mathcal{C}$ be a category for which every direct system has a direct limit, let A be an object of $\mathcal{C}$, and let A_X be the previously defined constant presheaf. Let $x \in X$. We claim that the stalk of A_X at x is

$$A_{X,x} = \varinjlim_{x \in U} A_X(U) = A.$$

More precisely, for each open subset U of X with $x \in U$, define

$$A_X(U) = A \longrightarrow A$$

to be the identity. Clearly, if U and V are open subsets of X with $x \in V \subset U$, then

$$\begin{array}{ccc} A_X(U) & \longrightarrow & A_X(V) \\ & \searrow \quad \swarrow & \\ & A & \end{array}$$

commutes as all of these morphisms are the identity. Next, suppose that $B \in \mathrm{Ob}(\mathcal{C})$, and there exist morphisms

$$A_X(U) = A \longrightarrow B$$

such that if U and V are open subsets of X with $x \in V \subset U$, then

$$\begin{array}{ccc} A_X(U) & \longrightarrow & A_X(V) \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

commutes. Since $A_X(U) = A = A_X(V)$ and $A_X(U) \to A_X(V)$ is the identity morphism for all open subsets U and V of X with $x \in V \subset U$, we see that the maps $A_X(U) \to B$, for U an open subset of X such that $x \in U$, are all the same morphism $\alpha : A \to B$. Evidently, if U is an open subset of X such that $x \in U$, then

$$\begin{array}{ccc} A_X(U) & \longrightarrow & A \\ & \searrow & \downarrow{\scriptstyle \alpha} \\ & & B \end{array}$$

commutes. Moreover, it is clear that α is the unique such morphism.

The stalks of a skyscraper presheaf. Let X be a topological space, let $A \in \mathrm{Ob}(\mathcal{Ab})$, and fix an element $x_0 \in X$. Let $\mathcal{S} = \mathcal{S}_A$ be the associated skyscraper presheaf defined above. Let C be the closure of the set $\{x_0\}$ in X. Let $x \in X$. We claim that the stalk $\mathcal{S}_x$ is given by

$$\mathcal{S}_x = \begin{cases} A & \text{if } x \in C, \\ 0 & \text{if } x \notin C. \end{cases}$$

Let $x \in C$. Let U be an open subset of X such that $x \in U$. Then $x_0 \in U$ (otherwise, $X - U$ is a closed set containing x_0, so that $x \in C \subset X - U$ because C is the closure of $\{x_0\}$, a contradiction). Hence, $\mathcal{S}(U) = A$. From this, we see that if U and V are open subsets of X such that $x \in V \subset U$, then the restriction homomorphism $\mathcal{S}(U) \to \mathcal{S}(V)$ is id_A. Arguing as in the case of the constant presheaf, we obtain

$$\mathcal{S}_x = \varinjlim_{x \in U} \mathcal{S}(U) = A.$$

Now assume that $x \notin C$. We need to prove that $\mathcal{S}_x = 0$. For this, it will suffice to prove that if $B \in \mathrm{Ob}(\mathcal{Ab})$, and for each open subset U of X such that $x \in U$,

$$\mathcal{S}(U) \longrightarrow B$$

is a homomorphism such that

$$\begin{array}{ccc} \mathcal{S}(U) & \longrightarrow & \mathcal{S}(V) \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

commutes for all open subsets U and V of X such that $x \in V \subset U$, then $\mathcal{S}(U) \to B$ is the zero homomorphism for all open subsets U of X such that $x \in U$. Let U be an open subset of X such that $x \in U$. Since $x \notin C$, there exists a closed subset D of X such that $x_0 \in D$ and $x \notin D$. Let $W = X - D$. Then $x \in W$ and $x_0 \notin W$. Define $V = W \cap U$. Then $x \in V \subset U$. As $x_0 \notin V$, we have $\mathcal{S}(V) = 0$; hence, the homomorphism $\mathcal{S}(U) \to \mathcal{S}(V)$ is the zero homomorphism. It follows that $\mathcal{S}(U) \to B$ is also the zero homomorphism, as desired.

The stalks of the pathological example. Let X be a T_1 topological space with at least two points, and let $\mathcal{P}$ be the pathological presheaf from above. Let $x \in X$. We claim that $\mathcal{P}_x = 0$, i.e.,

$$\varinjlim_{x \in U} \mathcal{P}(U) = 0.$$

To prove this, it will suffice to prove that if $B \in \mathrm{Ob}(\mathcal{Ab})$, and for each open subset U of X such that $x \in U$,

$$\mathcal{P}(U) \longrightarrow B$$

is a homomorphism such that

$$\begin{array}{ccc} \mathcal{P}(U) & \longrightarrow & \mathcal{P}(V) \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

commutes for all open subsets U and V of X such that $x \in V \subset U$, then $\mathcal{P}(U) \to B$ is the zero homomorphism for all open subsets U of X such that $x \in U$. Let U be an open subset of X such that $x \in U$. If $U \neq X$, then $\mathcal{P}(U) = 0$ by definition, so that $\mathcal{P}(U) \to B$ is the zero homomorphism. Assume that $U = X$. By our assumption, there exists $y \in X$ with $x \neq y$. Also, since X is T_1, there exists an open subset V of X such that $x \in V$ but $y \notin V$. The homomorphism

$$\mathcal{P}(U) = \mathbb{Z} \longrightarrow \mathcal{P}(V) = 0$$

is necessarily the zero homomorphism. This implies that $\mathcal{P}(U) \to B$ is also the zero homomorphism.

2.4 Morphisms of presheaves

Let X be a topological space, let $\mathcal{C}$ be a category, and let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on X with values in $\mathcal{C}$. A **morphism** of presheaves

$$\mathcal{F} \xrightarrow{f} \mathcal{G}$$

is a collection of morphisms

$$\left\{ \mathcal{F}(U) \xrightarrow{f(U)} \mathcal{G}(U) \right\}_{U \subset X \text{ open}}$$

such that for all open subsets U and V of X with $V \subset U$ the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f(U)} & \mathcal{G}(V) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \xrightarrow{f(V)} & \mathcal{G}(V) \end{array} \tag{2.2}$$

commutes. Evidently, the collection

$$\mathcal{F} \xrightarrow{\mathrm{id}_{\mathcal{F}}} \mathcal{F} = \left\{ \mathcal{F}(U) \xrightarrow{\mathrm{id}_{\mathcal{F}(U)}} \mathcal{F}(U) \right\}_{U \subset X \text{ open}}$$

is a morphism of presheaves. The composition of morphisms of presheaves is defined in the obvious way. We say that $f : \mathcal{F} \to \mathcal{G}$ is an **isomorphism** if there exists a morphism $g : \mathcal{G} \to \mathcal{F}$ such that $g \circ f = \mathrm{id}_{\mathcal{F}}$ and $f \circ g = \mathrm{id}_{\mathcal{G}}$. We note that the morphism $f : \mathcal{F} \to \mathcal{G}$ also induces morphisms of stalks. Let $x \in X$. If U and V are open subsets of X such that $x \in V \subset U$, then the diagram (2.2) commutes. By Lemma 1.3.4, this implies that there exists a unique morphism

$$\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U) \longrightarrow \mathcal{G}_x = \varinjlim_{x \in U} \mathcal{G}(U)$$

such that

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f(U)} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_x & \longrightarrow & \mathcal{G}_x \end{array}$$

commutes for all open subsets U of X such that $x \in U$.

Lemma 2.4.1. *Let X be a topological space, let $\mathcal{C}$ be a category, and let $\mathcal{F}$ and G be presheaves on X with values in $\mathcal{C}$. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. Then f is an isomorphism if and only if $f(U)$ is an isomorphism for every open subset U of X.*

Proof. Assume that f is an isomorphism, with inverse $g : \mathcal{G} \to \mathcal{F}$. Then $g \circ f = \mathrm{id}_{\mathcal{F}}$ and $f \circ g = \mathrm{id}_{\mathcal{G}}$. This means that for all open subsets U of X we have $g(U) \circ f(U) = \mathrm{id}_{\mathcal{F}(U)}$ and $f(U) \circ g(U) = \mathrm{id}_{\mathcal{G}(U)}$. It follows that $f(U)$ is an isomorphism for all open subsets U of X.

Conversely, assume that $f(U)$ is an isomorphism for all open subsets U of X. For each open subset U of X, let $g(U) : \mathcal{G}(U) \to \mathcal{F}(U)$ be the inverse of $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$. We claim that

$$\left\{ \mathcal{G}(U) \xrightarrow{g(U)} \mathcal{F}(U) \right\}_{U \subset X \text{ open}}$$

is a morphism of presheaves. Let U and V be open subsets of X with $V \subset U$. Since f is a morphism of presheaves

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f(U)} & \mathcal{G}(V) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ \mathcal{F}(V) & \xrightarrow{f(V)} & \mathcal{G}(V) \end{array}$$

commutes; here, we have given names to the restriction morphisms. We have:

$$\begin{aligned} \beta \circ f(U) &= f(V) \circ \alpha \\ \beta \circ f(U) \circ g(U) &= f(V) \circ \alpha \circ g(U) \\ \beta \circ \mathrm{id}_{\mathcal{G}(U)} &= f(V) \circ \alpha \circ g(U) \end{aligned}$$

$$\begin{aligned}
\beta &= f(V) \circ \alpha \circ g(U) \\
g(V) \circ \beta &= g(V) \circ f(V) \circ \alpha \circ g(U) \\
g(V) \circ \beta &= \mathrm{id}_{\mathcal{F}(V)} \circ \alpha \circ g(U) \\
g(V) \circ \beta &= \alpha \circ g(U).
\end{aligned}$$

Thus,

$$\begin{array}{ccc}
\mathcal{G}(U) & \xrightarrow{g(U)} & \mathcal{F}(V) \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\
\mathcal{G}(V) & \xrightarrow{g(V)} & \mathcal{F}(V)
\end{array}$$

commutes. It follows that

$$\mathcal{G} \xrightarrow{g} \mathcal{F} = \left\{ \mathcal{G}(U) \xrightarrow{g(U)} \mathcal{F}(U) \right\}_{U \subset X \text{ open}}$$

is a morphism of presheaves. Since $g \circ f = \mathrm{id}_{\mathcal{F}}$ and $f \circ g = \mathrm{id}_{\mathcal{G}}$, the morphism f is an isomorphism. □

Chapter 3

Sheaves

3.1 The definition

Let X be a topological space. Let U be an open subset of X. An **open cover** of U consists of a set I, and for each $i \in I$, an open subset U_i of U, such that $U = \cup_{i \in I} U_i$. Next, let $\mathcal{C}$ be a subcategory of $\mathcal{S}et$, and let $\mathcal{F}$ be a presheaf on X with values in $\mathcal{C}$. If U and V are open subsets of X such that $V \subset U$, then we will denote the restriction morphism $\mathcal{F}(U) \to \mathcal{F}(V)$ by $\rho_{U,V}$; note that since $\mathcal{C}$ is a subcategory of $\mathcal{S}et$, the morphism $\rho_{U,V}$ is actually a function between sets.

Given these circumstances, we consider two conditions. The first condition is called the **gluing condition** and is stated as follows:

(G) If U is an open subset of X, $\{U_i\}_{i \in I}$ is an open cover of U, $\{s_i\}_{i \in I}$ is such that $s_i \in \mathcal{F}(U_i)$ for $i \in I$, and for all $i, j \in I$ we have $\rho_{U_i, U_i \cap U_j}(s_i) = \rho_{U_j, U_i \cap U_j}(s_j)$, then there exists $s \in \mathcal{F}(U)$ such that $\rho_{U,U_i}(s) = s_i$ for all $i \in I$.

The gluing condition asserts that if a collection of local sections agree on overlaps, then these local sections are the restriction of a section. We also consider the following condition:

(L) If U is an open subset of X, $\{U_i\}_{i \in I}$ is an open cover of U, and $s, s' \in \mathcal{F}(U)$ are such that $\rho_{U,U_i}(s) = \rho_{U,U_i}(s')$ for all $i \in I$, then $s = s'$.

We will sometimes refer to condition (L) as the **locality condition**. If conditions (G) and (L) both hold, and if a collection of local sections agree on overlaps, then by (G) these local sections are the restriction of a section, and by (L) this section is unique. If $\mathcal{F}$ satisfies both (G) and (L), then we say that $\mathcal{F}$ is a **sheaf**. If $\mathcal{F}$ satisfies (L) (but possibly not (G)), then we say that $\mathcal{F}$ is a **monopresheaf** or a **separated presheaf**.

Let X be a topological space, and let $\mathcal{C}$ be a subcategory of $\mathcal{S}et$. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X with values in $\mathcal{C}$. In this case, we define a **morphism** $\mathcal{F} \to \mathcal{G}$ exactly as in the case of presheaves.

Proposition 3.1.1. *Let X be a topological space, let $\mathcal{C}$ be a subcategory of $\mathcal{S}et$, and let $\mathcal{F}$ be a monopresheaf on X with values in $\mathcal{C}$. Let U be an open subset of X, and let $s, s' \in \mathcal{F}(U)$. Then $s = s'$ if and only if $s_x = s'_x$ for all $x \in U$.*

Proof. It is clear that if $s = s'$, then $s_x = s'_x$ for all $x \in U$. Assume that $s_x = s'_x$ for all $x \in U$. Let $x \in U$. Since $s_x = s'_x$, there exists an open subset V_x of X such that $x \in V_x \subset U$ and $\rho_{U,V_x}(s) =$

$\rho_{U,V_x}(s')$. The collection $\{V_x\}_{x\in U}$ is an open cover of U. Applying the locality condition (L) to this cover and s and s' we conclude that $s = s'$. ◻

Corollary 3.1.2. *Let X be a topological space, let $\mathcal{C}$ be a subcategory of $\mathcal{S}et$, and let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on X with values in $\mathcal{C}$. Let $f, f' : \mathcal{F} \to \mathcal{G}$ be maps of presheaves. Assume further that $\mathcal{G}$ is a monopresheaf. If $f_x = f'_x$ for all $x \in X$, then $f = f'$.*

Proof. Let U be an open subset of X. We need to prove that $f(U), f'(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ are the same function. Let $s \in \mathcal{F}(U)$; we need to prove that $f(U)(s) = f'(U)(s)$. Let $x \in U$. Since the diagrams

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f(U)} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_x = \varinjlim_{x\in V} \mathcal{F}(V) & \xrightarrow{f_x} & \mathcal{G}_x = \varinjlim_{x\in V} \mathcal{G}(V) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f'(U)} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_x = \varinjlim_{x\in V} \mathcal{F}(V) & \xrightarrow{f'_x} & \mathcal{G}_x = \varinjlim_{x\in V} \mathcal{G}(V) \end{array}$$

commute, we have:

$$\begin{aligned} f(U)(s)_x &= f_x(s_x) \\ &= f'_x(s_x) \qquad (\text{because } f_x = f'_x) \\ &= f'(U)(s)_x. \end{aligned}$$

Since $f(U)(s)_x = f'(U)(s)_x$ for all $x \in U$, Proposition 3.1.1 implies that $f(U)(s) = f'(U)(s)$. ◻

Lemma 3.1.3. *Let X be a topological space, let $\mathcal{C}$ be a subcategory of $\mathcal{S}et$, and let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on X with values in $\mathcal{C}$. Let $f : \mathcal{F} \to \mathcal{G}$ be an isomorphism of presheaves. If $\mathcal{F}$ is a sheaf, then so is $\mathcal{G}$.*

Proof. We first verify the gluing condition (G). Let U be an open subset of X, let $\{U_i\}_{i\in I}$ be an open cover of U, let $\{s_i\}_{i\in I}$ be such that $s_i \in \mathcal{G}(U_i)$ for $i \in I$, and assume that for all $i, j \in I$ we have $\rho^{\mathcal{G}}_{U_i,U_i\cap U_j}(s_i) = \rho^{\mathcal{G}}_{U_j,U_i\cap U_j}(s_j)$. For $i \in I$, define $r_i = f(U)^{-1}(s_i) \in \mathcal{F}(U_i)$. Then for all $i, j \in I$ we have $\rho^{\mathcal{F}}_{U_i,U_i\cap U_j}(r_i) = \rho^{\mathcal{F}}_{U_j,U_i\cap U_j}(r_j)$. Since $\mathcal{F}$ is a sheaf, $\mathcal{F}$ satisfies the gluing condition (G). Hence, there exists $r \in \mathcal{F}(U)$ such that $\rho^{\mathcal{F}}_{U,U_i}(r) = r_i$ for $i \in I$. Set $s = f(U)(r) \in \mathcal{G}(U)$. Then $\rho^{\mathcal{G}}_{U,U_i}(s) = s_i$ for $i \in I$. This verifies the gluing condition for $\mathcal{G}$. The locality condition (L) for $\mathcal{G}$ is similarly verified. ◻

3.2 Which presheaf examples are sheaves?

We consider which of the examples of presheaves from Section 2.2 are sheaves.

Presheaves of continuous, differentiable, and analytic functions. The presheaves C^Y, C^r, and C^ω are all sheaves. For example, suppose X and Y are topological spaces, and consider the presheaf C^Y. To verify the gluing condition (G), let U be an open subset of X, let $\{U_i\}_{i\in I}$ be an open cover of U, and let $\{f_i\}_{i\in I}$ be such that $f_i \in C^Y(U_i)$ for $i \in I$ and for all $i, j \in I$, we have $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$. Define $f : U \to Y$ by $f(x) = f_i(x)$ for $x \in U$ and $i \in I$ such that $x \in U_i$. Then f is well-defined and continuous, and $f|_{U_i} = f_i$ for $i \in I$. This verifies the gluing condition (G). The locality condition (L) is similarly verified.

Constant presheaves. Let X be a topological space, and assume that $\mathcal{C}$ is a subcategory of $\mathcal{S}et$. Let A be an object of $\mathcal{C}$. In Section 2.2 we defined the constant sheaf A_X on X with values in $\mathcal{C}$. The presheaf A_X is not always a sheaf. For example, assume that A contains at least two elements s and s'. Then A_X does not satisfy the locality condition (L). To see this, let $U = \emptyset$. For an open cover of U we let $I = \emptyset$. We have $A_X(U) = A_X(\emptyset) = A$ so that $s, s' \in A_X(U)$. Then the condition in the locality condition (L) is trivially satisfied; however, $s \neq s'$.

Skyscraper presheaves. Let X be a non-empty topological space, let $A \in \mathrm{Ob}(\mathcal{A}b)$, and let $x_0 \in X$. In Section 2.2 we defined the associated skyscraper presheaf $\mathcal{S}_A$ on X with values in $\mathcal{A}b$. We claim that $\mathcal{S}$ is a sheaf.

To verify the gluing condition (G), let U be an open subset of X, let $\{U_i\}_{i\in I}$ be an open cover of U, and let $\{s_i\}_{i\in I}$ be such that $s_i \in \mathcal{S}(U_i)$ and $\rho_{U_i,U_i\cap U_j}(s_i) = \rho_{U_j,U_i\cap U_j}(s_j)$ for $i, j \in I$. Assume that $x_0 \notin U$. Then $\mathcal{S}(U_i) = 0$ for all $i \in I$ so that $s_i = 0$ for all $i \in I$; hence, setting $s = 0$, we have $\rho_{U,U_i}(s) = s_i$ for all $i \in I$. Assume that $x_0 \in U$. Let $k \in I$ be such that $x_0 \in U_k$. Then $\mathcal{S}(U_k) = A = \mathcal{S}(U)$. Define define $s = s_k$; note that $s \in \mathcal{S}(U)$. Let $i \in I$. If $x_0 \notin U_i$, then $\mathcal{S}(U_i) = 0$ so that $s_i = 0$, and we have $\rho_{U,U_i}(s) = s_i$. Assume that $x_0 \in U_i$. By assumption, we have $\rho_{U_i,U_i\cap U_k}(s_i) = \rho_{U_k,U_i\cap U_k}(s_k)$; since $\mathcal{S}(U_k) = \mathcal{S}(U_i \cap U_k) = A$ and $\rho_{U_k,U_i\cap U_k} = \rho_{U_i,U_i\cap U_k} = \mathrm{id}_A$, this implies that $s_i = s_k = s$. Similarly, $\rho_{U,U_i}(s) = \mathrm{id}_A(s) = s$. It follows that $\rho_{U,U_i}(s) = s_i$. This verifies the gluing condition (G).

To verify the locality condition (L), let U be an open subset of X, let $\{U_i\}_{i\in I}$ be an open cover of U, and let $s, s' \in \mathcal{S}(U)$ be such that $\rho_{U,U_i}(s) = \rho_{U,U_i}(s')$ for all $i \in I$. Assume first that $x_0 \notin U$. Then $\mathcal{S}(U) = 0$, and $s = s' = 0$. Assume that $x_0 \in U$. Then $x_0 \in U_k$ for some $k \in I$. We have $\mathcal{S}(U) = \mathcal{S}(U_k) = A$ and $\rho_{U,U_k} = \mathrm{id}_A$. Since $\rho_{U,U_k}(s) = \rho_{U,U_k}(s')$, we obtain $s = s'$. This verifies the locality condition (L).

The pathological example. Let X be a T_1 topological space with at least two points, and let $\mathcal{P}$ be the presheaf on X with values in $\mathcal{A}b$ defined in Section 2.2. We claim that $\mathcal{P}$ is *not* a sheaf. By our assumptions on X, there exists an open cover $\{U_i\}_{i\in I}$ of X such that U_i for $i \in I$ is a proper subset of X. Consider the elements $0, 1 \in \mathcal{P}(X) = \mathbb{Z}$. We have $\rho_{X,U_i}(0) = 0 = \rho_{X,U_i}(1)$ for all $i \in I$; however, $0 \neq 1$. Thus, $\mathcal{P}$ does not satisfy the locality condition (L).

3.3 Étalé spaces

Let X be a topological space. An **Étalé space** over X is a pair (E, p) where E is a topological space and $p\colon E \to X$ is a function that is a local homeomorphism (i.e., for every $a \in E$ there exists an open subset C of E and an open subset V of X such that $a \in C$, $p(C) = V$, and the function $p|_C\colon C \to V$ is a homeomorphism). Let (E, p) and (E', p') be étalé spaces over X. A **morphism** $f\colon (E, p) \to (E', p')$ is a continuous function $f\colon E \to E'$ such that

$$\begin{array}{ccc} E & \xrightarrow{\ f\ } & E' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & X & \end{array} \tag{3.1}$$

commutes.

Some topology

Lemma 3.3.1. *Let Y_1 and Y_2 be topological spaces and let $f: Y_1 \to Y_2$ be a local homeomorphism. Then f is continuous and open.*

Proof. To prove that f is continuous, let U be an open subset of Y_2; we need to prove that $f^{-1}(U)$ is open. Let $y \in f^{-1}(U)$. Since f is a local homeomorphism, there exists an open subset V_y of Y_1 such that $y \in V_y$, $f(V_y)$ is open, and $f|_{V_y}: V_y \to f(V_y)$ is a homeomorphism. Evidently, we have $f(y) \in f(V_y) \cap U$. Define $U_y = (f|_{V_y})^{-1}(f(V_y) \cap U)$. The set U_y is an open subset of Y_1, U_y is contained in $f^{-1}(U)$, and $y \in U_y$. Since $f^{-1}(U)$ is the union of the open sets U_y for $y \in f^{-1}(U)$, the set $f^{-1}(U)$ is open.

To see that f is open, let C be an open subset of Y_1. Since f is a local homeomorphism, the open set C admits an open cover $\{C_i\}_{i\in I}$ of open subsets of Y_1 such that $f(C_i)$ is open for $i \in I$. It follows that the set $f(C) = \cup_{i\in I} p(C_i)$ is open. □

Lemma 3.3.2. *Let X be a topological space, and let (E, p) and (E', p') be étalé spaces over X. Let $f: E \to E'$ be a function such that the diagram (3.1) commutes. The following are equivalent:*

(a) f is continuous.

(b) f is open.

(c) f is a local homeomorphism.

Proof. (c) $\Longrightarrow$ (a) and (b). This follows from Lemma 3.3.1.

(a) $\Longrightarrow$ (c). Assume that f is continuous. Let $a \in E$. Since p' is a local homeomorphism, there exists an open subset C' of E' such that $f(a) \in C'$, $p'(C')$ is an open subset of X, and the function $p'|_{C'}: C' \to p(C')$ is a local homeomorphism. Since f is continuous, there exists an open subset D of E such that $a \in D$ and $f(D) \subset C'$. Also, since p is a local homeomorphism, there exists an open subset C of E such that $a \in C \subset D$, $p(C)$ is an open subset of X, and $p|_C: C \to p(C)$ is a homeomorphism. Since $p = p' \circ f$ and $f(C) \subset C'$, we have $p(C) = p'(f(C)) \subset p'(C')$. Since $p(C) \subset p'(C')$, $p(C)$ is open in X, and $p'|_{C'}: C' \to p(C')$ is a homeomorphism, the set $(p'|_{C'})^{-1}(p(C))$ is an open subset of C'. Using $p = p' \circ f$ it is straightforward to verify that $f(C) = (p'|_{C'})^{-1}(p(C))$. It follows that $f(C)$ is an open subset of C'. We now have a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\ f|_C\ } & f(C) \\ & {\scriptstyle p|_C} \searrow \quad \swarrow {\scriptstyle p'|_{f(C)}} & \\ & p(C) = p'(f(C)) & \end{array}$$

where every set is open. Since $p|_C = (p'|_{f(C)}) \circ f|_C$ and $p|_C$ and $p'|_{f(C)}$ are homeomorphisms, the function $f|_C: C \to f(C)$ is also a homeomorphism.

(b) $\Longrightarrow$ (c). Assume that f is open. Let $a \in E$. Since p is a local homeomorphism, there exists an open set C such that $a \in C$, $p(C)$ is open, and $p|_C: C \to p(C)$ is a homeomorphism. Since f is open, the subset $f(C)$ of E' is open. Also, since $f(a)$ is contained in the open set $f(C)$ and p' is a local homeomorphism, there exists an open subset C' of E' such that $f(a) \in C' \subset f(C)$ and $p'|_{C'}: C' \to p(C')$ is a local homeomorphism. Define $V = p(C) \cap p(C')$, $D = (p|_C)^{-1}(V)$, and $D' = (p|_{C'})^{-1}(V)$. Then $p|_D: D \to V$ and $p'|_{D'}: D' \to V$ are homeomorphisms. We claim that $f(D) \subset D'$. Let $d \in D$. Then $p(d) \in p(D) = V = p'(D')$. Let $d' \in D'$ be such that $p'(d') = p(d)$. Since $D' \subset C' \subset f(C)$, there exists $c \in C$ such that $f(c) = d'$. Now $p'(d') = p'(f(c)) = p(c)$. It follows that $p(d) = p(c)$. Since $p|_C$ is injective, we obtain $c = d$. Since $f(c) = d'$, this yields that $f(d) = d'$. It follows that $f(D) \subset D'$. We now have a commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{f|_D} & D' \\ & {\scriptstyle p|_D}\searrow \quad \swarrow{\scriptstyle p'|_{D'}} & \\ & p(D) = p'(D') & \end{array}$$

where every set is open. Since $p|_D = (p'|_{D'}) \circ f|_D$ and $p|_D$ and $p'|_{D'}$ are homeomorphisms, the function $f|_D \colon D \to D'$ is also a homeomorphism. □

Lemma 3.3.3. *Let X be a topological space, and let (E, p) be an étale space over X.*

(a) Let U be an open subset of X, and let $s \colon U \to E$ be a continuous function such that

$$\begin{array}{ccc} & & E \\ & {\scriptstyle s}\nearrow & \downarrow{\scriptstyle p} \\ U & \hookrightarrow & X \end{array} \tag{3.2}$$

commutes. Then s is injective and is a local homeomorphism.

(b) Let $x \in X$ and assume that $a \in p^{-1}(\{x\})$. There exists an open subset V of X and a continuous function $t \colon V \to E$ such that $x \in V$, the diagram

$$\begin{array}{ccc} & & E \\ & {\scriptstyle t}\nearrow & \downarrow{\scriptstyle p} \\ V & \hookrightarrow & X \end{array} \tag{3.3}$$

commutes, and $t(x) = a$. The set $p^{-1}(\{x\})$ is a discrete subset of E.

(c) Let $x \in X$, and Let U_1 and U_2 be open subsets of X such that $x \in U_1$ and $x \in U_2$. Let $s_1 \colon U_1 \to E$ and $s_2 \colon U_2 \to E$ be continuous functions such that

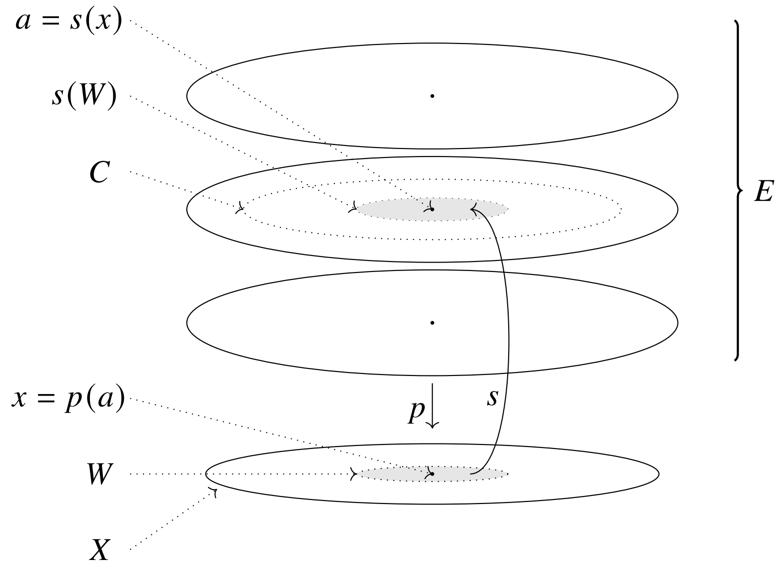
$$\begin{array}{ccc} & & E \\ & {\scriptstyle s_1}\nearrow & \downarrow{\scriptstyle p} \\ U_1 & \hookrightarrow & X \end{array} \quad \textit{and} \quad \begin{array}{ccc} & & E \\ & {\scriptstyle s_2}\nearrow & \downarrow{\scriptstyle p} \\ U_2 & \hookrightarrow & X \end{array} \tag{3.4}$$

commute. If $s_1(x) = s_2(x)$, then there exists an open subset W of X such that $x \in W \subset U_1 \cap U_2$ and $s_1(y) = s_2(y)$ for $y \in W$.

Proof. (a) Since $p(s(x)) = x$ for $x \in U$, the function s is injective. Let $x \in U$, and define $a = s(x)$. Since p is a local homeomorphism, there exists an open subset C of E and an open subset V of X such that $a \in C$, $p(C) = V$, and $i = p|_C \colon C \to V$ is a homeomorphism. Also, since s is continuous at x, there exists an open subset W of U such that $x \in W$ and $s(W) \subset C$. Since $s(W) \subset C$, we see that $W = i(s(W)) \subset i(C) = V$; since i^{-1} is a homeomorphism, the set $s(W) = i^{-1}(W)$ is open. We have the commutative diagram

$$\begin{array}{ccc} s(W) & \hookrightarrow & C \\ {\scriptstyle s}\uparrow\wr & & {\scriptstyle i}\downarrow\wr \\ W & \hookrightarrow & V \end{array}$$

We now see that $s\colon W \to s(W)$ is $(i|_W)^{-1}\colon W \to s(W)$ and is thus a homeomorphism.



(b) We have $p(a) = x$. Since p is a local homeomorphism, there exists an open subset C of E and an open subset V of X such that $a \in C$, $p(C) = V$, and $p|_C\colon C \to V$ is a homeomorphism. Define $t\colon V \to E$ by $t(y) = (p|_C)^{-1}(y)$ for $y \in V$. Then (3.3) commutes, and $t(x) = a$. To prove that $p^{-1}(\{x\})$ is a discrete subset of E, since a is an arbitrary element of $p^{-1}(\{x\})$, it will suffice to prove that the intersection of $p^{-1}(\{x\})$ with the open set C is $\{a\}$. Let $b \in C \cap p^{-1}(\{x\})$. Then $p(b) = x = p(a)$. Since $p|_C\colon C \to V$ is a bijection, we have $a = b$.

(c) Let $a = s_1(x) = s_2(x)$; then $a \in E$ and $p(a) = x$. Since p is a local homeomorphism, there exists an open subset C of E and an open subset V of X such that $a \in C$, $p(C) = V$, $x \in V$, and $p|_C\colon C \to V$ is a homeomorphism. Since s_1 and s_2 are continuous at x, there exists an open subset W of X such that $x \in W$, $W \subset U_1 \cap U_2 \cap C$, $s_1(W) \subset C$, and $s_2(W) \subset C$. Diagrammatically, for $i = 1$ and $i = 2$, we have

$$\begin{array}{ccc} & & C \\ & {\scriptstyle s_i}\nearrow & \wr\downarrow{\scriptstyle p} \\ W & \hookrightarrow & V \end{array}$$

Now let $y \in W$. Then $p(s_1(y)) = y = p(s_2(y))$; since $s_1(y), s_2(y) \in C$, and since $p|_C\colon C \to V$ is a homeomorphism, we must have $s_1(y) = s_2(y)$. □

From étalé spaces to sheaves

A sheaf is associated to every étalé space as follows. Let X be a topological space, and Let $p\colon E \to X$ be a étalé space over X. For each open subset U of X, we let $\Gamma(U, E)$ be the set of all continuous maps $s\colon U \to E$ such that

$$\begin{array}{ccc} & & E \\ & \overset{s}{\nearrow} & \downarrow p \\ U & \hookrightarrow & X \end{array}$$

commutes. If U and V are open subsets of X such that $V \subset U$, then we define

$$\Gamma(U, E) \xrightarrow{\rho_{U,V}} \Gamma(V, E)$$

by $\rho_{U,V}(s) = s|_V$ for $s \in \Gamma(U, E)$.

Lemma 3.3.4. *Let X be a topological space, and Let $p\colon E \to X$ be a étalé space over X. Then $\Gamma(\cdot, E)$ is a sheaf on X with values in $\mathcal{S}et$.*

Proof. It is easy to see that $\Gamma(\cdot, E)$ is a presheaf. To verify the gluing condition (G), let U be an open subset of X, let $\{U_i\}_{i\in I}$ be an open cover of U, and let $\{s_i\}_{i\in I}$ be such that $s_i \in \Gamma(U_i, E)$ for $i \in I$ and $\rho_{U_i,U_i\cap U_j}(s_i) = \rho_{U_j,U_i\cap U_j}(s_j)$ for all $i, j \in I$. Define $s : U \to E$ by $s(x) = s_i(x)$ if $i \in I$ is such that $x \in U_i$; since $\{s_i\}_{i\in I}$ is a cover of U there is at least one such i. We claim that s is well-defined. Assume that $i, j \in I$ are such that $s \in U_i$ and $s \in U_j$. As a consequence of our assumption, $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$. Hence, $s_i(x) = s_j(x)$, and s is well-defined. If $i \in I$ and $x \in U_i$, then $(p \circ s)(x) = \rho(s_i(x)) = x$; also, we see that if $i \in I$, then $\rho_{U,U_i}(s) = s|_{U_i} = s_i$ which proves that s is continuous. We conclude that $s \in \Gamma(U, E)$ and the gluing condition (G) holds. To verify the locality condition (L), let U be an open subset of X, let $\{U_i\}_{i\in I}$ be an open cover of U, and let $s, s' \in \Gamma(U, E)$ be such that $\rho_{U,U_i}(s) = \rho_{U,U_i}(s')$ for all $i \in I$. Then $s|_{U_i} = s'|_{U_i}$ for all $i \in I$. Since $\{U_i\}_{i\in I}$ is a cover of U, we obtain $s = s'$. □

Lemma 3.3.5. *Let X be a topological space, and let (E, p) be an étalé space over X. Let $x \in X$. If U is an open subset of X such that $x \in X$, then define $\Gamma(U, E) \to p^{-1}(\{x\})$ by $s \mapsto s(x)$ for $s \in \Gamma(U, E)$. Then, with these morphisms, $p^{-1}(\{x\})$ is a direct limit of $\{\Gamma(U, E)\}_{x\in U\subset X}$. The canonical isomorphism*

$$\Gamma(\cdot, E)_x = \varinjlim_{x\in U} \Gamma(U, E) \xrightarrow{\sim} p^{-1}(\{x\}) \tag{3.5}$$

sends $s_x = [s]$ to $s(x)$ for $s \in \Gamma(U, E)$ where U is an open subset of X such that $x \in U$ (here $\varinjlim \Gamma(U, E)$ is constructed as in the proof of Theorem 1.3.2).

Proof. Assume first that $p^{-1}(\{x\})$ is empty. In this case, it is easy to see that $\Gamma(U, E) = \emptyset$ for all open subsets U of X such that $x \in U$. It follows that $\Gamma(\cdot, E)_x = \emptyset = p^{-1}(\{x\})$. Now assume that $p^{-1}(\{x\})$ is non-empty. For U an open subset of X such that $x \in U$, define

$$\Gamma(U, E) \longrightarrow p^{-1}(\{x\})$$

by sending $s \in \Gamma(U, E)$ to $s(x)$. It is clear that if U and V are open subsets of X such that $x \in V \subset U$, then diagram

$$\begin{array}{ccc} \Gamma(U,E) & \longrightarrow & \Gamma(V,E) \\ & \searrow \quad \swarrow & \\ & p^{-1}(\{x\}) & \end{array}$$

commutes. We see that the conditions (a) and (b) from Lemma 1.3.3 follow from (b) and (c) of Lemma 3.3.3, respectively; Lemma 1.3.3 now implies the desired result. □

Lemma 3.3.6. *Let X be a topological space, and let $f\colon (E, p) \to (E', p')$ be a morphism of étalé spaces over X. Define*

$$\Gamma(\cdot, E) \xrightarrow{\Gamma f} \Gamma(\cdot, E')$$

by letting

$$\Gamma f = \left\{\Gamma(U, E) \xrightarrow{(\Gamma f)(U)} \Gamma(U, E')\right\}_{U \subset X \text{ open}}$$

where $(\Gamma f)(U)(s) = f(U) \circ s$ for U an open subset of X and $s \in \Gamma(U, E)$. Then Γf is a well-defined morphism of sheaves.

Proof. The proof of this lemma is straightforward and is left to the reader. □

From presheaves to étalé spaces

Let X be a topological space, and let $\mathcal{F}$ be a presheaf on X with values in $\mathcal{S}et$. We will attach an étalé space to $\mathcal{F}$. Define

$$L\mathcal{F} = \bigsqcup_{x \in X} \mathcal{F}_x$$

be the disjoint union of all the stalks of $\mathcal{F}$. Define

$$L\mathcal{F} \xrightarrow{p = p_{\mathcal{F}}} X$$

by setting $p(y) = x$ for $x \in X$ and $y \in \mathcal{F}_x$, so that p is the natural projection. We will define a topology on $L\mathcal{F}$ as follows. Let U be an open subset of X, and let $s \in \mathcal{F}(U)$. We define a function

$$U \xrightarrow{\hat{s}} L\mathcal{F}$$

by setting $\hat{s}(x) = s_x \in \mathcal{F}_x$ for $x \in U$. Evidently, $\hat{s}(U)$ is a subset of $L\mathcal{F}$, and the following diagram commutes

$$\begin{array}{ccc} & & L\mathcal{F} \\ & \overset{\hat{s}}{\nearrow} & \downarrow p \\ U & \hookrightarrow & X \end{array}$$

The following shows that the subsets $\hat{s}(U)$ of X define a topology with respect to which $(L\mathcal{F}, p)$ is an étalé space.

Theorem 3.3.7. *Let X be a topological space, and let $\mathcal{F}$ be a presheaf on X with values in $\mathcal{S}et$. Let B be the collection of sets $\hat{s}(U)$ as U ranges over the open subsets of X and s ranges over the elements of $\mathcal{F}(U)$. Then B is the basis for a topology on $L\mathcal{F}$ and:*

(a) If U is an open subset of X and $s \in \mathcal{F}(U)$, then the function $\hat{s}\colon U \to \hat{s}(U)$ is a homeomorphism.

(b) The pair $(L\mathcal{F}, p)$ is an étalé space.

Proof. To prove that B is the basis for a topology we have to prove two statements: (1) B covers X; (2) if $\hat{s}(U), \hat{t}(V)$ are in B, where U and V are open subsets of X, $s \in \mathcal{F}(U)$, and $t \in \mathcal{F}(V)$, and $e \in \hat{s}(U) \cap \hat{t}(V)$, then there exists an open subset W of X and $r \in \mathcal{F}(W)$ such that

$$e \in \hat{r}(W) \subset \hat{s}(U) \cap \hat{t}(V).$$

The statement (1) follows immediately from (a) of Lemma 2.3.1. To prove (2), let U and V be open subsets of X, let $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$, and assume that e is in $\hat{s}(U) \cap \hat{t}(V)$. Taking into account the involved definitions, it follows that there exists $x \in U \cap V$ such that $e = s_x = t_x$. Since $s_x = t_x$, there exists an open subset W of X such that $x \in W \subset U \cap V$ and $\rho_{U,W}(s) = \rho_{V,W}(t)$. We define $r = \rho_{U,W}(s) = \rho_{V,W}(t)$. We now claim that $e \in \hat{r}(W)$ and

$$\hat{r}(W) \subset \hat{s}(U) \cap \hat{t}(V). \tag{3.6}$$

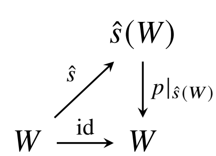
Since $r = \rho_{U,W}(s)$, the elements r and s define the same germ in $\mathcal{F}_x$, i.e., $r_x = s_x$. Since $s_x = e$, we have $r_x = e$; this implies that $e \in \hat{r}(W)$. Next, let $a \in \hat{r}(W)$. Let $y \in W$ be such that $r_y = a$. Since $y \in W \subset U \cap V$ and $r = \rho_{U,W}(s) = \rho_{V,W}(t)$, the elements r, s, and t all define the same germ in $\mathcal{F}_y$, i.e., $r_y = s_y = t_y$. Thus, $a = s_y = t_y \in \hat{s}(U) \cap \hat{t}(V)$. This proves (3.6).

(a) Let U be an open subset of X, and let $s \in \mathcal{F}(U)$. To prove that $\hat{s}\colon U \to \hat{s}(U)$ is a homeomorphism it will suffice to prove that this function is a continuous and open bijection. It is easy to see that our function is a bijection. To see it is continuous, let V be an open subset of X, and let $t \in \mathcal{F}(V)$; to prove that $\hat{s}$ is continuous it will suffice to prove that $\hat{s}^{-1}(\hat{t}(V))$ is open. Now

$$\begin{aligned} \hat{s}^{-1}(\hat{t}(V)) &= \{x \in U\colon \hat{s}(x) \in \hat{t}(V)\} \\ &= \{x \in U\colon s_x \in \hat{t}(V)\} \\ &= \{x \in U\colon x \in V \text{ and } s_x = t_x\} \\ &= \{x \in U \cap V\colon s_x = t_x\}. \end{aligned}$$

Let $x \in \hat{s}^{-1}(\hat{t}(V)) = \{x \in U \cap V\colon s_x = t_x\}$. By (b) of Lemma 2.3.1 there exists an open subset W of X such that $x \in W \subset U \cap V$ and $s_y = t_y$ for $y \in W$. It follows $x \in W \subset \hat{s}^{-1}(\hat{t}(V))$, proving that $\hat{s}^{-1}(\hat{t}(V))$ is open. Next, let V be an open subset of U; we need to prove that $\hat{s}(V)$ is open. Define $t = \rho_{U,V}(s) \in \mathcal{F}(V)$. Let $x \in V$. Then $\hat{t}(x) = t_x = s_x = \hat{s}(x)$. This implies that $\hat{s}(V) = \hat{t}(V)$. Since $\hat{t}(V)$ is in B, the set $\hat{s}(V) = \hat{t}(V)$ is open.

(b) To prove that p is a local homeomorphism, let $e \in L\mathcal{F}$. Then $e \in \mathcal{F}_x$ for some $x \in X$. By (a) of Lemma 2.3.1, there exists an open subset W of X and $s \in \mathcal{F}(W)$ such that $x \in W$ and $s_x = e$. There is a commutative diagram



Since $\hat{s}\colon W \to \hat{s}(W)$ is a homeomorphism by (a) and $\mathrm{id}\colon W \to W$ is a homeomorphism, so is $p|_{\hat{s}(W)}\colon \hat{s}(W) \to W$. □

Lemma 3.3.8. *Let X be a topological space. Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on X with values in $\mathcal{S}et$. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. Define*

$$(L\mathcal{F}, p_{\mathcal{F}}) \xrightarrow{Lf} (L\mathcal{G}, p_{\mathcal{G}})$$

by letting Lf be the function f_x on each stalk $\mathcal{F}_x$ for $x \in X$ (see Section 2.4). Then Lf is a well-defined morphism of étalé spaces.

Proof. We need to prove that Lf is continuous. By Lemma 3.3.2 it suffices to prove that Lf is open. Let U be an open subset of X, and let $s \in \mathcal{F}(U)$; to prove that Lf is open, it will suffice to prove that $(Lf)(\hat{s}(U))$ is open. Now

$$(Lf)(\hat{s}(U)) = \{f_x(s_x)\colon x \in U\}.$$

Let $x \in U$. We have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f(U)} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_x & \xrightarrow{f_x} & \mathcal{G}_x \end{array}$$

Therefore, $f_x(s_x) = (f(U)(s))_x$. It follows that

$$(Lf)(\hat{s}(U)) = \{f_x(s_x)\colon x \in U\} = \widehat{f(U)(s)}(U).$$

Hence, by the definition of the topology on $L\mathcal{G}$, the set $(Lf)(\hat{s}(U))$ is open. □

From étalé spaces to sheaves to étalé spaces

Theorem 3.3.9. *Let X be a topological space, and let (E, p) be an étalé space over X. Let $\mathcal{F} = \Gamma(\cdot, E)$. Recall from Lemma 3.3.5 that, for each $x \in X$, there exists a canonical bijection*

$$\mathcal{F}_x = \Gamma(\cdot, E)_x = \varinjlim_{x \in U} \Gamma(U, E) \xrightarrow{\sim} p^{-1}(\{x\})$$

that sends an equivalence class $s_x = [s]$ to $s(x)$ for $s \in \Gamma(U, E)$, where U is an open subset of X such that $x \in U$. The induced bijection

$$f\colon L\Gamma E = L\mathcal{F} = \bigsqcup_{x \in X} \mathcal{F}_x \xrightarrow{\sim} E = \bigsqcup_{x \in X} p^{-1}(\{x\})$$

is an isomorphism between the étalé spaces $(L\Gamma E, p_{\mathcal{F}}) = (L\mathcal{F}, p_{\mathcal{F}})$ and (E, p) over X.

Proof. It is clear that the diagram

$$\begin{array}{ccc} L\mathcal{F} & \xrightarrow{f} & E \\ & {\scriptstyle p_{\mathcal{F}}}\searrow \quad \swarrow{\scriptstyle p} & \\ & X & \end{array}$$

commutes. Since f is a bijection, by Lemma 3.3.1, to prove that f is an isomorphism of étalé spaces, it will suffice to prove that f is a local homeomorphism; for this, by Lemma 3.3.2, it will suffice to verify that f is open. Let U be an open subset of X, and let $s \in \Gamma(U, E) = \mathcal{F}(U)$; by the definition of the topology on $L\mathcal{F}$, to prove that f is open, it is sufficient to check that $f(\hat{s}(U))$ is open. Now

$$\begin{aligned} f(\hat{s}(U)) &= f(\{s_x : x \in U\}) \\ &= \{s(x) : x \in U\} \\ &= s(U). \end{aligned}$$

This is an open set by Lemma 3.3.3 and Lemma 3.3.1. □

From presheaves to étalé spaces to sheaves

Lemma 3.3.10. *Let X be a topological space, and let $\mathcal{F}$ be a presheaf on X with values in $\mathcal{S}et$. For each open subset U of X, define*

$$\mathcal{F}(U) \overset{n_{\mathcal{F}}(U)}{\longrightarrow} (\Gamma L\mathcal{F})(U)$$

by $n_{\mathcal{F}}(U)(s) = \hat{s}$ for $s \in \mathcal{F}(U)$. Then the collection of functions

$$\left\{\mathcal{F}(U) \overset{n_{\mathcal{F}}(U)}{\longrightarrow} (\Gamma L\mathcal{F})(U)\right\}_{U \subset X \text{ open}}$$

defines a morphism of presheaves

$$\mathcal{F} \overset{n_{\mathcal{F}}}{\longrightarrow} \Gamma L\mathcal{F}.$$

Proof. We first note that the maps $n_{\mathcal{F}}(U)$ for U an open subset of X are well-defined: in Theorem 3.3.7 we proved that if $s \in \mathcal{F}(U)$, then $\hat{s}\colon U \to L\mathcal{F}$ is a homeomorphism and is thus continuous. Next, let U and V be open subsets of X with $V \subset U$. To see that

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & (\Gamma L\mathcal{F})(U) \\ \big\downarrow & & \big\downarrow \\ \mathcal{F}(V) & \longrightarrow & (\Gamma L\mathcal{F})(V) \end{array} \tag{3.7}$$

commutes, let $s \in \mathcal{F}(U)$ and $x \in V$. Then image of s under the composition

$$\mathcal{F}(U) \longrightarrow (\Gamma L\mathcal{F})(U) \longrightarrow (\Gamma L\mathcal{F})(V)$$

is the function $V \to L\mathcal{F}$ that sends x to $s_x \in L\mathcal{F}$. The image of s under the composition

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(V) \longrightarrow (\Gamma L\mathcal{F})(V)$$

is the function that sends x to $(\rho_{U,V}s)_x \in L\mathcal{F}$. Since the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{F}(V) \\ & \searrow \quad \swarrow & \\ & \mathcal{F}_x & \end{array}$$

commutes, we have $s_x = (\rho_{U,V}s)_x$. Hence, (3.7) commutes. ◻

Lemma 3.3.11. *Let X be a topological space. Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on X with values in $\mathcal{S}et$. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. The diagram of morphisms of presheaves*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{f} & \mathcal{G} \\ \downarrow{\scriptstyle n_{\mathcal{F}}} & & \downarrow{\scriptstyle n_{\mathcal{G}}} \\ \Gamma L\mathcal{F} & \xrightarrow{\Gamma L f} & \Gamma L\mathcal{G} \end{array} \tag{3.8}$$

commutes.

Proof. Let U be an open subset of X and let $s \in \mathcal{F}(U)$. We need to prove that the elements

$$((\Gamma L f)(U))\,(n_{\mathcal{F}}(U)(s)) \quad \text{and} \quad n_{\mathcal{F}}(U)\,(f(U)(s))$$

of $\Gamma L\mathcal{G}$ are equal. Both of these elements are functions from U to $L\mathcal{G}$. Let $x \in U$. Then

$$\begin{aligned} \Big(((\Gamma L f)(U))\,(n_{\mathcal{F}}(U)(s))\Big)(x) &= \Big(((\Gamma L f)(U))\,(\hat{s})\Big)(x) \\ &= \Big((Lf) \circ \hat{s}\Big)(x) \\ &= (Lf)(\hat{s}(x)) \\ &= (Lf)(s_x) \\ &= f_x(s_x). \end{aligned}$$

And:

$$\begin{aligned} \Big(n_{\mathcal{G}}(U)\,(f(U)(s))\Big)(x) &= \widehat{f(U)(s)}(x) \\ &= \big(f(U)(s)\big)_x. \end{aligned}$$

Since the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f(U)} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_x & \xrightarrow{f_x} & \mathcal{G}_x \end{array}$$

commutes, we have $f_x(s_x) = \big(f(U)(s)\big)_x$. This is the desired result. ◻

Lemma 3.3.12. *Let X be a topological space, and let $\mathcal{F}$ be a presheaf on X with values in $\mathcal{S}et$. The morphism*

$$\mathcal{F} \xrightarrow{n_{\mathcal{F}}} \Gamma L\mathcal{F}$$

is an isomorphism of presheaves if and only if $\mathcal{F}$ is a sheaf.

Proof. Assume that $n_{\mathcal{F}}$ is an isomorphism of presheaves. By Lemma 3.3.4, $\Gamma L\mathcal{F}$ is a sheaf. Lemma 3.1.3 now implies that $\mathcal{F}$ is a sheaf.

Assume that $\mathcal{F}$ is a sheaf. To show that $n_{\mathcal{F}}$ is an isomorphism it will suffice to prove that $n_{\mathcal{F}}(U)$ is an isomorphism, i.e., bijection, for every open subset U of X (see Lemma 2.4.1). Let U be an open subset of X. We first prove that $n_F(U)\colon \mathcal{F}(U) \to (\Gamma L\mathcal{F})(U)$ is injective. Let $s, s' \in \mathcal{F}(U)$ and assume that $n_{\mathcal{F}}(U)(s) = n_{\mathcal{F}}(U)(s')$. Then $s_x = s'_x$ for all $x \in U$. By Proposition 3.1.1 we have $s = s'$, and $n_{\mathcal{F}}(U)$ is injective.

To prove that $n_{\mathcal{F}}(U)$ is surjective, let $t \in (\Gamma L\mathcal{F})(U)$. Then, by defintion, $t\colon U \to L\mathcal{F}$ is a continuous function such that

$$\begin{array}{ccc} & & L\mathcal{F} \\ & \overset{t}{\nearrow} & \downarrow{\scriptstyle p_{\mathcal{F}}} \\ U & \hookrightarrow & X \end{array}$$

commutes. By Lemma 3.3.3, t is an injective local homeomorphism. By Lemma 3.3.1, t is open. It follows that the set $t(U)$ is an open subset of $L\mathcal{F}$. By the definition of the topology on $L\mathcal{F}$, for each $x \in U$, there exists an open subset U_x of U and $s^x \in \mathcal{F}(U_x)$ such that $x \in U_x$, $t(x) \in \widehat{s^x}(U_x)$, and $\widehat{s^x}(U_x) \subset t(U)$. We consider the open cover $\{U_x\}_{x\in U}$ of U and the collection $\{s^x\}_{x\in U}$. Let $x_1, x_2 \in U$; we claim that

$$\rho_{U_{x_1}, U_{x_1}\cap U_{x_2}}(s^{x_1}) = \rho_{U_{x_2}, U_{x_1}\cap U_{x_2}}(s^{x_2}). \tag{3.9}$$

To see this, let

$$s_1 = \rho_{U_{x_1}, U_{x_1}\cap U_{x_2}}(s^{x_1}), \qquad s_2 = \rho_{U_{x_2}, U_{x_1}\cap U_{x_2}}(s^{x_1}).$$

Let $z \in U_{x_1} \cap U_{x_2}$. Then s_1 and s^{x_1} define the same germ in $\mathcal{F}_z$, and s_2 and s^{x_2} define the same germ in $\mathcal{F}_z$. That is,

$$s_{1,z} = (s^{x_1})_z, \qquad s_{2,z} = (s^{x_2})_z.$$

We also have, by construction,

$$\widehat{s^{x_1}}(U_{x_1}), \widehat{s^{x_2}}(U_{x_2}) \subset t(U).$$

Considering the definition of $L\mathcal{F}$, this implies that

$$\widehat{s^{x_1}}(z) = t(z) = \widehat{s^{x_2}}(z)$$

so that $(s^{x_1})_z = (s^{x_2})_z$. By Lemma 3.1.1 we conclude that $s_1 = s_2$, as claimed. Since $\mathcal{F}$ is a sheaf, by the gluing condition (G), there exists $r \in \mathcal{F}(U)$ such that $\rho_{U,U_x}(r) = s^x$ for all $x \in U$. Let $x \in U$. Then $t(x) = (s^x)_x = r_x$. This implies that $n_{\mathcal{F}}(U)(r) = t$, proving that $n_{\mathcal{F}}(U)$ is surjective. □

We can use Lemma 3.3.12 to again prove that the constant presheaf is not a sheaf if A has at least two elements. Let X be a topological space and let A be a set. By definition, the constant presheaf $\mathcal{F} = A_X$ is such that $\mathcal{F}(U) = A$ for all open subsets U of X. Thus, $\mathcal{F}(\emptyset) = A$. We also have

$$(\Gamma L\mathcal{F})(\emptyset) = \Gamma(\emptyset, L\mathcal{F}) = \{\emptyset\}.$$

We see that if A has at least two elements, then the function

$$\mathcal{F}(\emptyset) = A \overset{n_{\mathcal{F}}(\emptyset)}{\longrightarrow} (\Gamma L\mathcal{F})(\emptyset) = \{\emptyset\}$$

cannot be a bijection.

Sheafification of presheaves

The ΓL construction will allow us to canonically associate a sheaf to every presheaf. For this, we need the following lemma.

Lemma 3.3.13. *Let X be a topological space, and let $\mathcal{F}$ be a presheaf on X with values in $\mathcal{S}et$. If $x \in X$, then the induced morphism*

$$\mathcal{F}_x \overset{(n_{\mathcal{F}})_x}{\longrightarrow} (\Gamma L\mathcal{F})_x$$

is an isomorphism.

Proof. Let $x \in X$. The function $(n_{\mathcal{F}})_x$ is such that

$$((n_{\mathcal{F}})_x)\,(s_x) = (((n_{\mathcal{F}}(U))(s))_x = (\hat{s})_x$$

for $s \in \mathcal{F}(U)$ where U is an open subset of X such that $x \in U$ (see the remark after Lemma 1.3.4). Next, we recall the function

$$(\Gamma L\mathcal{F})_x \overset{f}{\longrightarrow} p_{\mathcal{F}}^{-1}(\{x\}) = \mathcal{F}_x$$

from Lemma 3.3.5. This function is a bijection, and satisfies

$$f((\hat{s})_x) = (\hat{s})(x) = s_x$$

for $s \in \mathcal{F}(U)$, where U is an open subset of X such that $x \in U$. Since f is a bijection, to prove the lemma it will suffice to prove that

$$f \circ (n_{\mathcal{F}})_x = \mathrm{id}_{\mathcal{F}_x}.$$

Let U be an open subset of X such that $x \in U$. Let $s \in \mathcal{F}(U)$. Then

$$\begin{aligned}(f \circ (n_{\mathcal{F}})_x)\,(s_x) &= f\,((n_{\mathcal{F}})_x(s_x)) \\ &= f\,((\hat{s})_x) \\ &= s_x \\ &= \mathrm{id}_{\mathcal{F}_x}\,(s_x)\,.\end{aligned}$$

Since every element of $\mathcal{F}_x$ has the form s_x for some open subset U of X such that $x \in U$ and $s \in \mathcal{F}(U)$, we conclude that $f \circ (n_{\mathcal{F}})_x = \mathrm{id}_{\mathcal{F}_x}$. □

Theorem 3.3.14 (Sheafification)**.** *Let X be a topological space, and let $\mathcal{F}$ be a presheaf on X with values in $\mathcal{S}et$. Let $\mathcal{G}$ be a sheaf on X with values in $\mathcal{S}et$. Assume that there exists a morphism*

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \tag{3.10}$$

of presheaves. Then there exists a unique morphism

$$\Gamma L\mathcal{F} \xrightarrow{g} \mathcal{G}$$

of sheaves such that

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{n_{\mathcal{F}}} & \Gamma L\mathcal{F} \\ & {}_{f}\searrow & \downarrow g \\ & & \mathcal{G} \end{array} \tag{3.11}$$

commutes.

Proof. By Lemma 3.3.11, we have the following commutative diagram of presheaves:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{n_{\mathcal{F}}} & \Gamma L\mathcal{F} \\ \downarrow f & & \downarrow \Gamma L f \\ \mathcal{G} & \xrightarrow{n_{\mathcal{G}}} & \Gamma L\mathcal{G} \end{array}$$

By Lemma 3.3.12, the morphism $n_{\mathcal{G}}$ is an isomorphism of sheaves. It follows that if $g\colon \Gamma L\mathcal{F} \to \mathcal{G}$ is $n_{\mathcal{G}}^{-1} \circ \Gamma L f$, then (3.11) commutes. This proves the existence of the desired g. To prove the uniqueness of g, assume that $g, g'\colon \Gamma L\mathcal{F} \to \mathcal{G}$ are two sheaf morphisms such that

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{n_{\mathcal{F}}} & \Gamma L\mathcal{F} \\ & {}_{f}\searrow & \downarrow g \\ & & \mathcal{G} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{F} & \xrightarrow{n_{\mathcal{F}}} & \Gamma L\mathcal{F} \\ & {}_{f}\searrow & \downarrow g' \\ & & \mathcal{G} \end{array}$$

commute. Let $x \in X$. Then $g_x \circ (n_{\mathcal{F}})_x = f_x$. By Lemma 3.3.13, the function $(n_{\mathcal{F}})_x$ is an isomorphism; hence, we have $g_x = f_x \circ (n_{\mathcal{F}})_x^{-1}$. Similarly, $g'_x = f_x \circ (n_{\mathcal{F}})_x^{-1}$, so that $g_x = g'_x$. By Corollary 3.1.2 we now have $g = g'$. □

If the notation is as in Theorem 3.3.14, then we refer to $\Gamma L\mathcal{F}$ as the **sheafification** of $\mathcal{F}$. We see that if $\mathcal{F}$ is a sheaf, then $\Gamma L\mathcal{F}$ is isomorphic to $\mathcal{F}$ via the isomorphism $n_{\mathcal{F}}\colon \mathcal{F} \to \Gamma L\mathcal{F}$ (see Lemma 3.3.12).

The constant sheaf

We consider the sheafication of the constant presheaf. Let X be a topological space, and let A be a set. The constant presheaf $\mathcal{F} = A_X$ is defined by $\mathcal{F}(U) = A$ for all open subsets U of X, and all the restriction maps for this presheaf are the identity function $\mathrm{id}_A : A \to A$. We will calculate $\Gamma L\mathcal{F}$. First of all, we have:

$$L\mathcal{F} = \bigsqcup_{x \in X} \mathcal{F}_x = \bigsqcup_{x \in X} A.$$

In fact, we may naturally identify this with $X \times A$:

$$L\mathcal{F} \cong X \times A.$$

We have

$$p_{\mathcal{F}}(x, a) = x$$

for $(x, a) \in L\mathcal{F} \cong X \times A$. The topology on $L\mathcal{F}$ is defined to be the topology that has as base the sets $\hat{s}(U)$ where U is an open subset of X and $s \in \mathcal{F}(U) = A$; here, $\hat{s}\colon U \to L\mathcal{F}$ is defined by $\hat{s}(x) = s_x \in \mathcal{F}_x$ for $x \in U$, and

$$\hat{s}(U) = \{s_x : x \in U\} = U \times \{s\}$$

because the germ s_x in $\mathcal{F}_x = A$ is s (see the discussion about the stalks of the constant presheaf on p. 18). Let us now endow A with the discrete topology. We see then that the topology on $L\mathcal{F} = X \times A$ is the product topology. Next, let U be an open subset of X. We have

$$\begin{aligned}(\Gamma L\mathcal{F})(U) &= \Gamma(U, L\mathcal{F}) \\ &= \{s\colon U \to L\mathcal{F} = X \times A\colon s \text{ is continuous}, (p_{\mathcal{F}} \circ s)(x) = x, x \in U\} \\ &= \{s\colon U \to L\mathcal{F} = X \times A\colon s(x) = (x, r(x)), r\colon U \to A \text{ continuous}\} \\ &\cong \{r\colon U \to A\colon r \text{ is continuous}\} \\ &= \{r\colon U \to A\colon r \text{ is locally constant}\}.\end{aligned}$$

We will refer to $\Gamma L A_X$ as the **constant sheaf**.

3.4 Abelian considerations

In this section we will show that the results of Section 3.3 still hold if we assume that the involved presheaves, sheaves, and étalé spaces are abelian. We will not repeat the previous section; instead, we will just provide the required additional definitions and arguments.

Abelian étalé spaces

We first develop the concept of an étalé space of abelian groups.

Proposition 3.4.1. *Let X be a topological space, and let (E, p) be an étalé space over X. Assume that for every $x \in X$ the set $p^{-1}(\{x\})$ is an abelian group. The following are equivalent:*

(a) For every open subset U of X, the set $\Gamma(U, E)$ is an abelian group under pointwise addition of functions.

(b) Define

$$E\pi E = \{(e, e') \in E \times E\colon p(e) = p(e')\}.$$

The map

$$m\colon E\pi E \longrightarrow E$$

defined by $m(e, e') = e - e'$ for $(e, e') \in E\pi E$ is continuous.

Proof. (a) $\implies$ (b). Assume that (a) holds. Let $(e, e') \in E\pi E$; we will prove that m is continuous at (e, e'). Let $x = p(e) = p(e')$. Also, let $a = e - e'$. Let C be an open subset of E such that $a \in C$. By (b) of Lemma 3.3.3, there exists an open subset U of X such that $x \in U$ and $s, s' \in \Gamma(U, E)$ such that $s(x) = e$, $s(x) = e'$, $s(U)$ and $s'(U)$ are open in E, and $s\colon U \to s(U)$ and $s'\colon U \to s'(U)$ are homeomorphisms. Consider the composition

$$U \xrightarrow{(s,s')} E\pi E \xrightarrow{m} E$$

where (s, s') sends $y \in U$ to $(s(y), s'(y))$. This composition is $s - s'$. By (a), $s - s'$ is in $\Gamma(U, E)$, and is hence continuous. It follows that there exists an open subset V of X such that $x \in V \subset U$ and

$$(m \circ (s, s'))(V) = m\left((s, s')(V)\right) \subset C.$$

Now

$$\begin{aligned}(s, s')(V) &= \{(s(y), s'(y))\colon y \in V\}\\ &= (E\pi E) \cap (s(V) \times s'(V)).\end{aligned}$$

Since $s\colon U \to s(U)$ and $s'\colon U \to s'(U)$ are homeomorphisms, the sets $s(V)$ and $s'(V)$ are open in E; hence $s(V) \times s'(V)$ is open in $E \times E$. It follows that $(s, s')(V) = (E\pi E) \cap (s(V) \times s'(V))$ is open in $E\pi E$. Also, $(e, e') \in (E\pi E) \cap (s(V) \times s'(V))$. It follows that m is continuous at (e, e').

(b) $\implies$ (a). Assume that (b) holds. Let U be an open subset of X. If U is empty, then $\Gamma(U, E)$ contains exactly one element, the empty function from U to E (see p. 3), so that $\Gamma(U, E)$ is the trivial group. Assume that U is non-empty. Let $f, g \in \Gamma(U, E)$; to prove that $\Gamma(U, E)$ is an abelian group under pointwise addition it will suffice to prove that $f - g \in \Gamma(U, E)$. This amounts to proving that $f - g$ is continuous. Consider the function

$$U \xrightarrow{(f,g)} E\pi E$$

that sends $x \in U$ to $(f(x), g(x))$. We claim that this function is continuous. Let C and D be open subsets of E; to prove that (f, g) is continuous, it will suffice to prove that $(f, g)^{-1}((E\pi E) \cap (C \times D))$ is open. Now

$$\begin{aligned}(f, g)^{-1}((E\pi E) \cap (C \times D)) &= \{x \in U\colon (f(x), g(x)) \in (E\pi E) \cap (C \times D)\}\\ &= \{x \in U\colon (f(x), g(x)) \in C \times D\}\\ &= \{x \in U\colon f(x) \in C\} \cap \{x \in U\colon g(x) \in D\}\\ &= f^{-1}(C) \cap g^{-1}(D).\end{aligned}$$

Since f and g are continuous, the sets $f^{-1}(C)$ and $g^{-1}(D)$ are open; hence, $f^{-1}(C) \cap g^{-1}(D)$ is open, proving that (f, g) is continuous. Since (f, g) is continuous, so is the composition

$$U \xrightarrow{(f,g)} E\pi E \xrightarrow{m} E.$$

This composition is $f - g$; hence, $f - g$ is continuous. □

Let X be a topological space, and let (E, p) be an étalé space over X. We say that (E, p) is an **abelian étalé space** or a **étalé space of abelian groups** if:

(a) For every $x \in X$, the set $p^{-1}(\{x\})$ is an abelian group.

(b) The minus map $m\colon E\pi E \to E$ from Proposition 3.4.1 is continuous.

We note that if (E, p) is an abelian étalé space over X, then the set $p^{-1}(\{x\})$ is non-empty for every $x \in X$ (since, by definition, a group is a non-empty set). Assume that (E, p) and (E', p') are abelian étalé spaces over X. A **morphism** $(E, p) \to (E', p')$ is a continuous function $E \to E'$ such that

$$\begin{array}{ccc} E & \longrightarrow & E' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & X & \end{array}$$

commutes, and, for all $x \in X$, the induced function

$$p^{-1}(\{x\}) \longrightarrow p'^{-1}(\{x\})$$

is a homomorphism of abelian groups.

From abelian étalé spaces to abelian sheaves

Let X be a topological space. By Proposition 3.4.1, if (E, p) is an abelian étalé space over X, then for all open subsets U of X, $\Gamma(U, E)$ is an abelian group under pointwise addition of functions, and $\Gamma E = \Gamma(\cdot, E)$ is a sheaf with values in $\mathcal{Ab}$, i.e., an abelian sheaf. If $x \in X$, then the canonical bijection

$$\Gamma(\cdot, E)_x \xrightarrow{\sim} p^{-1}(\{x\})$$

from Lemma 3.3.5 is an isomorphism of abelian groups. Let $f\colon (E, p) \to (E', p')$ be a morphism of abelian étalé spaces. Consider (E, p) and (E', p') as just étalé spaces; then f is a morphism of étalé spaces. In Lemma 3.3.6 we noted that, as a morphism of étalé spaces, the morphism f induces a morphism

$$\Gamma(\cdot, E) \xrightarrow{\Gamma f} \Gamma(\cdot, E')$$

of sheaves. It is straightforward to verify that Γf is, in fact, a morphism of abelian sheaves on X.

From abelian presheaves to abelian étalé spaces

Proposition 3.4.2. *Let X be a topological space, and let $\mathcal{F}$ be presheaf on X with values in $\mathcal{Ab}$. Then $(L\mathcal{F}, p_{\mathcal{F}})$ is an abelian étalé space.*

Proof. If $x \in X$, then $p_{\mathcal{F}}^{-1}(\{x\}) = \mathcal{F}_x$, by definition, and this is an abelian group. To complete the proof we need to verify that the minus map $m\colon L\mathcal{F}\pi L\mathcal{F} \to L\mathcal{F}$ is continuous. Let $(e, e') \in L\mathcal{F}\pi L\mathcal{F}$, and let $x = p_{\mathcal{F}}(e) = p_{\mathcal{F}}(e')$. Let C be an open subset of $L\mathcal{F}$ that contains $m(e, e') = e - e'$. We need to find an open subset D of $L\mathcal{F}\pi L\mathcal{F}$ such that $(e, e') \in D$ and $m(D) \subset C$. Using the definition of the topology on $L\mathcal{F}$, we may assume that $C = \hat{r}(W)$, where W is an open subset of X such that $x \in W$ and $r \in \mathcal{F}(W)$. By (a) of Lemma 2.3.1 there exists and open subset V of X suh that $x \in X$ and there exist $s, s' \in \mathcal{F}(V)$ such that $s_x = e$ and $s'_x = e'$. We may assume that $V \subset W$.

Now $s - s' \in \mathcal{F}(V)$ and $(s - s')_x = s_x - s'_x = e - e' = r_x$. By (b) of Lemma 2.3.1, we may assume that $(s - s')_y = r_y$ for $y \in V$. We now claim that

$$m\left((\hat{s}(V) \times \hat{s}'(V)) \cap (L\mathcal{F} \pi L\mathcal{F})\right) \subset \hat{r}(W).$$

Let $(g, g') \in (\hat{s}(V) \times \hat{s}'(V)) \cap (L\mathcal{F} \pi L\mathcal{F})$. Then there exists $y \in V$ such that $g, g' \in \mathcal{F}_y$, $s_y = g$, and $s'_y = g'$. We have

$$m(g, g') = g - g' = s_y - s'_y = (s - s')_y = r_y \in \hat{r}(W).$$

Also, it is evident that $(e, e') \in (\hat{s}(V) \times \hat{s}'(V)) \cap (L\mathcal{F} \pi L\mathcal{F})$. This completes the proof that m is continuous at (e, e'). □

Let X be a topological space, and let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on X with values in $\mathcal{Ab}$. Let $f \colon \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. The induced morphism $Lf \colon L\mathcal{F} \to L\mathcal{G}$ of étalé spaces is evidently a morphism of abelian étalé spaces.

From abelian étalé spaces to abelian sheaves to abelian étalé spaces

Theorem 3.3.9 holds in the abelian setting, with the same proof.

From abelian presheaves to abelian étalé spaces to abelian sheaves

Lemma 3.3.10, Lemma 3.3.11, Lemma 3.3.12 all hold in the abelian setting, with the same proofs (it is straightforward to verify that $n_{\mathcal{F}}$ is a morphism of abelian presheaves).

Sheafication of abelian presheaves

Lemma 3.3.13 and Theorem 3.3.14 both hold in the abelian setting.

Chapter 4

Morphisms

In this chapter we will prove some essential properties about morphisms between presheaves and sheaves. One of the results of this chapter is that the category of presheaves and the category of sheaves are both abelian categories, and we begin by defining this concept.

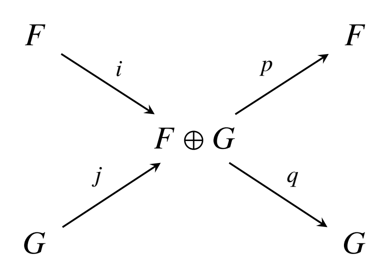
4.1 Abelian categories

In this section we define the concept of an **abelian category**. Let $\mathscr{A}$ be a category. Let $F, G \in \mathrm{Ob}(\mathscr{A})$ and assume that $f \in \mathrm{Mor}(F, G)$. We say that f is a **monomorphism** if, for all $H \in \mathrm{Ob}(\mathscr{A})$ and $g, h \in \mathrm{Mor}(H, F)$, if $f \circ g = f \circ h$, then $g = h$. We say that f is an **epimorphism** if, for all $H \in \mathrm{Ob}(\mathscr{A})$ and $g, h \in \mathrm{Mor}(G, H)$, if $g \circ f = h \circ f$, then $g = h$. Assume that for every pair of objects $F, G \in \mathrm{Ob}(\mathscr{A})$ the set $\mathrm{Mor}(F, G)$ is an abelian group (in particular, this implies that $\mathrm{Mor}(F, G)$ is non-empty). For $\mathscr{A}$ to be an **abelian category** the category $\mathscr{A}$ must satisfy four axioms:

(AB1) There exists an object $0 \in \mathrm{Ob}(\mathscr{A})$ such that $\mathrm{Mor}(F, 0)$ and $\mathrm{Mor}(0, F)$ are the trivial group for all $F \in \mathrm{Ob}(\mathscr{A})$. Also, for all objects $F, F', G, G' \in \mathrm{Ob}(\mathscr{A})$, $p \in \mathrm{Mor}(F', F)$, $f, g \in \mathrm{Mor}(F, G)$, and $q \in \mathrm{Mor}(G, G')$, we have

$$(f + g) \circ p = f \circ p + g \circ p, \qquad q \circ (f + g) = q \circ f + q \circ g. \tag{4.1}$$

(AB2) (Biproducts) Let $F, G \in \mathrm{Ob}(\mathscr{A})$. There exists an object $F \oplus G \in \mathrm{Ob}(\mathscr{A})$ and morphisms



such that:

(a) For any $H \in \mathrm{Ob}(\mathcal{A})$ and morphisms

$$\begin{array}{ccc} F & \searrow & \\ & & H \\ G & \nearrow & \end{array}$$

there exists a unique morphism $F \oplus G \to H$ such that

$$\begin{array}{ccccc} F & & & & \\ & \overset{i}{\searrow} & & \searrow & \\ & & F \oplus G & \longrightarrow & H \\ & \underset{j}{\nearrow} & & \nearrow & \\ G & & & & \end{array}$$

commutes.

(b) For any object $H \in \mathrm{Ob}(\mathcal{A})$ and morphisms

$$\begin{array}{ccc} & \nearrow & F \\ H & & \\ & \searrow & G \end{array} \tag{4.2}$$

there exists a unique morphism $H \to F \oplus G$ such that

$$\begin{array}{ccccc} & & & & F \\ & \nearrow & & \overset{p}{\nearrow} & \\ H & \longrightarrow & F \oplus G & & \\ & \searrow & & \underset{q}{\searrow} & \\ & & & & G \end{array} \tag{4.3}$$

commutes.

(AB3) Let $F, G \in \mathrm{Ob}(\mathcal{A})$ and $f \in \mathrm{Mor}(F, G)$.

(a) (Kernels) There exists $K \in \mathrm{Ob}(\mathcal{A})$ and a morphism $i\colon K \to F$ such that the composition

$$K \xrightarrow{\ i\ } F \xrightarrow{\ f\ } G$$

is the zero morphism in $\mathrm{Mor}(K, G)$, and for any $H \in \mathrm{Ob}(\mathcal{A})$ and $g \in \mathrm{Mor}(H, F)$ such that the composition

$$H \xrightarrow{\ g\ } F \xrightarrow{\ f\ } G$$

is the zero morphism in $\mathrm{Mor}(H, G)$, there exists a unique morphism

$$H \longrightarrow K$$

such that

$$\begin{array}{ccc} H & & \\ \downarrow & \searrow^{g} & \\ K & \xrightarrow{i} & F \end{array}$$

commutes.

(b) (Cokernels) There exists $C \in \mathrm{Ob}(\mathcal{A})$ and a morphism $p\colon G \to C$ such that the composition

$$F \xrightarrow{f} G \xrightarrow{p} C$$

is the zero morphism in $\mathrm{Mor}(F, C)$, and for any $H \in \mathrm{Ob}(\mathcal{A})$ and $g \in \mathrm{Mor}(G, H)$ such that

$$F \xrightarrow{f} G \xrightarrow{g} H$$

is the zero morphism in $\mathrm{Mor}(F, H)$, there exists a unique morphism

$$C \longrightarrow H$$

such that

$$\begin{array}{ccc} & & H \\ & \nearrow^{g} & \uparrow \\ G & \xrightarrow{p} & C \end{array}$$

commutes. In (a), we say that $i\colon K \to F$ is a **kernel** of $f\colon F \to G$, and in (b), we say that $p\colon G \to C$ is a **cokernel** of $f\colon F \to G$.

(AB4) Let $F, G \in \mathrm{Ob}(\mathcal{A})$ and $f \in \mathrm{Mor}(F, G)$. If f is a monomorphism, then f is a kernel of some morphism. If f is an epimorphism, then f is a cokernel of some morphism.

Let the notation be as in the definition of an abelian category. Then the statement (AB3) is equivalent to following statement (AB3)'. This is proven in the subsequent lemma.

(AB3)' Let $F, G \in \mathrm{Ob}(\mathcal{A})$ and $f \in \mathrm{Mor}(F, G)$.

(a) (Kernels) There exists $K \in \mathrm{Ob}(\mathcal{A})$ and a morphism $i\colon K \to F$ such that for all objects $X \in \mathrm{Ob}(\mathcal{A})$,

$$0 \longrightarrow \mathrm{Mor}(X, K) \xrightarrow{i\circ\cdot} \mathrm{Mor}(X, F) \xrightarrow{f\circ\cdot} \mathrm{Mor}(X, G) \tag{4.4}$$

is an exact sequence of abelian groups.

(b) (Cokernels) There exists $C \in \mathrm{Ob}(\mathcal{A})$ and a morphism $p\colon G \to C$ such that for all objects $Y \in \mathrm{Ob}(\mathcal{A})$,

$$0 \longrightarrow \mathrm{Mor}(C, Y) \xrightarrow{\cdot\circ p} \mathrm{Mor}(G, Y) \xrightarrow{\cdot\circ f} \mathrm{Mor}(F, Y) \tag{4.5}$$

is an exact sequence of abelian groups.

Lemma 4.1.1. *Let the notation be as in the definition of an abelian category. The conditions (AB3) and (AB3)' are equivalent.*

Proof. Assume that (AB3) holds. Let $F, G \in \mathrm{Ob}(\mathcal{A})$ and $f \in \mathrm{Mor}(F, G)$. Let $i\colon K \to F$ be as in (a) of (AB3). Let $X \in \mathrm{Ob}(\mathcal{A})$. We need to prove that (4.4) is exact. Assume that $h \in \mathrm{Mor}(X, K)$ is such that $i \circ h = 0$. We then have $f \circ (i \circ h) = 0$. By the universal property of i, there exists a unique morphism $X \to K$ such that

$$\begin{array}{ccc} X & & \\ \downarrow & \searrow^{i \circ h = 0} & \\ K & \xrightarrow{i} & F \end{array}$$

commutes. The morphisms 0 and h from X to K both make this diagram commute; by uniqueness, we obtain $h = 0$. It follows that (4.4) is exact at $\mathrm{Mor}(X, K)$. Clearly, the composition of the second and third maps of (4.4) is zero. Let $g \in \mathrm{Mor}(X, F)$ and assume that $f \circ g = 0$. By the universal property of i, there exists a unique morphism $X \to K$ such that

$$\begin{array}{ccc} X & & \\ \downarrow & \searrow^{g} & \\ K & \xrightarrow{i} & F \end{array}$$

commutes. Thus, g is in the image of the second map of (4.4). This completes the proof that (4.4) is exact so that (a) of (AB3)' holds. Next, let $p\colon G \to C$ be as in (b) of (AB3). Let $Y \in \mathrm{Ob}(\mathcal{A})$. We need to prove that (4.5) is exact. Let $h \in \mathrm{Mor}(C, Y)$ and assume that $h \circ p = 0$. Then $(h \circ p) \circ f = 0$. By the universal property of p, there exists a unique morphism $C \to Y$ such that

$$\begin{array}{ccc} & & Y \\ & {}^{h \circ p = 0}\nearrow & \uparrow \\ G & \xrightarrow{p} & C \end{array}$$

commutes. The morphisms 0 and h both make this diagram commute; by uniqueness, $h = 0$, and (4.5) is exact at $\mathrm{Mor}(C, Y)$. Clearly, the composition of the second and third maps in (4.5) is zero. Let $g \in \mathrm{Mor}(G, Y)$ be such that $g \circ f = 0$. By the universal property of p, there exists a unique morphism $C \to Y$ such that

$$\begin{array}{ccc} & & Y \\ & {}^{g}\nearrow & \uparrow \\ G & \xrightarrow{p} & C \end{array}$$

commutes. It follows that g is in the image of the second map of (4.5). Hence, (4.5) is exact, and (b) of (AB3)' holds.

Now assume that (AB3)' holds. Let $i\colon K \to F$ be as in (a) of (AB3)'. Letting $X = K$ in (4.4) yields the following exact sequence:

$$0 \longrightarrow \mathrm{Mor}(K, K) \xrightarrow{i \circ \cdot} \mathrm{Mor}(K, F) \xrightarrow{f \circ \cdot} \mathrm{Mor}(K, G)$$

Now $i \circ \mathrm{id}_K = i$ is in the image of the second map of this sequence; hence, the third maps sends i to zero, i.e., $f \circ i = 0$. Assume that $H \in \mathrm{Ob}(\mathcal{A})$ and $g \in \mathrm{Mor}(H, F)$ is such that $f \circ g = 0$. Since

$$0 \longrightarrow \mathrm{Mor}(H, K) \xrightarrow{i \circ \cdot} \mathrm{Mor}(H, F) \xrightarrow{f \circ \cdot} \mathrm{Mor}(H, G)$$

is exact, there exists a unique morphism $H \to K$ such that

$$\begin{array}{ccc} H & & \\ \downarrow & \searrow^{g} & \\ K & \xrightarrow{i} & F \end{array}$$

commutes. This proves that (a) of (AB3) holds. The proof that (b) of (AB3) holds is similar. This completes the proof that (AB3) holds. □

Proposition 4.1.2. *Let $\mathcal{A}$ be a category and assume that $\mathcal{A}$ satisfies (AB1). Assume that* $0, 0' \in \mathrm{Ob}(\mathcal{A})$ *are such that* $\mathrm{Mor}(F, 0) = \mathrm{Mor}(F, 0') = \mathrm{Mor}(0, F) = \mathrm{Mor}(0', F) = 0$ *for all* $F \in \mathrm{Ob}(\mathcal{A})$. *There exists a unique isomorphism* $i\colon 0 \to 0'$.

Proof. By assumption, both $\mathrm{Mor}(0, 0')$ and $\mathrm{Mor}(0', 0)$ are the trivial group. Let i be the unique element of $\mathrm{Mor}(0, 0')$ and let j be the unique element of $\mathrm{Mor}(0', 0)$. Consider $i \circ j \in \mathrm{Mor}(0', 0')$. Since $\mathrm{Mor}(0', 0')$ is the trivial group, this set contains a unique element; since $\mathrm{id}_{0'} \in \mathrm{Mor}(0', 0')$ we must have $i \circ j = \mathrm{id}_{0'}$. Similarly, $j \circ i = \mathrm{id}_0$. This proves the assertion. □

Proposition 4.1.3. *Let $\mathcal{A}$ be a category and assume that $\mathcal{A}$ satisfies (AB1). Let* $F \in \mathrm{Ob}(\mathcal{A})$.

(a) Let $f \in \mathrm{Mor}(F, 0)$. *If f is a monomorphism, then F is a zero object.*

(b) Let $f \in \mathrm{Mor}(0, F)$. *If f is an epimorphism, then F is a zero object.*

Proof. (a). Let $G \in \mathrm{Ob}(\mathcal{A})$. We need to prove that $\mathrm{Mor}(F, G) = \mathrm{Mor}(G, F) = 0$. Let $h \in \mathrm{Mor}(G, F)$. Then $f \circ h = f \circ 0$, where $0 \in \mathrm{Mor}(G, F)$. Since f is a monomorphism, we have $h = 0$. We have proven that $\mathrm{Mor}(G, F) = 0$. Next, let $h \in \mathrm{Mor}(F, G)$. Then $h = h \circ \mathrm{id}_F$. However, $\mathrm{Mor}(F, F) = 0$ by what we have already proven. Hence, $\mathrm{id}_F = 0$. This implies that $h = 0$, and so $\mathrm{Mor}(F, G) = 0$.

(b). The proof of this statement is similar to the proof of (a). □

Proposition 4.1.4. *Let $\mathcal{A}$ be a category and assume that $\mathcal{A}$ satisfies (AB1) and (AB3). Let* $F, G \in \mathrm{Ob}(\mathcal{A})$ *and* $f \in \mathrm{Mor}(F, G)$. *Let* $K, C \in \mathrm{Ob}(\mathcal{A})$, $i \in \mathrm{Mor}(K, F)$ *and* $p \in \mathrm{Mor}(G, C)$ *be such that i is a kernel of f and p is a cokernel of f. Then:*

(a) f is a monomorphism if and only if $K = 0$.

(b) f is an epimorphism if and only if $C = 0$.

Proof. (a). Assume that f is a monomorphism. We have $f \circ 0 = 0$ where the first 0 is in $\mathrm{Mor}(0, F)$. Assume that $H \in \mathrm{Ob}(\mathcal{A})$ and $g \in \mathrm{Mor}(H, F)$ is such that $f \circ g = 0$. We also have $f \circ 0 = 0$ where the first 0 is in $\mathrm{Mor}(H, F)$. Since f is a monomorphism we conclude that $g = 0$. This implies that

$$\begin{array}{ccc} H & & \\ \downarrow{\scriptstyle 0} & \searrow^{g} & \\ 0 & \xrightarrow{0} & F \end{array}$$

commutes. Since $\mathrm{Mor}(H, 0) = 0$, 0 is the unique morphism in $\mathrm{Mor}(H, 0)$ for which this diagram commutes. It follows that $0 \to F$ is a kernel for f, i.e., $K = 0$.

Now assume that $K = 0$. Let $H \in \mathrm{Ob}(\mathcal{A})$ and $g_1, g_2 \in \mathrm{Mor}(H, F)$ be such that $f \circ g_1 = f \circ g_2$. Then $f \circ (g_1 - g_2) = 0$. By the universal property of i, there exists a unique element $t \in \mathrm{Mor}(H, K)$ such that

$$\begin{array}{ccc} H & & \\ \downarrow{\scriptstyle t} & \searrow^{g_1 - g_2} & \\ K & \xrightarrow{i} & F \end{array}$$

commutes. Since $K = 0$ we have $t = 0$. It follows that $g_1 - g_2 = 0$, i.e., $g_1 = g_2$. Thus, f is a monomorphism.

The proof of (b) is similar. □

Proposition 4.1.5. *Let $\mathcal{A}$ be an abelian category. Every kernel of $\mathcal{A}$ is a monomorphism, and every cokernel of $\mathcal{A}$ is an epimorphism.*

Proof. Let $F, G \in \mathrm{Ob}(\mathcal{A})$ and $f \in \mathrm{Mor}(F, G)$. Let $K \in \mathrm{Ob}(\mathcal{A})$ and $i \in \mathrm{Mor}(K, F)$ be such that i is a kernel of f. Let $H \in \mathrm{Ob}(\mathcal{A})$ and $h_1, h_2 \in \mathrm{Mor}(H, F)$ be such that $i \circ h_1 = i \circ h_2$; we need to prove that $h_1 = h_2$. Since $i \circ h_1 = i \circ h_2$ we have $i \circ (h_1 - h_2) = 0$. The composition

$$H \xrightarrow{0} F \xrightarrow{f} G$$

is the zero morphism in $\mathrm{Mor}(H, G)$. Since i is a kernel of f, there exists a unique morphism $t \in \mathrm{Mor}(H, K)$ such that

$$\begin{array}{ccc} H & & \\ \downarrow{\scriptstyle t} & \searrow^{0} & \\ K & \xrightarrow{i} & F \end{array}$$

commutes. If $0 \in \mathrm{Mor}(H, K)$ or $h_1 - h_2 \in \mathrm{Mor}(H, K)$ is substituted for t, then this diagram commutes; by the uniqueness property of t we have $0 = t = h_1 - h_2$. It follows that $h_1 = h_2$. A similar argument proves that every cokernel of $\mathcal{A}$ is an epimorphism. □

Proposition 4.1.6. *Let $\mathcal{A}$ be an abelian category. Let $F, G \in \mathrm{Ob}(\mathcal{A})$ and $f \in \mathrm{Mor}(F, G)$. The morphism f is an isomorphism if and only if f is a monomorphism and an epimorphism.*

Proof. Assume that f is an isomorphism with inverse $g \in \mathrm{Mor}(G, F)$. Let $H \in \mathrm{Ob}(\mathcal{A})$ and let $h_1, h_2 \in \mathrm{Mor}(H, F)$ be such that $f \circ h_1 = f \circ h_2$. Then

$$h_1 = \mathrm{id}_F \circ h_1 = g \circ f \circ h_1 = g \circ f \circ h_2 = \mathrm{id}_F \circ h_2 = h_2.$$

It follows that f is an epimorphism. A similar argument shows that f is a monomorphism.

Assume that f is a monomorphism and an epimorphism. By (AB4), there exists $H \in \mathrm{Ob}(\mathcal{A})$ and $p \in \mathrm{Mor}(G, H)$ such that f is a kernel for p. The composition

$$F \xrightarrow{f} G \xrightarrow{0} 0$$

is zero. Thus, we have $0 \circ f = p \circ f$. Since f is an epimorphism we obtain $0 = p$. Since $p = 0$ the composition

$$G \xrightarrow{\mathrm{id}_G} G \xrightarrow{p} H$$

is zero. Since f is a kernel for p, there exists $g \in \mathrm{Mor}(G, F)$ such that

$$\begin{array}{ccc} G & & \\ \downarrow g & \searrow^{\mathrm{id}_G} & \\ F & \xrightarrow{f} & G \end{array}$$

commutes. Thus, $f \circ g = \mathrm{id}_G$. We also have

$$\begin{aligned} f \circ g &= \mathrm{id}_G \\ f \circ g \circ f &= \mathrm{id}_G \circ f \\ f \circ g \circ f &= f \\ f \circ g \circ f &= f \circ \mathrm{id}_F \\ g \circ f &= \mathrm{id}_F, \end{aligned}$$

where the last equality follows the assumption that f is an epimorphism. □

Assume that $\mathcal{A}$ is an abelian category. Let $F, G \in \mathrm{Ob}(\mathcal{A})$ and let $f \in \mathrm{Mor}(F, G)$. Since $\mathcal{A}$ is an abelian category, the morphism f admits a cokernel:

$$G \xrightarrow{p} \mathrm{coker} f.$$

We define an **image** of f to be a kernel of p:

$$\mathrm{im} f = \ker\left(G \xrightarrow{p} \mathrm{coker} f \right).$$

From the involved definitions, the following composition is zero:

$$\mathrm{im} f \xrightarrow{i} G \xrightarrow{p} \mathrm{coker} f. \tag{4.6}$$

Lemma 4.1.7. *Let $\mathcal{A}$ be an abelian category, let $F, G, H \in \mathrm{Ob}(\mathcal{A})$, and let $f \in \mathrm{Mor}(F, G)$ and $g \in \mathrm{Mor}(G, H)$. Assume that the composition*

$$F \xrightarrow{f} G \xrightarrow{g} H \tag{4.7}$$

is zero. There exists a unique morphism

$$\mathrm{im} f \xrightarrow{t} \ker g \tag{4.8}$$

such that

$$\begin{array}{ccc} \mathrm{im} f & & \\ \downarrow t & \searrow^{i} & \\ \ker g & \longrightarrow & G \end{array} \tag{4.9}$$

commutes. The morphism $t\colon \mathrm{im} f \to \ker g$ is a monomorphism.

Proof. We first prove that the composition

$$\operatorname{im} f \xrightarrow{\ i\ } G \xrightarrow{\ g\ } H \tag{4.10}$$

is zero. Since $g \circ f = 0$, i.e., the composition (4.7) is zero, there exists a morphism

$$\operatorname{coker} f \xrightarrow{\ r\ } H$$

such that

$$\begin{array}{ccc} & & H \\ & \overset{g}{\nearrow} & \uparrow r \\ G & \xrightarrow{\ p\ } & \operatorname{coker} f \end{array}$$

commutes. Since $\operatorname{im} f = \ker p$ (by definition), we have $p \circ i = 0$ (see (4.6)). Thus,

$$\begin{array}{ccc} & & H \\ & \overset{g}{\nearrow} & \uparrow r \\ G & \xrightarrow{\ p\ } & \operatorname{coker} f \\ i \uparrow & \overset{0}{\nearrow} & \\ \operatorname{im} f & & \end{array}$$

commutes. We now deduce that $g \circ i = 0$. Since $g \circ i = 0$, from the universal property of $\ker g$, there exists a morphism as in (4.8) such that (4.9) commutes. Finally, by Proposition 4.1.5, the morphism i is a monomorphism (recall the definition of $\operatorname{im} f$). This implies that t is also a monomorphism. □

Let $\mathcal{A}$ be an abelian category, let $F, G, H \in \operatorname{Ob}(\mathcal{A})$, and let $f \in \operatorname{Mor}(F, G)$ and $g \in \operatorname{Mor}(G, H)$. Assume that the composition

$$F \xrightarrow{\ f\ } G \xrightarrow{\ g\ } H \tag{4.11}$$

is zero. From Lemma 4.1.7, there is a canonical monomorphism

$$\operatorname{im} f \longrightarrow \ker g$$

We say that the sequence (4.11) is **exact** at G if this monomorphism is an isomorphism (by Proposition 4.1.6 this amounts to asserting that our monomorphism is also an epimorphism).

4.2 The first axiom

Let X be a topological space. We define two categories. The first category is called PS_X. We define the objects of PS_X to be

$$\operatorname{Ob}(\mathrm{PS}_X) = \{ \text{ abelian presheaves on } X \ \}.$$

The morphisms of PS_X consist of the morphisms of abelian presheaves on X as defined in Section 2.4. The second category is called SH_X. The objects of SH_X are

$$\mathrm{Ob}(\mathrm{SH}_X) = \{ \text{ abelian sheaves on } X \ \}.$$

The morphisms of SH_X are the morphisms of abelian sheaves on X; as mentioned in Section 3.1, these morphisms are defined exactly as in the case of presheaves. It is straightforward to verify that PS_X and SH_X are categories. In this chapter we will prove that PS_X and SH_X are, in fact, abelian categories as defined in Section 4.1.

For this, we first define an addition on $\mathrm{Mor}(\mathcal{F},\mathcal{G})$ so that $\mathrm{Mor}(\mathcal{F},\mathcal{G})$ is an abelian group for $\mathcal{F},\mathcal{G} \in \mathrm{Ob}(\mathrm{PS}_X)$ or $\mathcal{F},\mathcal{G} \in \mathrm{Ob}(\mathrm{SH}_X)$. Let $\mathcal{F},\mathcal{G} \in \mathrm{Ob}(\mathrm{PS}_X)$ or $\mathcal{F},\mathcal{G} \in \mathrm{Ob}(\mathrm{SH}_X)$. Let $f, g \in \mathrm{Mor}(\mathcal{F},\mathcal{G})$. Let U be an open subset of X. We define

$$\mathcal{F}(U) \xrightarrow{(f+g)(U)} \mathcal{G}(U)$$

by

$$(f+g)(U)(x) = f(U)(x) + g(U)(x)$$

for $x \in \mathcal{F}(U)$. Evidently, $(f+g)(U)$ is a homomorphism from the abelian group $\mathcal{F}(U)$ to the abelian group $\mathcal{G}(U)$. We define

$$\mathcal{F} \xrightarrow{f+g} \mathcal{G}$$

to be the collection

$$\left\{ \ \mathcal{F}(U) \xrightarrow{(f+g)(U)} \mathcal{G}(U) \ \right\}_{U \subset X \text{ open}}$$

A verification shows that $f+g$ is morphism and that the set $\mathrm{Mor}(\mathcal{F},\mathcal{G})$ is an abelian group with this definition.

Turning now to the first axiom for abelian categories, we let 0_X be the constant presheaf corresponding to trivial abelian group 0 (the group with one element). Thus, $0_X(U) = 0$ for all open subsets U of X. See Section 2.2. In fact, 0_X is also a sheaf (see p. 37) since the sheafification of 0_X is 0_X (see p. 37). This is true because the trivial group contains a single element. We have $\mathrm{Mor}(0_X,\mathcal{F}) = 0$ and $\mathrm{Mor}(\mathcal{F},0_X) = 0$ for $\mathcal{F} \in \mathrm{PS}_X$ and for $\mathcal{F} \in \mathrm{SH}_X$. Also, it is easy to see that the bilinearity condition (4.1) holds. Thus, with these defintions, both of the categories PS_X and SH_X satisfy axiom (AB1).

4.3 The second axiom

Let X be a topological space. We will now verify that the categories PS_X and SH_X satisfy the second axiom (AB2) of abelian categories. Let $\mathcal{F},\mathcal{G} \in \mathrm{PS}_X$. Define

$$\mathcal{F} \oplus \mathcal{G} \colon \mathrm{Open}(X) \longrightarrow \mathcal{A}b$$

in the following way. If U is an open subset of X, then we define

$$(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$$

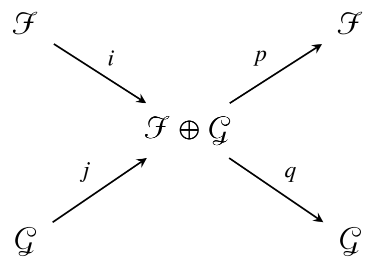
If U and V are open subsets of X with $V \subset U$, then we define

$$(\mathcal{F} \oplus \mathcal{G})(U) \longrightarrow (\mathcal{F} \oplus \mathcal{G})(V)$$

as the direct sum of the two homomorphisms

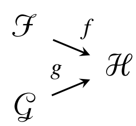
$$\mathcal{F}(U) \longrightarrow \mathcal{F}(V) \quad \text{and} \quad \mathcal{G}(U) \longrightarrow \mathcal{G}(V)\ .$$

It is straightforward to verify that $\mathcal{F} \oplus \mathcal{G}$ is a presheaf. We also have natural morphisms of presheaves as follows:



Proposition 4.3.1. *Let X be a topological space, and let $\mathcal{F}, \mathcal{G} \in \mathrm{PS}_X$. The presheaf $\mathcal{F} \oplus \mathcal{G}$, with the above morphisms, satisfies the assertions of (AB2).*

Proof. Let $\mathcal{H} \in \mathrm{PS}_X$ and let



be morphisms. Let U be an open subset of X. We define

$$(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U) \xrightarrow{h(U)} \mathcal{H}(U)$$

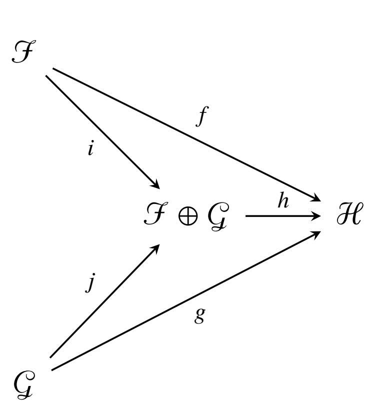
by

$$h(U)(a, b) = f(U)(a) + g(U)(b)$$

for $a \in \mathcal{F}(U)$ and $b \in \mathcal{G}(U)$. The collection

$$\left\{ \ (\mathcal{F} \oplus \mathcal{G})(U) \xrightarrow{h(U)} \mathcal{H}(U)\ \right\}_{U \subset X \text{ open}}$$

is a morphism of presheaves. We claim that



(4.12)

commutes. Let U be a open subset of X and let $a \in \mathcal{F}(U)$. Then

$$\begin{aligned}(h(U) \circ i(U))(a) &= h(U)(a, 0) \\ &= f(U)(a) + h(U)(0) \\ &= f(U)(a).\end{aligned}$$

Hence, $h(U) \circ i(U) = f(U)$. Similarly, $h(U) \circ j(U) = g(U)$. This proves our claim that (4.12) commutes. Next, suppose that

$$\mathcal{F} \oplus \mathcal{G} \xrightarrow{h'} \mathcal{H}$$

is a morphism of presheaves such that (4.12) commutes with h' in place of h; we will prove that $h = h'$. Let U be an open subset of X. Let $a \in \mathcal{F}(U)$ and $b \in \mathcal{G}(U)$. Then

$$\begin{aligned}h'(U)(a, b) &= h'(U)((a, 0) + (0, b)) \\ &= h'(U)(a, 0) + h'(U)(0, b) \\ &= h'(U)(i(U)(a)) + h'(U)(j(U)(b)) \\ &= (h'(U) \circ i(U))(a) + (h'(U) \circ j(U))(b) \\ &= f(U)(a) + g(U)(b) \\ &= h(U)(a, b).\end{aligned}$$

It follows that $h' = h$. We have proven that $\mathcal{F} \oplus \mathcal{G}$, along with the morphisms i, j, p, and q, satisfy (a) of (AB2). Similar arguments prove that (b) of (AB2) is also satisfied. □

Proposition 4.3.2. *Let X be a topological space, and let $\mathcal{F}, \mathcal{G} \in \mathrm{SH}_X$. Then $\mathcal{S} \oplus \mathcal{G}$ is a sheaf.*

Proof. We need to verify that $\mathcal{F} \oplus \mathcal{G}$ verifies the gluing condition (G) and the locality condition (L). To verify (G), let U be an open subset of X, let $\{U_i\}_{i \in I}$ be an open cover of U, and let $\{s_i\}_{i \in I}$ be such that $s_i \in (\mathcal{F} \oplus \mathcal{G})(U_i)$ for $i \in I$ and for all $i, j \in I$ we have $\rho_{U_i, U_i \cap U_j}(s_i) = \rho_{U_j, U_i \cap U_j}(s_j)$. From the definition of $\mathcal{F} \oplus \mathcal{G}$, for each $i \in I$ we have $s_i = (a_i, b_i)$ where $a_i \in \mathcal{F}(U_i)$ and $b_i \in \mathcal{G}(U_i)$. Also, we see that for all $i, j \in I$ we have $\rho_{U_i, U_i \cap U_j}(a_i) = \rho_{U_j, U_i \cap U_j}(a_j)$ and $\rho_{U_i, U_i \cap U_j}(b_i) = \rho_{U_j, U_i \cap U_j}(b_j)$. Since $\mathcal{F}$ and $\mathcal{G}$ satisfy (G), there exist $a \in \mathcal{F}(U)$ and $b \in \mathcal{G}(U)$ such that $\rho_{U, U_i}(a) = a_i$ and $\rho_{U, U_i}(b) = b_i$ for $i \in I$. Taking the definitions into account, we have $\rho_{U, U_i}(a, b) = (\rho_{U, U_i}(a), \rho_{U, U_i}(b)) = (a_i, b_i) = s_i$ for $i \in I$. This verifies (G) for $\mathcal{F} \oplus \mathcal{G}$. The locality condition is similarly verified. □

Corollary 4.3.3. *Let X be a topological space. The categories PS_X and SH_X satisfy axiom (AB2) of the definition of an abelian category.*

Proof. The category PS_X satisfies axiom (AB2) by Proposition 4.3.1. The category PS_X satisfies axiom (AB2) by Proposition 4.3.1 and Proposition 4.3.2. □

4.4 The third axiom

Let X be a topological space. In this section we will prove that PS_X and SH_X satisfy the third axiom (AB3) of abelian categories.

Kernels

Let $\mathcal{F}, \mathcal{G} \in \mathrm{PS}_X$, and let $f \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$. If U is an open subset of X, then we define

$$(\ker f)(U) = \{s \in \mathcal{F}(U) \colon f(U)(s) = 0\}.$$

Evidently, $(\ker f)(U)$ is an subgroup of $\mathcal{F}(U)$ for every open subset of X. Let U and V be open subsets of X with $V \subset U$. Since f is a morphism, the following diagram commutes:

$$\begin{CD} \mathcal{F}(U) @>{f(U)}>> \mathcal{G}(U) \\ @V{\rho^{\mathcal{F}}_{U,V}}VV @VV{\rho^{\mathcal{G}}_{U,V}}V \\ \mathcal{F}(V) @>{f(V)}>> \mathcal{G}(V) \end{CD}$$

Here, the vertical arrows are the restriction maps for $\mathcal{F}$ and $\mathcal{G}$. Let $s \in (\ker f)(U)$. Then

$$f(V)(\rho^{\mathcal{F}}_{U,V}(s)) = \rho^{\mathcal{G}}_{U,V}(f(U)(s)) = \rho^{\mathcal{G}}_{U,V}(0) = 0.$$

It follows that

$$\rho^{\mathcal{F}}_{U,V}((\ker f)(U)) \subset (\ker f)(V).$$

We now see that $\ker f$, equipped with the restrictions of the $\rho^{\mathcal{F}}_{U,V}$, is a presheaf on X with values in $\mathcal{Ab}$, i.e., $\ker f \in \mathrm{PS}_X$. For each open subset U of X, let

$$(\ker f)(U) \xrightarrow{i(U)} \mathcal{F}(U)$$

be the inclusion function. It is straightforward to verify that the collection

$$i = \left\{ (\ker f)(U) \xrightarrow{i(U)} \mathcal{F}(U) \right\}_{U \subset X \text{ open}}$$

is an element of $\mathrm{Mor}(\ker f, \mathcal{F})$.

Proposition 4.4.1. *Let X be a topological space. Let $\mathcal{F}, \mathcal{G} \in \mathrm{PS}_X$ and $f \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$. The composition*

$$\ker f \xrightarrow{i} \mathcal{F} \xrightarrow{f} \mathcal{G}$$

is the zero morphism in $\mathrm{Mor}(\ker f, \mathcal{G})$*, and for any $\mathcal{H} \in \mathrm{PS}_X$ and $g \in \mathrm{Mor}(\mathcal{H}, \mathcal{F})$ such that the composition*

$$\mathcal{H} \xrightarrow{g} \mathcal{F} \xrightarrow{f} \mathcal{G}$$

is the zero morphism in $\mathrm{Mor}(\mathcal{H}, \mathcal{G})$*, there exists a unique morphism*

$$\mathcal{H} \xrightarrow{j} \ker f$$

such that

$$\begin{array}{ccc} \mathcal{H} & & \\ \Big\downarrow{\scriptstyle j} & \searrow^{g} & \\ \ker f & \xrightarrow{i} & \mathcal{F} \end{array}$$

commutes.

Proof. It is clear that $f \circ i$ is the zero morphism in $\mathrm{Mor}(\ker f, \mathcal{G})$. Let $\mathcal{H} \in \mathrm{PS}_X$ and $g \in \mathrm{Mor}(\mathcal{H}, \mathcal{F})$ be such that $f \circ g$ is the zero morphism in $\mathrm{Mor}(\mathcal{H}, \mathcal{G})$. Let U be an open subset of X. Since $f \circ g = 0$, we have $f(U) \circ g(U) = 0$. This implies that $g(U)(\mathcal{H}(U)) \subset (\ker f)(U)$. It follows that the function

$$\mathcal{H}(U) \xrightarrow{j(U)} (\ker f)(U)$$

that sends $s \in \mathcal{H}(U)$ to $g(U)(s)$ is well-defined. Also, it is easy to check that

$$\mathcal{H} \xrightarrow{j} \ker f \quad = \left\{ \mathcal{H}(U) \xrightarrow{j(U)} (\ker f)(U) \right\}_{U \subset X \text{ open}}$$

is a morphism of presheaves and is thus in $\mathrm{Mor}(\mathcal{H}, \ker f)$. We see that $g = i \circ j$. Finally, assume that $j' \in \mathrm{Mor}(\mathcal{H}, \ker f)$ is also such that $g = i \circ j'$. Let U be an open subset of X, and let $s \in \mathcal{H}(U)$. Then

$$g(U)(s) = i(U)(j'(U)(s)) = j'(U)(s).$$

Since $j(U)(s) = g(U)(s)$, we obtain $j'(U) = j(U)$; it follows that $j' = j$. □

Lemma 4.4.2. *Let X be a topological space. Let $\mathcal{F}, \mathcal{G} \in \mathrm{SH}_X$, and let $f \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$. The presheaf* $\ker f$ *is a sheaf.*

Proof. We first verify the gluing condition (G) for $\ker f$. Let U be an open subset of X. Let $\{U_i\}_{i \in I}$ of U, let $\{s_i\}_{i \in I}$ be such that $s_i \in (\ker f)(U_i)$ for $i \in I$, and assume that for all $i, j \in I$ we have $\rho^{\mathcal{F}}_{U_i, U_i \cap U_j}(s_i) = \rho^{\mathcal{F}}_{U_j, U_i \cap U_j}(s_j)$. Since $\mathcal{F}$ satisfies the gluing condition (G), there exists $s \in \mathcal{F}(U)$ such that $\rho^{\mathcal{F}}_{U, U_i}(s) = s_i$ for all $i \in I$. We need to prove that $s \in (\ker f)(U)$, i.e., $f(U)(s) = 0$. Let $i \in I$. The following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f(U)} & \mathcal{G}(U) \\ \downarrow{\scriptstyle \rho^{\mathcal{F}}_{U,U_i}} & & \downarrow{\scriptstyle \rho^{\mathcal{G}}_{U,U_i}} \\ \mathcal{F}(U_i) & \xrightarrow{f(U_i)} & \mathcal{G}(U_i) \end{array}$$

Therefore,

$$\begin{aligned} \rho^{\mathcal{G}}_{U,U_i}(f(U)(s)) &= f(U_i)(\rho^{\mathcal{F}}_{U,U_i}(s)) \\ &= f(U_i)(s_i) \\ &= 0. \end{aligned}$$

Of course, we also have $\rho^{\mathcal{G}}_{U,U_i}(0) = 0$. By the locality condition (L) for $\mathcal{G}$ we have $f(U)(s) = 0$. This verifies the gluing condition (G) for $\ker f$. The locality condition for $\ker f$ is proved similarly. □

Cokernels

Again let X be a topological space. Let $\mathcal{F}, \mathcal{G} \in \mathrm{PS}_X$ and $f \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$. If U is an open subset of X, then we define

$$(\mathrm{pcok} f)(U) = \mathcal{G}(U)/\mathrm{im}(f(U)).$$

Let U and V be open subsets of X such that $V \subset U$. Since f is a morphism, the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f(U)} & \mathcal{G}(U) \\ \downarrow \rho^{\mathcal{F}}_{U,V} & & \downarrow \rho^{\mathcal{G}}_{U,V} \\ \mathcal{F}(V) & \xrightarrow{f(V)} & \mathcal{G}(V) \end{array}$$

commutes. It follows that

$$\rho^{\mathcal{G}}_{U,V}(\mathrm{im}(f(U))) \subset \mathrm{im}(f(V)). \tag{4.13}$$

Define

$$(\mathrm{pcok} f)(U) = \mathcal{G}(U)/\mathrm{im}(f(U)) \longrightarrow (\mathrm{pcok} f)(V) = \mathcal{G}(V)/\mathrm{im}(f(V)) \tag{4.14}$$

by

$$r \mapsto \rho^{\mathcal{G}}_{U,V}(r) + \mathrm{im}(f(V))$$

for $r \in \mathcal{G}(U)$; since we have the inclusion (4.13), this map is a well-defined homomorphism. It is straightforward to verify that the assignment $U \mapsto (\mathrm{pcok} f)(U)$ for U an open subset of X, along with the restriction maps (4.14), is an element of PS_X. For each open subset U of X, let

$$\mathcal{G}(U) \xrightarrow{p(U)} (\mathrm{pcok} f)(U) = \mathcal{G}(U)/\mathrm{im}(f(U))$$

be the natural projection. The collection

$$p = \left\{ \mathcal{G}(U) \xrightarrow{p(U)} (\mathrm{pcok} f)(U) \right\}_{U \subset X \text{ open}} \tag{4.15}$$

is an element of $\mathrm{Mor}(\mathcal{G}, \mathrm{pcok} f)$.

Proposition 4.4.3. *Let X be a topological space. Let $\mathcal{F}, \mathcal{G} \in \mathrm{PS}_X$ and $f \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$. The composition*

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{p} \mathrm{pcok} f$$

is the zero morphism in $\mathrm{Mor}(\mathcal{F}, \mathrm{pcok} f)$*, and for any $\mathcal{H} \in \mathrm{PS}_X$ and $g \in \mathrm{Mor}(\mathcal{G}, \mathcal{H})$ such that the composition*

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$$

is the zero morphism in $\mathrm{Mor}(\mathcal{F}, \mathcal{H})$*, there exists a unique morphism*

$$\mathrm{pcok} f \xrightarrow{q} \mathcal{H}$$

such that

$$\begin{array}{ccc} & & \mathcal{H} \\ & \overset{g}{\nearrow} & \uparrow q \\ \mathcal{G} & \xrightarrow{p} & \mathrm{pcok} f \end{array}$$

commutes.

Proof. It is clear that $p \circ f$ is the zero morphism in $\mathrm{Mor}(\mathcal{F}, \mathrm{pcok} f)$. Let $\mathcal{H} \in \mathrm{PS}_X$ and $g \in \mathrm{Mor}(\mathcal{G}, \mathcal{H})$ be such that $g \circ f$ is the zero morphism in $\mathrm{Mor}(\mathcal{F}, \mathcal{H})$. Let U be an open subset of X. Since $g \circ f = 0$, we have $g(U) \circ f(U) = 0$. This implies that $g(U)(\mathrm{im}(f(U))) = 0$. It follows that the function

$$(\mathrm{pcok} f)(U) = \mathcal{G}(U)/\mathrm{im}(f(U)) \xrightarrow{q(U)} \mathcal{H}(U)$$

that sends $s + \mathrm{im}(f(U))$ for $s \in \mathcal{G}(U)$ to $g(U)(s)$ is well-defined. Also, it is easy to check that

$$\mathrm{pcok} f \xrightarrow{q} \mathcal{H} \ = \left\{ \ (\mathrm{pcok} f)(U) \xrightarrow{q(U)} \mathcal{H}(U) \ \right\}_{U \subset X \text{ open}}$$

is a morphism of presheaves and is thus in $\mathrm{Mor}(\mathrm{pcok} f, \mathcal{H})$. We see that $g = q \circ p$. Finally, assume that $q' \in \mathrm{Mor}(\mathrm{pcok} f, \mathcal{H})$ is also such that $g = q' \circ p$. Let U be an open subset of X, and let $s \in \mathcal{G}(U)$. Then

$$g(U)(s) = q'(U)(p(U)(s)) = q'(U)(s + (\mathrm{im} f)(U)).$$

Similarly, $g(U)(s) = q(U)(s + (\mathrm{im} f)(U))$, so that $q'(U)(s + (\mathrm{im} f)(U)) = q(U)(s + (\mathrm{im} f)(U))$. This implies that $q'(U) = q(U)$ and hence $q' = q$. □

Now assume that $\mathcal{F}, \mathcal{G} \in \mathrm{SH}_X$ and $f \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$. Above, we defined the presheaf $\mathrm{pcok} f \in \mathrm{PS}_X$. We now define

$$\mathrm{scok} f = \text{sheafification of } \mathrm{pcok} f = \Gamma L(\mathrm{pcok} f).$$

We recall that there exists a canonical morphism of presheaves

$$\mathrm{pcok} f \xrightarrow{n = n_{\mathrm{pcok} f}} \mathrm{scok} f$$

as in (3.10). See Theorem 3.3.14 (note that we verified in Section 3.4 that this theorem also holds in the abelian setting). Let

$$\mathcal{G} \xrightarrow{p} \mathrm{pcok} f$$

be the morphism (4.15), and which has the universal property as in Proposition 4.4.3. We let

$$\mathcal{G} \xrightarrow{p_s} \mathrm{scok} f$$

be the composition

$$\mathcal{G} \xrightarrow{p} \mathrm{pcok} f \xrightarrow{n} \mathrm{scok} f$$

We have the following result.

Proposition 4.4.4. *Let X be a topological space. Let $\mathcal{F}, \mathcal{G} \in \mathrm{SH}_X$ and $f \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$. The composition*

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{p_s} \mathrm{scok} f$$

is the zero morphism in $\mathrm{Mor}(\mathcal{F}, \mathrm{scok} f)$*, and for any* $\mathcal{H} \in \mathrm{SH}_X$ *and* $g \in \mathrm{Mor}(\mathcal{G}, \mathcal{H})$ *such that the composition*

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$$

is the zero morphism in $\mathrm{Mor}(\mathcal{F}, \mathcal{H})$*, there exists a unique morphism*

$$\mathrm{scok} f \xrightarrow{m} \mathcal{H}$$

such that

$$\begin{array}{ccc} & & \mathcal{H} \\ & \overset{g}{\nearrow} & \uparrow m \\ \mathcal{G} & \xrightarrow{p_s} & \mathrm{scok} f \end{array}$$

commutes.

Proof. In this proof we will use Theorem 3.3.14 (note that we verified in Section 3.4 that this theorem also holds in the abelian setting). We first note that $p_s \circ f = m \circ p \circ f = m \circ 0 = 0$, that is, $p_s \circ f$ is the zero morphism in $\mathrm{Mor}(\mathcal{F}, \mathrm{scok} f)$. Let $\mathcal{H} \in \mathrm{SH}_X$ and $g \in \mathrm{Mor}(\mathcal{G}, \mathcal{H})$ be such that $g \circ f$ is the zero morphism in $\mathrm{Mor}(\mathcal{F}, \mathcal{H})$. By Proposition 4.4.3, there exists a unique morphism

$$\mathrm{pcok} f \xrightarrow{q} \mathcal{H}$$

such that

$$\begin{array}{ccc} & & \mathcal{H} \\ & \overset{g}{\nearrow} & \uparrow q \\ \mathcal{G} & \xrightarrow{p} & \mathrm{pcok} f \end{array}$$

commutes. By Theorem 3.3.14 there exists a unique morphism $m \in \mathrm{Mor}(\mathrm{scok} f, \mathcal{H})$ such that

$$\begin{array}{ccc} \mathcal{H} & & \\ q \uparrow & \overset{m}{\nwarrow} & \\ \mathrm{pcok} f & \xrightarrow{n} & \mathrm{scok} f \end{array}$$

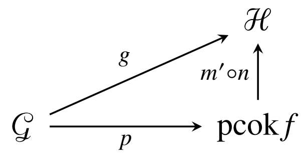
commutes. Putting the last two diagrams together, we obtain the following commutative diagram:

$$\begin{array}{ccc} & & \mathcal{H} \\ & \overset{g}{\nearrow} & \uparrow m \\ \mathcal{G} & \xrightarrow[p_s = n \circ p]{} & \mathrm{scok} f \end{array}$$

Finally, assume that $m' \in \mathrm{Mor}(\mathrm{scok} f, \mathcal{H})$ is such that

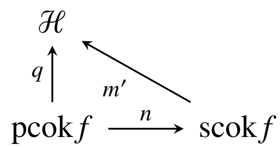
$$\begin{array}{ccc} & & \mathcal{H} \\ & \overset{g}{\nearrow} & \uparrow m' \\ \mathcal{G} & \xrightarrow[p_s = n \circ p]{} & \mathrm{scok} f \end{array}$$

commutes. We can rewrite this commutative diagram as the following commutative diagram:



By the uniqueness property of q we have $q = m' \circ n$ so that the following diagram commutes:

and



By the uniqueness property of m, we conclude that $m' = m$. ◻

Corollary 4.4.5. *Let X be a topological space. The categories* PS_X *and* SH_X *satisfy axiom (AB3) of the definition of an abelian category.*

Proof. The category PS_X satisfies axiom (AB3) by Proposition 4.4.1 and Proposition 4.4.3. The category SH_X satisfies axiom (AB3) by Proposition 4.4.1, Lemma 4.4.2 and Proposition 4.4.4. ◻

Corollary 4.4.6. *Let X be a topological space. Let $\mathcal{F}, \mathcal{G} \in \mathrm{PS}_X$ and $f \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$. Then*

(a) f is a monomorphism if and only if $\ker f(U) = 0$ for all open subsets U of X.

(b) f is an epimorphism if and only if $\mathrm{im} f(U) = \mathcal{G}(U)$ for all open subsets U of X.

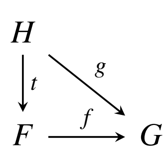
Proof. To begin, we note that we have already verified that PS_X satisfies (AB1) and (AB3) (see Section 4.2 and Corollary 4.4.5).

(a). By Proposition 4.1.4, f is a monomorphism if and only if $\ker f = 0$. The definition of $\ker f$ implies that $\ker f = 0$ if and only if $\ker f(U) = 0$ for all open subsets of U.

(b). By Proposition 4.1.4, f is an epimorphism if and only if $\mathrm{pcok} f = 0$. The definition of $\mathrm{pcok} f$ implies that $\mathrm{pcok} f = 0$ if and only if $\mathrm{im} f(U) = \mathcal{G}(U)$ for all open subsets of U. ◻

4.5 The fourth axiom

Lemma 4.5.1. *Let F, G, and H be abelian groups, let $f\colon F \to G$ be a monomorphism, and let $g\colon H \to G$ be a homomorphism such that $g(H) \subset f(F)$. There exists a unique homomorphism $t\colon H \to F$ such that*



commutes.

Proof. Since f is injective, there exists a homomorphism $f'\colon f(F) \to F$ such that $f(f'(x)) = x$ for $x \in F$. Define $t\colon H \to F$ by $t(y) = f'(g(y))$ for $y \in H$. Then t is a homomorphism and the above diagram commutes. Since f is a monomorphism, t is unique. ◻

Proposition 4.5.2. *Let X be a topological space. Let $\mathcal{F}, \mathcal{G} \in \mathrm{PS}_X$, and let $f \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$. Assume that f is a monomorphism. Then f is a kernel for the morphism $p\colon \mathcal{G} \to \mathrm{pcok} f$.*

Proof. Since $p\colon \mathcal{G} \to \mathrm{pcok} f$ is a cokernel for f the composition

$$\mathcal{F} \xrightarrow{\ f\ } \mathcal{G} \xrightarrow{\ p\ } \mathrm{pcok} f$$

is the zero morphism in $\mathrm{Mor}(\mathcal{F}, \mathrm{pcok} f)$. Assume that $\mathcal{H} \in \mathrm{PS}_X$ and $g \in \mathrm{Mor}(\mathcal{H}, \mathcal{G})$ is such that

$$\mathcal{H} \xrightarrow{\ g\ } \mathcal{G} \xrightarrow{\ p\ } \mathrm{pcok} f$$

is the zero morphism in $\mathrm{Mor}(\mathcal{H}, \mathrm{pcok} f)$. Let U be an open subset of X. By Lemma 4.5.1, there exists a unique homomorphism $t(U)\colon \mathcal{H}(U) \to \mathcal{F}(U)$ such that

$$\begin{array}{ccc} \mathcal{H}(U) & & \\ \Big\downarrow{\scriptstyle t(U)} & \searrow^{g(U)} & \\ \mathcal{F}(U) & \xrightarrow{f(U)} & \mathcal{G}(U) \end{array}$$

commutes. The collection

$$t = \left\{ \mathcal{H}(U) \xrightarrow{t(U)} \mathcal{F}(U) \right\}_{U \subset X \text{ open}}$$

is an element of $\mathrm{Mor}(\mathcal{H}, \mathcal{F})$ and the diagram

$$\begin{array}{ccc} \mathcal{H} & & \\ \Big\downarrow{\scriptstyle t} & \searrow^{g} & \\ \mathcal{F} & \xrightarrow{f} & \mathcal{G} \end{array}$$

commutes. The uniqueness of t follows from the uniqueness of $t(U)$ for U an open subset of X as in Lemma 4.5.1. □

Lemma 4.5.3. *Let F, G, and H be abelian groups. Let $f\colon F \to G$ be an epimorphism, and let $g\colon F \to H$ be a homomorphism such that $g(\ker f) = 0$. There exists a unique homomorphism $t\colon G \to H$ such that*

$$\begin{array}{ccc} F & \xrightarrow{g} & H \\ & \searrow_{f} & \Big\uparrow{\scriptstyle t} \\ & & G \end{array}$$

commutes.

Proof. Define $t\colon G \to H$ by $t(y) = g(x)$ for $y \in G$, where $x \in F$ is such that $f(x) = y$; since f is surjective, such an x exists. Since $g(\ker f) = 0$, the function t is well-defined. It is straightforward to verify that t is a homomorphism, that the diagram commutes, and that t is unique. □

Proposition 4.5.4. *Let X be a topological space. Let $\mathcal{F}, \mathcal{G} \in \mathrm{PS}_X$, and let $f \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$. Assume that f is an epimorphism. Then f is a cokernel for the morphism $i\colon \ker f \to \mathcal{F}$.*

Proof. Since $i\colon \ker f \to \mathcal{F}$ is a kernel for f, the composition

$$\ker f \xrightarrow{\ i\ } \mathcal{F} \xrightarrow{\ f\ } \mathcal{G}$$

is zero. Assume that $\mathcal{H} \in \mathrm{PS}_X$ and $g \in \mathrm{Mor}(\mathcal{F}, \mathcal{H})$ is such that the composition

$$\ker f \xrightarrow{\ i\ } \mathcal{F} \xrightarrow{\ g\ } \mathcal{H}$$

is zero. Let U be an open subset of X. Then $g(U)(\ker f(U)) = 0$. By Lemma 4.5.3, there exists a unique homomorphism $t(U)\colon \mathcal{G}(U) \to \mathcal{H}(U)$ such that

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{g(U)} & \mathcal{H}(U) \\ & \searrow^{f(U)} & \uparrow{\scriptstyle t(U)} \\ & & \mathcal{G}(U) \end{array}$$

The collection

$$t = \left\{ \mathcal{G}(U) \xrightarrow{t(U)} \mathcal{H}(U) \right\}_{U \subset X \text{ open}}$$

is an element of $\mathrm{Mor}(\mathcal{G}, \mathcal{H})$ such that the diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{g} & \mathcal{H} \\ & \searrow^{f} & \uparrow{\scriptstyle t} \\ & & \mathcal{G} \end{array}$$

commutes. The uniqueness of t follows from the uniqueness of $t(U)$ for U an open subset of X as in Lemma 4.5.3. □

Lemma 4.5.5. *Let X be a topological space. Let $\mathcal{F}, \mathcal{G} \in \mathrm{PS}_X$ and $f \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$. If $\mathcal{F}$ is a monopresheaf, and f_x is a monomorphism for all $x \in X$, then f is a monomorphism.*

Proof. Assume that $\mathcal{F}$ is a monopresheaf and f_x is a monomorphism for all $x \in X$. Let $\mathcal{H} \in \mathrm{PS}_X$ and let $h_1, h_2 \in \mathrm{Mor}(\mathcal{H}, \mathcal{F})$ be such that $f \circ h_1 = f \circ h_2$; we need to prove that $h_1 = h_2$. If $x \in X$, then $f_x \circ h_{1,x} = f_x \circ h_{2,x}$ and so $h_{1,x} = h_{2,x}$ since f_x is a monomorphism. By Corollary 3.1.2, since $\mathcal{F}$ is a monopresheaf, we conclude that $h_1 = h_2$. □

Lemma 4.5.6. *Let X be a topological space. Let $\mathcal{F}, \mathcal{G} \in \mathrm{SH}_X$ and $f \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$. If f is a monomorphism, then the presheaf $\mathrm{pcok} f$ satisfies the locality condition (L) and is thus a monopresheaf.*

Proof. Let U be an open subset of X and let $\{U_i\}_{i \in I}$ be an open cover X. Let $r, r' \in (\mathrm{pcok} f)(U)$ be such that $\rho^{\mathrm{pcok} f}_{U,U_i}(r) = \rho^{\mathrm{pcok} f}_{U,U_i}(r')$ for all $i \in I$; we need to prove that $r = r'$. Now for $i \in I$ we have $(\mathrm{pcok} f)(U) = \mathcal{G}(U)/\mathrm{im} f(U)$; hence, there exists $s, s' \in \mathcal{G}(U)$ such that $r = s + \mathrm{im} f(U)$

and $r' = s' + \operatorname{im} f(U)$. To prove that $r = r'$ it will suffice to prove that $s - s' \in \operatorname{im} f(U)$. By our assumption, $\rho^{\mathcal{G}}_{U,U_i}(s - s') \in \operatorname{im} f(U_i)$ for all $i \in I$; for each $i \in I$, let $t_i \in \mathcal{F}(U_i)$ be such that

$$f(U_i)(t_i) = \rho^{\mathcal{G}}_{U,U_i}(s - s'). \tag{4.16}$$

Let $i, j \in I$. The following diagram commutes

$$\begin{array}{ccc} \mathcal{F}(U_i) & \xrightarrow{f(U_i)} & \mathcal{G}(U_i) \\ \downarrow{\scriptstyle \rho^{\mathcal{F}}_{U_i,U_i\cap U_j}} & & \downarrow{\scriptstyle \rho^{\mathcal{G}}_{U_i,U_i\cap U_j}} \\ \mathcal{F}(U_i \cap U_j) & \xrightarrow{f(U_i\cap U_j)} & \mathcal{G}(U_i \cap U_j). \end{array}$$

Hence,

$$\begin{aligned} f(U_i \cap U_j)(\rho^{\mathcal{F}}_{U_i,U_i\cap U_j}(t_i)) &= \rho^{\mathcal{G}}_{U_i,U_i\cap U_j}(f(U_i)(t_i)) \\ &= \rho^{\mathcal{G}}_{U_i,U_i\cap U_j}(\rho^{\mathcal{G}}_{U,U_i}(s - s')) \\ &= \rho^{\mathcal{G}}_{U,U_i\cap U_j}(s - s'). \end{aligned}$$

Similarly,

$$f(U_i \cap U_j)(\rho^{\mathcal{F}}_{U_j,U_i\cap U_j}(t_i)) = \rho^{\mathcal{G}}_{U,U_i\cap U_j}(s - s').$$

Since the right-hand sides of these equations are the same, we have

$$f(U_i \cap U_j)(\rho^{\mathcal{F}}_{U_i,U_i\cap U_j}(t_i)) = f(U_i \cap U_j)(\rho^{\mathcal{F}}_{U_j,U_i\cap U_j}(t_i)).$$

Since f is a monomorphism, by Proposition 4.1.4 we have $\ker f = 0$. It follows that $f(U_i \cap U_j)$ is injective. Hence,

$$\rho^{\mathcal{F}}_{U_i,U_i\cap U_j}(t_i) = \rho^{\mathcal{F}}_{U_j,U_i\cap U_j}(t_i).$$

Since $\mathcal{F}$ satisfies the gluing condition (G), there exists $t \in \mathcal{F}(U)$ such that

$$\rho^{\mathcal{F}}_{U,U_i}(t) = t_i \tag{4.17}$$

for $i \in I$. Let $i \in I$. Then by (4.16) and (4.17)

$$\begin{aligned} \rho^{\mathcal{G}}_{U,U_i}(s - s') &= f(U_i)(t_i) \\ &= f(U_i)(\rho^{\mathcal{F}}_{U,U_i}(t)) \\ &= \rho^{\mathcal{G}}_{U,U_i}(f(U)(t)). \end{aligned}$$

Here, the last equality follows because

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f(U)} & \mathcal{G}(U) \\ \downarrow{\scriptstyle \rho^{\mathcal{F}}_{U,U_i}} & & \downarrow{\scriptstyle \rho^{\mathcal{G}}_{U,U_i}} \\ \mathcal{F}(U_i) & \xrightarrow{f(U_i)} & \mathcal{G}(U_i) \end{array}$$

commutes. We have proven that

$$\rho^{\mathcal{G}}_{U,U_i}(s - s') = \rho^{\mathcal{G}}_{U,U_i}(f(U)(t))$$

for all $i \in I$. Since $\mathcal{G}$ satisfies the locality condition (L) we conclude that $s - s' = f(U)(t)$. This is the desired result. □

Proposition 4.5.7. *Let X be a topological space. Let $\mathcal{F}, \mathcal{G} \in \mathrm{SH}_X$ and $f \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$. Assume that f is a monomorphism. The canonical morphism*

$$\mathrm{pcok} f \xrightarrow{n = n_{\mathrm{pcok} f}} \mathrm{scok} f = \Gamma L(\mathrm{pcok} f)$$

is a monomorphism.

Proof. By Lemma 3.3.13, for every $x \in X$ the morphism n_x is an isomorphism and hence a monomorphism. By Lemma 4.5.6, the presheaf $\mathrm{pcok} f$ is a monopresheaf. Lemma 4.5.5 now implies that n is a monomorphism. □

Corollary 4.5.8. *Let X be a topological space. Let $\mathcal{F}, \mathcal{G} \in \mathrm{SH}_X$ and $f \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$.*

(a) If f is a monomorphism, then f is a kernel for the morphism $p_s\colon \mathcal{G} \to \mathrm{scok} f$.

(b) If f is an epimorphism, then f is a cokernel for the morphism $i\colon \ker f \to \mathcal{F}$.

Proof. (a) Assume that f is a monomorphism. By Proposition 4.4.4, the composition

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{p_s} \mathrm{scok} f$$

is zero. Let $\mathcal{H} \in \mathrm{SH}_X$ and $g \in \mathrm{Mor}(\mathcal{H}, \mathcal{G})$ be such that the composition

$$\mathcal{H} \xrightarrow{g} \mathcal{G} \xrightarrow{p_s} \mathrm{scok} f$$

is zero. Since p_s is the composition

$$\mathcal{G} \xrightarrow{p} \mathrm{pcok} f \xrightarrow{n} \mathrm{scok} f = \Gamma L(\mathrm{pcok} f)$$

we have $n \circ p \circ g = 0$. By Proposition 4.5.7, the morphism n is a monomorphism. Hence, $p \circ g = 0$. By Proposition 4.5.2 there exists a unique morphism $t \in \mathrm{Mor}(\mathcal{H}, \mathcal{F})$ such that

$$\begin{array}{ccc} \mathcal{H} & & \\ \downarrow t & \searrow g & \\ \mathcal{F} & \xrightarrow{f} & \mathcal{G} \end{array}$$

commutes. This proves that f is a kernel for p_s.

(b) This follows from Proposition 4.5.4. □

Corollary 4.5.9. *Let X be a topological space. The categories PS_X and SH_X satisfy axiom (AB4) of the definition of an abelian category.*

Proof. This follows from Proposition 4.5.2, Proposition 4.5.4, and Corollary 4.5.8. □

Theorem 4.5.10. *Let X be a topological space. The categories PS_X and SH_X are abelian.*

Proof. This follows from Section 4.2, Corollary 4.3.3, Corollary 4.4.5, and Corollary 4.5.9. □

Corollary 4.5.11. *Let X be a topological space. Let $\mathcal{F}, \mathcal{G} \in \mathrm{SH}_X$ and $f \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$. Then*

(a) f is a monomorphism if and only if $\ker f(U) = 0$ for all open subsets U of X.

(b) f is an epimorphism if and only if $\operatorname{im} f(U) = \mathcal{G}(U)$ for all open subsets U of X.

Proof. We have already verified that SH_X satisfies (AB1) and (AB3) (see Section 4.2 and Corollary 4.4.5).

(a). By Proposition 4.1.4, f is a monomorphism if and only if $\ker f = 0$. The definition of $\ker f$ implies that $\ker f = 0$ if and only if $\ker f(U) = 0$ for all open subsets of U.

(b). By Proposition 4.1.4, f is an epimorphism if and only if $\operatorname{scok} f = 0$. Assume that $\operatorname{scok} f = 0$. By Proposition 4.5.7, the morphism $n\colon \operatorname{pcok} f \to \operatorname{scok} f$ is a monomorphism. Since $\operatorname{scok} f = 0$, by Proposition 4.1.3 we have $\operatorname{pcok} f = 0$. This implies that $\operatorname{im} f(U) = \mathcal{G}(U)$ for all open subsets U of X. Finally, assume that $\operatorname{im} f(U) = \mathcal{G}(U)$ for all open subsets U of X. Then $\operatorname{pcok} f = 0$. Since $\operatorname{scok} f = \Gamma L(\operatorname{pcok} f)$, we obtain $\operatorname{scok} f = 0$. □

4.6 Exact sequences

Let X be a topological space. Let $\mathcal{F}, \mathcal{G} \in \mathrm{PS}_X$, and let $f \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$. Since PS_X is an abelian category, the morphism f admits an image, which will denote by $\operatorname{pim} f$ and refer to as the **presheaf image** of f. Thus,

$$\operatorname{pim} f = \ker\left(\mathcal{G} \xrightarrow{\ p\ } \operatorname{pcok} f \right). \tag{4.18}$$

Assume further that $\mathcal{F}$ and $\mathcal{G}$ are sheaves, i.e., $\mathcal{F}, \mathcal{G} \in \mathrm{SH}_X$. Since the category SH_X is also abelian, the morphism f admits an image, which will denote by $\operatorname{sim} f$ and refer to as the **sheaf image** of f. We have

$$\operatorname{sim} f = \ker\left(\mathcal{G} \xrightarrow{\ p_s\ } \operatorname{scok} f \right). \tag{4.19}$$

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