Theta Series

Brooks Roberts

University of Idaho

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Contents

List of Tables

Chapter 1

Background

1.1 Dirichlet characters

Let N be a positive integer. A **Dirichlet character** modulo N is a homomorphism

$$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times.$$

If N is a positive integer and χ is a Dirichlet character modulo N, then we associate to χ a function

$$\mathbb{Z} \longrightarrow \mathbb{C},$$

also denoted by χ, by the formula

$$\chi(a) = \begin{cases} \chi(a + N\mathbb{Z}) & \text{if } (a, N) = 1, \\ 0 & \text{if } (a, N) > 1 \end{cases}$$

for $a \in \mathbb{Z}$. We refer to this function as the **extension** of χ to $\mathbb{Z}$. It is easy to verify that the following properties hold for the extension of χ to $\mathbb{Z}$:

1. $\chi(1) = 1$;
2. if $a_1, a_2 \in \mathbb{Z}$, then $\chi(a_1 a_2) = \chi(a_1)\chi(a_2)$;
3. if $a \in \mathbb{Z}$ and $(a, N) > 1$, then $\chi(a) = 0$;
4. if $a_1, a_2 \in \mathbb{Z}$ and $a_1 \equiv a_2 \pmod{N}$, then $\chi(a_1) = \chi(a_2)$.

Let N be a positive integer, and let χ be a Dirichlet character modulo N. We have $\chi(a)^{\phi(N)} = 1$ for $a \in \mathbb{Z}$ with $(a, N) = 1$; in particular, $\chi(a)$ is a $\phi(N)$-th root of unity. Here, $\phi(N)$ is the number of integers a such that $(a, N) = 1$ and $1 \leq a \leq N$.

If $N = 1$, then there exists exactly one Dirichlet character χ modulo N; the extension of χ to $\mathbb{Z}$ satisfies $\chi(a) = 1$ for all $a \in \mathbb{Z}$.

Let N be a positive integer. The Dirichlet character η modulo N that sends every element of $(\mathbb{Z}/N\mathbb{Z})^\times$ to 1 is called the **principal character** modulo N. The extension of η to $\mathbb{Z}$ is given by

$$\eta(a) = \begin{cases} 1 \text{ if } (a, N) = 1, \\ 0 \text{ if } (a, N) > 1 \end{cases}$$

for $a \in \mathbb{Z}$.

Let $f : \mathbb{Z} \to \mathbb{C}$ be a function, let N be a positive integer, and let χ be a Dirichlet character modulo N. We say that f **corresponds** to χ if f is the extension of χ, i.e., $f(a) = \chi(a)$ for all $a \in \mathbb{Z}$.

Let $f : \mathbb{Z} \to \mathbb{C}$, and assume that there exists a positive integer N and a Dirichlet character χ modulo N such that f corresponds to χ. Assume $N > 1$. Then there exist infinitely many positive integers N' and Dirichlet characters χ' modulo N' such that f corresponds to χ'. For example, let N' be any positive integer such that $N|N'$ and N' has the same prime divisors as N. Let χ' be the Dirichlet character modulo N' that is the composition

$$(\mathbb{Z}/N'\mathbb{Z})^\times \longrightarrow (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times,$$

where the first map is the natural surjective homomorphism. The extension of χ' to $\mathbb{Z}$ is the same as the extension of χ to $\mathbb{Z}$, namely f. Thus, f also corresponds to χ'.

Lemma 1.1.1. *Let $f : \mathbb{Z} \to \mathbb{C}$ be a function and let N be a positive integer. Assume that f satisifes the following conditions:*

1. *$f(1) \neq 0$;*
2. *if $a_1, a_2 \in \mathbb{Z}$, then $f(a_1 a_2) = f(a_1) f(a_2)$;*
3. *if $a \in \mathbb{Z}$ and $(a, N) > 1$, then $f(a) = 0$;*
4. *if $a \in \mathbb{Z}$, then $f(a + N) = f(a)$.*

There exists a unique Dirichlet character χ modulo N such that f corrsponds to χ.

Proof. Assume that f satisfies 1, 2, 3, and 4. Since $1 = 1 \cdot 1$, we have $f(1) = f(1)f(1)$, so that $f(1) = 1$. Next, we claim that $f(a_1) = f(a_2)$ for $a_1, a_2 \in \mathbb{Z}$ with $a_1 \equiv a_2 \pmod{N}$, or equivalently, if $a \in \mathbb{Z}$ and $x \in \mathbb{Z}$ then $f(a + xN) = f(a)$. Let $a \in \mathbb{Z}$ and $x \in \mathbb{Z}$. Write $x = \epsilon z$, where $\epsilon \in \{1, -1\}$ and z is positive. Then

$$\begin{aligned} f(a + xN) &= \chi(\epsilon(\epsilon a + zN)) \\ &= f(\epsilon)\chi(\epsilon a + zN) \\ &= f(\epsilon)\chi(\epsilon a + \underbrace{N + \cdots + N}_{z}) \end{aligned}$$

$$\begin{aligned} &= f(\epsilon)\chi(\epsilon a) \\ &= f(a). \end{aligned}$$

Now let $a \in Z$ with $(a, N) = 1$; we assert that $f(a) \neq 0$. Since $(a, N) = 1$, there exists $b \in \mathbb{Z}$ such that $ab = 1 + kN$ for some $k \in \mathbb{Z}$. We have $1 = f(1) = f(1 + kN) = f(ab) = f(a)f(b)$. It follows that $f(a) \neq 0$. We now define a function $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ by $\chi(a + N\mathbb{Z}) = f(a)$ for $a \in \mathbb{Z}$ with $(a, N) = 1$. By what we have already proven, α is a well-defined function. It is also clear that χ is a homomorphism. Finally, it is evident that the extension of χ to $\mathbb{Z}$ is f, so that f corresponds to χ. The uniqueness assertion is clear. □

Let p be an odd prime. For $m \in \mathbb{Z}$ define the **Legendre symbol** by

$$\left(\frac{m}{p}\right) = \begin{cases} 0 & \text{if } p \text{ divides } m, \\ -1 & \text{if } (m,p) = 1 \text{ and } x^2 \equiv m \pmod{p} \text{ has no solution } x \in \mathbb{Z}, \\ 1 & \text{if } (m,p) = 1 \text{ and } x^2 \equiv m \pmod{p} \text{ has a solution } x \in \mathbb{Z}. \end{cases}$$

The function $\left(\frac{\cdot}{p}\right) : \mathbb{Z} \to \mathbb{C}$ satisfies the conditions of Lemma 1.1.1 with $N = p$. We will also denote the Dirichlet character modulo p to which $\left(\frac{\cdot}{p}\right)$ corresponds by $\left(\frac{\cdot}{p}\right)$. We note that $\left(\frac{\cdot}{p}\right)$ is **real valued**, i.e., takes values in $\{-1, 0, 1\}$.

Let β be a Dirichlet character modulo M. We can construct other Dirichlet characters from β by forgetting information, as follows. Let N be a positive multiple of M. Since M divides N, there is a natural surjective homomorphism

$$(\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow (\mathbb{Z}/M\mathbb{Z})^\times,$$

and we can form the composition χ

$$(\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow (\mathbb{Z}/M\mathbb{Z})^\times \xrightarrow{\beta} \mathbb{C}^\times.$$

Then χ is a Dirichlet character modulo N, and we say that χ is **induced** from the Dirichlet character β modulo M. If N is a positive integer and χ is a Dirichlet character modulo N, and χ is not induced from any Dirichlet character β modulo M for a proper divisor M of N, then we say that χ is **primitive**.

Let N be a positive integer, and let χ be a Dirichlet character. Consider the set of positive integers N_1 such that $N_1 | N$ and

$$\chi(a) = 1$$

for $a \in \mathbb{Z}$ such that $(a, N) = 1$ and $a \equiv 1 \pmod{N_1}$. This set is non-empty since it contains N; we refer to the smallest such N_1 as the **conductor** of χ and denote it by $f(\chi)$.

Lemma 1.1.2. *Let N be positive integer, and let χ be a Dirichlet character modulo N. Let N_1 be a positive integer such that $N_1 | N$ and $\chi(a) = 1$ for $a \in \mathbb{Z}$ such that $(a, N) = 1$ and $a \equiv 1 \pmod{N_1}$. Then $f(\chi) | N_1$.*

Proof. We may assume that $N > 1$. Let $M = \gcd(f(\chi), N_1)$. We will prove that $\chi(a) = 1$ for $a \in \mathbb{Z}$ such that $(a, N) = 1$ and $a \equiv 1 \pmod{M}$; by the minimality of $f(\chi)$ this will imply that $M = f(\chi)$, so that $f(\chi)|N_1$. Let

$$N = p_1^{e_1} \cdots p_t^{e_t}$$

be the prime factorization of $r(\chi)$ into positive powers $e_1, \ldots, e_t$ of the distinct primes $p_1, \ldots, p_t$. Also, write

$$f(\chi) = p_1^{\ell_1} \cdots p_t^{\ell_t}, \qquad N_1 = p_1^{k_1} \cdots p_t^{k_t}.$$

By definition,

$$M = p_1^{\min(\ell_1, k_1)} \cdots p_t^{\min(\ell_t, k_t)}.$$

Let $a \in \mathbb{Z}$ be such that $(a, N) = 1$ and $a \equiv 1 \pmod{M}$. By the Chinese remainder theorem, there exists an integer b such that

$$b \equiv \begin{cases} 1 \pmod{p_i^{\ell_i}} \text{ if } \ell_i \geq k_i, \\ a \pmod{p_i^{k_i}} \text{ if } \ell_i < k_i \end{cases}$$

for $i \in \{1, \ldots, t\}$, and $(b, r(\chi)) = 1$. Let c be an integer such that $(c, N) = 1$ and $a \equiv bc \pmod{N}$. Evidently, $b \equiv 1 \pmod{p_i^{\ell_i}}$ and $c \equiv 1 \pmod{p_i^{k_i}}$ for $i \in \{1, \ldots, t\}$, so that $b \equiv 1 \pmod{f(\chi)}$ and $c \equiv 1 \pmod{N_1}$. It follows that $\chi(a) = \chi(bc) = \chi(b)\chi(c) = 1$. □

Lemma 1.1.3. *Let N be a positive integer, and let χ be a Dirichlet character modulo N. Then χ is primitive if and only if $f(\chi) = N$.*

Proof. Assume that χ is primitive. By Lemma 1.1.2 $f(\chi)$ is a divisor of N. By the definition of $f(\chi)$, the character χ is trivial on the kernel of the natural map

$$(\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow (\mathbb{Z}/f(\chi)\mathbb{Z})^\times.$$

This implies that χ factors through this map. Since χ is primitive, $f(\chi)$ is not a proper divisor of N, so that $f(\chi) = N$. The converse statement has a similar proof. □

Evidently, the conductor of $\left(\frac{\cdot}{p}\right)$ is also p, so that $\left(\frac{\cdot}{p}\right)$ is primitive.

Lemma 1.1.4. *Let N_1 and N_2 be positive integers, and let χ_1 and χ_2 be Dirichlet characters modulo N_1 and N_2, respectively. Let N be the least common multiple of N_1 and N_2. The function $f : \mathbb{Z} \to \mathbb{C}$ defined by $f(a) = \chi_1(a)\chi_2(a)$ for $a \in \mathbb{Z}$ corresponds to a unique Dirichlet χ character modulo N.*

Proof. It is clear that f satisfies properties 1, 2 and 4 of Lemma 1.1.1. To see that f satisfies property 3, assume that $a \in \mathbb{Z}$ and $(a, N) > 1$. We need to prove that $f(a) = 0$. There exists a prime p such that $p|a$ and $p|N$. Write $a = pb$ for some $b \in \mathbb{Z}$. Since $f(a) = f(p)f(b)$ it will suffice to prove that $f(p) = 0$, i.e, $\chi_1(p) = 0$ or $\chi_2(p) = 0$. Since $p|N$, we have $p|N_1$ or $p|N_2$. This implies that $\chi_1(p) = 0$ or $\chi_2(p) = 0$. □

Let the notation be as in Lemma 1.1.4. We refer to the Dirichlet character χ modulo N as the **product** of χ_1 and χ_2, and we write $\chi_1\chi_2$ for χ.

Lemma 1.1.5. *Let N_1 and N_2 be positive integers such that $(N_1, N_2) = 1$, and let χ_1 and χ_2 be Dirichlet characters modulo N_1 and modulo N_2, respectively. Let $\chi = \chi_1\chi_2$, the product of χ_1 and χ_2; this is a Dirichlet character modulo $N = N_1N_2$. The conductor of χ is $f(\chi) = f(\chi_1)f(\chi_2)$. Moreover, χ is primitive if and only if χ_1 and χ_2 are primitive.*

Proof. By Lemma 1.1.2 we have $f(\chi_1)|N_1$ and $f(\chi_2)|N_2$. Since $N = N_1N_2$, we obtain $f(\chi_1)f(\chi_2)|N$. Assume that $a \in \mathbb{Z}$ is such that $(a, N) = 1$ and $a \equiv 1 \pmod{f(\chi_1)f(\chi_2)}$. Then $(a, N_1) = (a, N_2) = 1$, $a \equiv 1 \pmod{f(\chi_1)}$, and $a \equiv 1 \pmod{f(\chi_2)}$. Therefore, $\chi_1(a) = \chi_2(a) = 1$, so that $\chi(a) = \chi_1(a)\chi_2(a) = 1$. By Lemma 1.1.2 it follows that we have $f(\chi)|f(\chi_1)f(\chi_2)$. Write $f(\chi) = M_1M_2$ where M_1 and M_2 are relatively prime positive integers such that $M_1|f(\chi_1)$ and $M_2|f(\chi_2)$. We need to prove that $M_1 = f(\chi_1)$ and $M_2 = f(\chi_2)$. Let $a \in \mathbb{Z}$ be such that $(a, N_1) = 1$ and $a \equiv 1 \pmod{M_1}$. By the Chinese remainder theorem, there exists an integer b such that $b \equiv a \pmod{M_1}$, $b \equiv 1 \pmod{f(\chi_2)}$, and $(b, N) = 1$. Evidently, $b \equiv 1 \pmod{f(\chi)}$. Hence, $1 = \chi(b) = \chi_1(b)\chi_2(b) = \chi_1(a)$. By the minimality of $f(\chi_1)$ we must now have $M_1 = f(\chi_1)$. Similarly, $M_2 = f(\chi_2)$. The final assertion of the lemma is straightforward. □

Lemma 1.1.6. *Let p be an odd prime. The Legendre symbol $\left(\frac{\cdot}{p}\right)$ is the only real valued primitive Dirichlet character modulo p. If e is a positive integer with $e > 1$, then there exist no real valued primitive Dirichlet characters modulo p^e.*

Proof. We have already remarked that $\left(\frac{\cdot}{p}\right)$ is a real valued primitive Dirichlet character modulo p. To prove the remaining assertions, let e be a positive integer, and assume that χ is a real valued primitive Dirichlet character modulo p^e; we will prove that $\chi = \left(\frac{\cdot}{p}\right)$ if $e = 1$ and obtain a contradiction if $e > 1$. Consider $(\mathbb{Z}/p^e\mathbb{Z})^\times$. It is known that this group is cyclic; let $x \in Z$ be such that $(x, p) = 1$ and $x + p^e\mathbb{Z}$ is a generator of $(\mathbb{Z}/p^e\mathbb{Z})^\times$. Since χ has conductor p^e, and since $x + p^e\mathbb{Z}$ is a generator of $(\mathbb{Z}/p^e\mathbb{Z})^\times$, we must have $\chi(x) \neq 1$. Since χ is real valued we obtain $\chi(x) = -1$. On the other hand, the function $\left(\frac{\cdot}{p}\right)$ is also a real valued Dirichlet character modulo p^e such that $\left(\frac{a}{p}\right) = -1$ for some $a \in \mathbb{Z}$; since $x + p^e\mathbb{Z}$ is a generator of $(\mathbb{Z}/p^e\mathbb{Z})^\times$, this implies that $\left(\frac{x}{p}\right) = -1$, so that $\chi(x) = \left(\frac{x}{p}\right)$. Since $x + p^e\mathbb{Z}$ is a generator of $(\mathbb{Z}/p^e\mathbb{Z})^\times$ and $\chi(x) = -1 = \chi'(x)$ we must have $\chi = \left(\frac{\cdot}{p}\right)$. We see that if $e = 1$, then the Legendre symbol $\left(\frac{\cdot}{p}\right)$ is the only real valued primitive Dirichlet character modulo p. Assume that $e > 1$. It is easy to verify that the conductor of the Dirichlet character $\left(\frac{\cdot}{p}\right)$ modulo p^e is p; this is a contradiction since by Lemma 1.1.3 the conductor of χ is p^e. □

Lemma 1.1.7. *There are no primitive characters modulo 2. There exists a unique primitive Dirichlet character ε_4 modulo $4 = 2^2$ which is defined by*

$$\varepsilon_4(1) = 1,$$
$$\varepsilon_4(3) = -1.$$

There exist two primitive Dirichlet characters ε_8' and ε_8'' modulo $8 = 2^3$ which are defined by

$$\begin{aligned} \varepsilon_8'(1) &= 1, \\ \varepsilon_8'(3) &= -1, \\ \varepsilon_8'(5) &= -1, \\ \varepsilon_8'(7) &= 1, \end{aligned} \qquad \text{and} \qquad \begin{aligned} \varepsilon_8''(1) &= 1, \\ \varepsilon_8''(3) &= 1, \\ \varepsilon_8''(5) &= -1, \\ \varepsilon_8''(7) &= -1. \end{aligned}$$

There exist no real valued primitive Dirichlet characters modulo p^e for $e \geq 4$.

Proof. We have $(\mathbb{Z}/2\mathbb{Z})^\times = \{1\}$. It follows that the unique Dirichlet character modulo 2 has conductor conductor 1; by Lemma 1.1.3, this character is not primitive.

We have $(\mathbb{Z}/4\mathbb{Z})^\times = \{1, 3\}$. Hence, there exist two Dirichlet characters modulo 4. The non-principal Dirichlet character modulo 4 is ε_4; since $\varepsilon_4(1+2) = -1$, it follows that the conductor of ε_4 is 4. By Lemma 1.1.3, ε_4 is primitive.

We have

$$(\mathbb{Z}/8\mathbb{Z})^\times = \{1, 3, 5, 7\} = \{1, 3\} \times \{1, 5\}$$

The non-principal Dirichlet characters modulo 8 are $\varepsilon_8', \varepsilon_8''$ and $\varepsilon_8'\varepsilon_8''$. Since $\varepsilon_8'(1+4) = \varepsilon_8''(1+4) = -1$ we have $f(\varepsilon_8') = f(\varepsilon_8'') = 8$. Since $(\varepsilon_8'\varepsilon_8'')(1+4) = 1$ we have $f(\varepsilon_8'\varepsilon_8'') = 4$. Hence, by Lemma 1.1.3, ε_8' and ε_8'' are primitive, and $\varepsilon_8'\varepsilon_8''$ is not primitive.

Finally, assume that $e \geq 4$ and let χ be a real valued Dirichlet character modulo p^e. Let $n \in \mathbb{Z}$ be such that $(n, 2) = 1$ and $n \equiv 1 \pmod 8$. It is known that there exists $a \in \mathbb{Z}$ such that $n \equiv a^2 \pmod{p^e}$. We obtain $\chi(n) = \chi(a^2) = \chi(a)^2 = 1$ because $\chi(a) = \pm 1$ (since χ is real valued). By Lemma 1.1.2 the conductor $f(\chi)$ divides 8. By Lemma 1.1.3, χ is not primitive. □

1.2 Fundamental discriminants

Let D be a non-zero integer. We say that D is a **fundamental discriminant** if

$D \equiv 1 \pmod 4$ and D is square-free,

or

$D \equiv 0 \pmod 4$, $D/4$ is square-free, and $D/4 \equiv 2$ or $3 \pmod 4$.

We say that D is a **prime fundamental discriminant** if

$D = -8$ or $D = -4$ or $D = 8$,

or

$D = -p$ for p a prime such that $p \equiv 3 \pmod 4$,

or

$D = p$ for p a prime such that $p \equiv 1 \pmod 4$.

it is clear that if D is a prime fundamental discriminant, then D is a fundamental discriminant.

Lemma 1.2.1. *Let D_1 and D_2 be relatively prime fundamental discriminants. Then D_1D_2 is a fundamental discriminant.*

Proof. The proof is straightforward. Note that since D_1 and D_2 are relatively prime, at most one of D_1 and D_2 is divisible by 4. □

Lemma 1.2.2. *Let D be a fundamental discriminant such that $D \neq 1$. There exist prime fundamental discriminants $D_1, \ldots, D_k$ such that*

$$D = D_1 \cdots D_k$$

and $D_1, \ldots, D_k$ are pairwise relatively prime.

Proof. Assume that $D < 0$ and $D \equiv 1 \pmod 4$. We may write $D = -p_1 \cdots p_t$ for a non-empty collection of distinct primes $p_1, \ldots, p_t$. Since D is odd, each of $p_1, \ldots, p_t$ is odd and is hence congruent to 1 or 3 mod 4. Let r be the number of the primes p from $p_1, \ldots, p_t$ such that $p \equiv 3 \pmod 4$. We have

$$\begin{aligned} 1 &\equiv D \pmod 4 \\ &\equiv (-1)3^r \pmod 4 \\ 1 &\equiv (-1)^{r+1} \pmod 4. \end{aligned}$$

It follows that r is odd. Hence,

$$\begin{aligned} D &= -\prod_{p\in\{p_1,\ldots,p_t\}} p \\ &= -\Big(\prod_{\substack{p\in\{p_1,\ldots,p_t\},\\ p\equiv 1 \pmod 4}} p\Big) \times \Big(\prod_{\substack{p\in\{p_1,\ldots,p_t\},\\ p\equiv 3 \pmod 4}} p\Big) \\ D &= \Big(\prod_{\substack{p\in\{p_1,\ldots,p_t\},\\ p\equiv 1 \pmod 4}} p\Big) \times \Big(\prod_{\substack{p\in\{p_1,\ldots,p_t\},\\ p\equiv 3 \pmod 4}} -p\Big). \end{aligned}$$

Each of the factors in the last equation is a prime fundamental discriminant, which proves the lemma in this case.

Assume that $D < 0$ and $D \equiv 0 \pmod 4$. If $D = -4$, then D is a prime fundamental discriminant. Assume that $D \neq -4$. We may write $D = -4p_1 \cdots p_t$ for a non-empty collection of distinct primes $p_1, \ldots, p_t$ such that $-p_1 \cdots p_t \equiv 2$ or $3 \pmod 4$. Assume first that $-p_1 \cdots p_t \equiv 2 \pmod 4$. Then exactly one of $p_1, \ldots, p_t$ is even, say $p_1 = 2$. Let r be the number of the primes p from $p_2, \ldots, p_t$ such that $p \equiv 3 \pmod 4$. We have

$$D = -4 \prod_{p\in\{p_1,\ldots,p_t\}} p$$

$$
\begin{aligned}
D &= -8 \prod_{p \in \{p_2,\ldots,p_t\}} p \\
&= -8\Big(\prod_{\substack{p \in \{p_2,\ldots,p_t\},\\ p \equiv 1 \pmod 4}} p\Big) \times \Big(\prod_{\substack{p \in \{p_2,\ldots,p_t\},\\ p \equiv 3 \pmod 4}} p\Big) \\
D &= ((-1)^{r+1} 8) \times \Big(\prod_{\substack{p \in \{p_2,\ldots,p_t\},\\ p \equiv 1 \pmod 4}} p\Big) \times \Big(\prod_{\substack{p \in \{p_2,\ldots,p_t\},\\ p \equiv 3 \pmod 4}} -p\Big).
\end{aligned}
$$

Each of the factors in the last equation is a prime fundamental discriminant, which proves the lemma in this case. Now assume that $-p_1 \cdots p_t \equiv 3 \pmod 4$. Then $p_1, \ldots, p_t$ are all odd. Let r be the number of the primes p from $p_1, \ldots, p_t$ such that $p \equiv 3 \pmod 4$. We have

$$
\begin{aligned}
3 &\equiv -p_1 \cdots p_t \pmod 4 \\
-1 &\equiv (-1)3^r \pmod 4 \\
1 &\equiv (-1)^r \pmod 4.
\end{aligned}
$$

It follows that r is even. Hence,

$$
\begin{aligned}
D &= -4 \prod_{p \in \{p_1,\ldots,p_t\}} p \\
&= -4\Big(\prod_{\substack{p \in \{p_1,\ldots,p_t\},\\ p \equiv 1 \pmod 4}} p\Big) \times \Big(\prod_{\substack{p \in \{p_1,\ldots,p_t\},\\ p \equiv 3 \pmod 4}} p\Big) \\
D &= (-4) \times \Big(\prod_{\substack{p \in \{p_1,\ldots,p_t\},\\ p \equiv 1 \pmod 4}} p\Big) \times \Big(\prod_{\substack{p \in \{p_1,\ldots,p_t\},\\ p \equiv 3 \pmod 4}} -p\Big).
\end{aligned}
$$

Each of the factors in the last equation is a prime fundamental discriminant, proving the lemma in this case.

Assume that $D > 0$ and $D \equiv 1 \pmod 4$. Since $D \neq 1$ by assumption, we have $D = p_1 \cdots p_t$ for a non-empty collection of distinct odd primes $p_1, \ldots, p_t$. Let r be the number of the primes p from $p_1, \ldots, p_t$ such that $p \equiv 3 \pmod 4$. We have

$$
\begin{aligned}
1 &\equiv D \pmod 4 \\
&\equiv 3^r \pmod 4 \\
1 &\equiv (-1)^r \pmod 4.
\end{aligned}
$$

We see that r is even. Therefore,

$$
\begin{aligned}
D &= \prod_{p \in \{p_1,\ldots,p_t\}} p \\
&= \Big(\prod_{\substack{p \in \{p_1,\ldots,p_t\},\\ p \equiv 1 \pmod 4}} p\Big) \times \Big(\prod_{\substack{p \in \{p_1,\ldots,p_t\},\\ p \equiv 3 \pmod 4}} p\Big)
\end{aligned}
$$

$$D = \Big(\prod_{\substack{p\in\{p_1,\ldots,p_t\},\\ p\equiv 1 \pmod 4}} p\Big) \times \Big(\prod_{\substack{p\in\{p_1,\ldots,p_t\},\\ p\equiv 3 \pmod 4}} -p\Big).$$

Each of the factors in the last equation is a prime fundamental discriminant, proving the lemma in this case.

Finally, assume that $D > 0$ and $D \equiv 0 \pmod 4$. We may write $D = 4p_1 \cdots p_t$ for a non-empty collection of distinct primes $p_1, \ldots, p_t$ such that $p_1 \cdots p_t \equiv 2$ or $3 \pmod 4$. Assume first that $p_1 \cdots p_t \equiv 2 \pmod 4$. Then exactly one of $p_1, \ldots, p_t$ is even, say $p_1 = 2$. Let r be the number of the primes p from $p_2, \ldots, p_t$ such that $p \equiv 3 \pmod 4$. We have

$$\begin{aligned}
D &= 4 \prod_{p\in\{p_1,\ldots,p_t\}} p \\
D &= 8 \prod_{p\in\{p_2,\ldots,p_t\}} p \\
&= 8\Big(\prod_{\substack{p\in\{p_2,\ldots,p_t\},\\ p\equiv 1 \pmod 4}} p\Big) \times \Big(\prod_{\substack{p\in\{p_2,\ldots,p_t\},\\ p\equiv 3 \pmod 4}} p\Big) \\
D &= ((-1)^r 8) \times \Big(\prod_{\substack{p\in\{p_2,\ldots,p_t\},\\ p\equiv 1 \pmod 4}} p\Big) \times \Big(\prod_{\substack{p\in\{p_2,\ldots,p_t\},\\ p\equiv 3 \pmod 4}} -p\Big).
\end{aligned}$$

Each of the factors in the last equation is a prime fundamental discriminant, which proves the lemma in this case. Now assume that $p_1 \cdots p_t \equiv 3 \pmod 4$. Then $p_1, \ldots, p_t$ are all odd. Let r be the number of the primes p from $p_1, \ldots, p_t$ such that $p \equiv 3 \pmod 4$. We have

$$\begin{aligned}
3 &\equiv p_1 \cdots p_t \pmod 4 \\
-1 &\equiv 3^r \pmod 4 \\
-1 &\equiv (-1)^r \pmod 4 \\
1 &\equiv (-1)^{r+1} \pmod 4
\end{aligned}$$

It follows that r is odd. Hence,

$$\begin{aligned}
D &= 4 \prod_{p\in\{p_1,\ldots,p_t\}} p \\
&= 4\Big(\prod_{\substack{p\in\{p_1,\ldots,p_t\},\\ p\equiv 1 \pmod 4}} p\Big) \times \Big(\prod_{\substack{p\in\{p_1,\ldots,p_t\},\\ p\equiv 3 \pmod 4}} p\Big) \\
D &= (-4) \times \Big(\prod_{\substack{p\in\{p_1,\ldots,p_t\},\\ p\equiv 1 \pmod 4}} p\Big) \times \Big(\prod_{\substack{p\in\{p_1,\ldots,p_t\},\\ p\equiv 3 \pmod 4}} -p\Big).
\end{aligned}$$

Each of the factors in the last equation is a prime fundamental discriminant, proving the lemma in this case. □

The fundamental discriminants between -1 and -100 are listed in Table A.1 and the fundamental discriminants between 1 and 100 are listed in Table A.2.

Let D be a fundamental discriminant. We define a function

$$\chi_D : \mathbb{Z} \longrightarrow \mathbb{C}$$

in the following way. First, let p be a prime. We define

$$\chi_D(p) = \begin{cases} \left(\dfrac{D}{p}\right) & \text{if } p \text{ is odd,} \\ 1 & \text{if } p = 2 \text{ and } D \equiv 1 \pmod 8, \\ -1 & \text{if } p = 2 \text{ and } D \equiv 5 \pmod 8, \\ 0 & \text{if } p = 2 \text{ and } D \equiv 0 \pmod 4. \end{cases}$$

Note that since D is a fundamental discriminant, we have $D \not\equiv 3 \pmod 8$ and $D \not\equiv 7 \pmod 8$. If n is a positive integer, and

$$n = p_1^{e_1} \cdots p_t^{e_t}$$

is the prime factorization of n, where $p_1, \ldots, p_t$ are primes, then we define

$$\chi_D(n) = \chi_D(p_1)^{e_1} \cdots \chi_D(p_t)^{e_t}. \tag{1.1}$$

This defines $\chi_D(n)$ for all positive integers n. We also define

$$\chi_D(-n) = \chi_D(-1)\chi_D(n)$$

for all positive integers n, where we define

$$\chi_D(-1) = \begin{cases} 1 & \text{if } D > 0, \\ -1 & \text{if } D < 0. \end{cases}$$

Finally, we define

$$\chi_D(0) = \begin{cases} 0 & \text{if } D \neq 1, \\ 1 & \text{if } D = 1. \end{cases}$$

We note that if $D = 1$, then $\chi_1(a) = 1$ for $a \in \mathbb{Z}$. Thus, χ_1 is the unique Dirichlet character modulo 1 (which has conductor 1, and is thus primitive).

Lemma 1.2.3. *Let D_1 and D_2 be relatively prime fundamental discriminants. Then*

$$\chi_{D_1D_2}(a) = \chi_{D_1}(a)\chi_{D_2}(a)$$

for all $a \in \mathbb{Z}$.

Proof. It is easy to verify that $\chi_{D_1D_2}(p) = \chi_{D_1}(p)\chi_{D_2}(p)$ for all primes p, $\chi_{D_1D_2}(-1) = \chi_{D_1}(-1)\chi_{D_2}(-1)$, and $\chi_{D_1D_2}(0) = 0 = \chi_{D_1}(0)\chi_{D_2}(0)$. The assertion of the lemma now follows from the definitions of χ_D, χ_{D_1} and χ_{D_2} on composite numbers. □

Lemma 1.2.4. *Let D be a fundamental discriminant. The function χ_D corresponds to a primitive Dirichlet character modulo $|D|$.*

Proof. By Lemma 1.2.2 we can write

$$D = D_1 \cdots D_k$$

where $D_1, \dots, D_k$ are prime fundamental discriminants and $D_1, \dots, D_k$ are pairwise relatively prime. By Lemma 1.2.3,

$$\chi_D(a) = \chi_{D_1}(a) \cdots \chi_{D_k}(a)$$

for $a \in \mathbb{Z}$. Lemma 1.1.4 and Lemma 1.1.5 now imply that we may assume that D is a prime fundamental discriminant. For the following argument we recall the Dirichlet characters ε_4, ε_8' and ε_8'' from Lemma 1.1.7.

Assume first that $D = -8$ so that $|D| = 8$. Let p be an odd prime. Then

$$\begin{aligned}
\chi_{-8}(p) &= \left(\frac{-8}{p}\right) \\
&= \left(\frac{-2}{p}\right)^3 \\
&= \left(\frac{-2}{p}\right) \\
&= \left(\frac{-1}{p}\right)\left(\frac{2}{p}\right) \\
&= (-1)^{\frac{p-1}{2}}(-1)^{\frac{p^2-1}{8}} \\
&= \begin{cases} 1 & \text{if } p \equiv 1,3 \pmod 8 \\ -1 & \text{if } p \equiv 5,7 \pmod 8 \end{cases}.
\end{aligned}$$

Also,

$$\chi_{-8}(2) = 0.$$

We see that $\chi_{-8}(p) = \varepsilon_8''(p)$ for all primes p. Also, $\chi_{-8}(-1) = -1 = \varepsilon_8''(-1)$ and $\chi_{-8}(0) = 0 = \varepsilon_8''(0)$. Since χ_{-8} and ε_8'' are multiplicative, it follows that

$$\chi_{-8} = \varepsilon_8'',$$

so that χ_{-8} corresponds to a primitive Dirichlet character mod $|-8| = 8$.

Assume that $D = -4$ so that $|D| = 4$. Let p be an odd prime. Then

$$\begin{aligned}
\chi_{-4}(p) &= \left(\frac{-4}{p}\right) \\
&= \left(\frac{-1}{p}\right)\left(\frac{2}{p}\right)^2 \\
&= \left(\frac{-1}{p}\right)
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{\frac{p-1}{2}} \\
&= \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4, \\ -1 & \text{if } p \equiv 3 \pmod 4. \end{cases}
\end{aligned}
$$

Also, $\chi_{-4}(2) = 0$, $\chi_{-4}(-1) = -1$, and $\chi_{-4}(0) = 0$. We see that $\chi_{-4}(p) = \varepsilon_4(p)$ for all primes p. Also, $\chi_{-4}(-1) - 1 = \varepsilon_4(-1)$ and $\chi_{-4}(0) = 0 = \varepsilon_4(0)$. Since χ_{-4} and ε_4 are multiplicative, it follows that

$$\chi_{-4} = \varepsilon_4,$$

so that χ_{-4} corresponds to a primitive Dirichlet character mod $|-4| = 4$.

Assume that $D = 8$. Let p be an odd prime. Then

$$
\begin{aligned}
\chi_8(p) &= \left(\frac{8}{p}\right) \\
&= \left(\frac{2}{p}\right)^3 \\
&= \left(\frac{2}{p}\right) \\
&= (-1)^{\frac{p^2-1}{8}} \\
&= \begin{cases} 1 & \text{if } p \equiv 1, 7 \pmod 8, \\ -1 & \text{if } p \equiv 3, 5 \pmod 8. \end{cases}
\end{aligned}
$$

Also, $\chi_8(2) = 0$, $\chi_8(-1) = 1$, and $\chi_8(0) = 0$. We see that $\chi_8(p) = \varepsilon_8'(p)$ for all primes p. Also, $\chi_8(-1) = 1 = \varepsilon_8'(-1)$ and $\chi_8(0) = 0 = \varepsilon_8'(0)$. Since χ_8 and ε_8' are multiplicative, it follows that

$$\chi_8 = \varepsilon_8',$$

so that χ_8 corresponds to a primitive Dirichlet character mod $|8| = 8$.

Assume that $D = -q$ for a prime q such that $q \equiv 3 \pmod 4$. Let p be an odd prime. Then

$$
\begin{aligned}
\chi_D(p) &= \left(\frac{-q}{p}\right) \\
&= \left(\frac{-1}{p}\right)\left(\frac{q}{p}\right) \\
&= (-1)^{\frac{p-1}{2}}(-1)^{\frac{p-1}{2}\frac{q-1}{2}}\left(\frac{p}{q}\right) \\
&= (-1)^{\frac{p-1}{2}}((-1)^{\frac{q-1}{2}})^{\frac{p-1}{2}}\left(\frac{p}{q}\right) \\
&= (-1)^{\frac{p-1}{2}}(-1)^{\frac{p-1}{2}}\left(\frac{p}{q}\right) \\
&= (-1)^{p-1}\left(\frac{p}{q}\right)
\end{aligned}
$$

$$= \left(\frac{p}{q}\right).$$

Also,

$$\begin{aligned}\chi_D(2) &= \begin{cases} 1 & \text{if } -q \equiv 1 \pmod 8, \\ -1 & \text{if } -q \equiv 5 \pmod 8 \end{cases} \\ &= \begin{cases} 1 & \text{if } q \equiv 7 \pmod 8, \\ -1 & \text{if } q \equiv 3 \pmod 8 \end{cases} \\ &= (-1)^{\frac{q^2-1}{8}} \\ &= \left(\frac{2}{q}\right),\end{aligned}$$

and

$$\begin{aligned}\chi_D(-1) &= -1 \\ &= (-1)^{\frac{q-1}{2}} \\ &= \left(\frac{-1}{q}\right).\end{aligned}$$

Since $\left(\frac{\cdot}{q}\right)$ and χ_D are multiplicative, it follows that $\left(\frac{a}{q}\right) = \chi_D(a)$ for all $a \in \mathbb{Z}$. Since $\left(\frac{\cdot}{q}\right)$ is a primitive Dirichlet character modulo q, it follows that χ_D corresponds to a primitive Dirichlet character modulo $q = |-q| = |D|$.

Assume that $D = q$ for a prime q such that $q \equiv 1 \pmod 4$. Let p be an odd prime. Then

$$\begin{aligned}\chi_D(p) &= \left(\frac{q}{p}\right) \\ &= (-1)^{\frac{p-1}{2}\frac{q-1}{2}}\left(\frac{p}{q}\right) \\ &= (-1)^{\frac{p-1}{2}\cdot 2}\left(\frac{p}{q}\right) \\ &= \left(\frac{p}{q}\right).\end{aligned}$$

Also,

$$\begin{aligned}\chi_D(2) &= \begin{cases} 1 & \text{if } q \equiv 1 \pmod 8, \\ -1 & \text{if } q \equiv 5 \pmod 8 \end{cases} \\ &= (-1)^{\frac{q^2-1}{8}} \\ &= \left(\frac{2}{q}\right),\end{aligned}$$

and

$$\chi_D(-1) = 1$$

$$= (-1)^{\frac{q-1}{2}}$$
$$= \left(\frac{-1}{q}\right).$$

Since $\left(\frac{\cdot}{q}\right)$ and χ_D are multiplicative, it follows that $\left(\frac{a}{q}\right) = \chi_D(a)$ for all $a \in \mathbb{Z}$. Since $\left(\frac{\cdot}{q}\right)$ is a primitive Dirichlet character modulo q, it follows that χ_D corresponds to a primitive Dirichlet character modulo $q = |q| = |D|$. ☐

From the proof of Lemma 1.2.4 we see that if D is a prime fundamental discriminant with $D > 1$, then

$$\chi_D = \begin{cases} \varepsilon_8'' & \text{if } D = -8, \\ \varepsilon_4 & \text{if } D = -4, \\ \varepsilon_8' & \text{if } D = 8, \\ \left(\frac{\cdot}{p}\right) & \text{if } D = -p \text{ is a prime with } p \equiv 3 \pmod 4, \\ \left(\frac{\cdot}{p}\right) & \text{if } D = p \text{ is a prime with } p \equiv 1 \pmod 4. \end{cases} \tag{1.2}$$

Proposition 1.2.5. *Let N be a positive integer, and let χ be a Dirichlet character modulo N. Assume that χ is primitive and real valued (i.e., $\chi(a) \in \{0, 1, -1\}$ for $a \in \mathbb{Z}$). Then there exists a fundamental discriminant D such that $|D| = N$ and $\chi = \chi_D$.*

Proof. If $N = 1$, then χ is the unique Dirichlet character modulo 1; we have already remarked that χ_1 is also the unique Dirichlet character modulo 1. Assume that $N > 1$. Let
$$N = p_1^{e_1} \cdots p_t^{e_t}$$
be the prime factorization of N into positive powers $e_1, \ldots, e_t$ of the distinct primes $p_1, \ldots, p_t$. We have

$$(\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\sim} (\mathbb{Z}/p_1^{e_1}\mathbb{Z})^\times \times \cdots \times (\mathbb{Z}/p_t^{e_t}\mathbb{Z})^\times$$

where the isomorphism sends $x + N\mathbb{Z}$ to $(x + p_1^{e_1}\mathbb{Z}, \ldots, x + p_t^{e_t}\mathbb{Z})$ for $x \in \mathbb{Z}$. Let $i \in \{1, \ldots, t\}$. Let χ_i be the character of $(\mathbb{Z}/p_i^{e_i}\mathbb{Z})^\times$ which is the composition

$$(\mathbb{Z}/p_i^{e_i}\mathbb{Z})^\times \hookrightarrow (\mathbb{Z}/p_1^{e_1}\mathbb{Z})^\times \times \cdots \times (\mathbb{Z}/p_t^{e_t}\mathbb{Z})^\times \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times,$$

where the first map is inclusion. We have

$$\chi(a) = \chi_1(a) \cdots \chi_t(a)$$

for $a \in \mathbb{Z}$. By Lemma 1.1.5 the Dirichlet characters $\chi_1, \ldots, \chi_t$ are primitive. Also, it is clear that $\chi_1, \ldots, \chi_t$ are all real valued. Again let $i \in \{1, \ldots, t\}$.

Assume first that p_i is odd. Since χ_i is primitive, Lemma 1.1.6 implies that $e_i = 1$, and that $\chi_i = \left(\frac{\cdot}{p_i}\right)$, the Legendre symbol. By (1.2), $\chi_i = \chi_{D_i}$ where

$$D_i = \begin{cases} p_i & \text{if } p_i \equiv 1 \pmod 4, \\ -p_i & \text{if } p_i \equiv 3 \pmod 4. \end{cases}$$

Evidently, $|-D_i| = p_i^{e_i}$. Next, assume that $p_i = 2$. By Lemma 1.1.7 we see that $e_i = 2$ or $e_i = 3$ with $\chi_i = \varepsilon_4$ if $e_i = 2$, and $\chi_i = \varepsilon_8'$ or ε_8'' if $e_i = 3$. By (1.2), $\chi_i = \chi_{D_i}$, where

$$D_i = \begin{cases} -4 & \text{if } e_i = 2, \\ 8 & \text{if } e_i = 3 \text{ and } \chi_i = \varepsilon_8', \\ -8 & \text{if } e_i = 3 \text{ and } \chi_i = \varepsilon_8''. \end{cases}$$

Clearly, $|-D_i| = p_i^{e_i}$. To now complete the proof, we note that by Lemma 1.2.1 the product $D = D_1 \cdots D_t$ is a fundamental discriminant, and by Lemma 1.2.3 we have $\chi_D = \chi_{D_1} \cdots \chi_{D_t}$. Since $\chi_{D_1} \cdots \chi_{D_t} = \chi_1 \cdots \chi_t = \chi$ and $|D| = N$, this completes the proof. □

1.3 Quadratic extensions

Proposition 1.3.1. *The map*

$$\{\text{quadratic extensions } K \text{ of } \mathbb{Q}\} \xrightarrow{\sim} \{\text{fundamental discriminants } D,\, D \neq 1\}$$

that sends K to its discriminant $\mathrm{disc}(K)$ is a well-defined bijection. Let K be a quadratic extension of $\mathbb{Q}$, and let p be a prime. Then the prime factorization of the ideal (p) generated by p in $\mathfrak{o}_K$ is given as follows:

$$(p) = \begin{cases} \mathfrak{p}^2 & (p \text{ is ramified}) & \text{if } \chi_D(p) = 0, \\ \mathfrak{p} \cdot \mathfrak{p}' & (p \text{ splits}) & \text{if } \chi_D(p) = 1, \\ \mathfrak{p} & (p \text{ is inert}) & \text{if } \chi_D(p) = -1. \end{cases}$$

Here, in the first and third case, $\mathfrak{p}$ is the unique prime ideal of $\mathfrak{o}_K$ lying over (p), and in the second case, $\mathfrak{p}$ and $\mathfrak{p}'$ are the two distinct prime ideals of $\mathfrak{o}_K$ lying over (p).

Proof. Let K be a quadratic extension of $\mathbb{Q}$. There exists a square-free integer d such that $K = \mathbb{Q}(\sqrt{d})$. Let $\mathfrak{o}_K$ be the ring of integers of K. It is known that

$$\mathfrak{o}_K = \begin{cases} \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \sqrt{d} & \text{if } d \equiv 2, 3 \pmod 4, \\ \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \dfrac{1 + \sqrt{d}}{2} & \text{if } d \equiv 1 \pmod 4. \end{cases}$$

By the definition of disc(K), we have

$$\operatorname{disc}(K) = \begin{cases} \det(\begin{bmatrix} 1 & \sqrt{d} \\ 1 & -\sqrt{d} \end{bmatrix})^2 & \text{if } d \equiv 2,3 \pmod 4, \\ \det(\begin{bmatrix} 1 & \dfrac{1+\sqrt{d}}{2} \\ 1 & \dfrac{1-\sqrt{d}}{2} \end{bmatrix})^2 & \text{if } d \equiv 1 \pmod 4 \end{cases}$$
$$= \begin{cases} 4d & \text{if } d \equiv 2,3 \pmod 4, \\ d & \text{if } d \equiv 1 \pmod 4. \end{cases}$$

It follows that the map is well-defined, and a bijection. For a proof of the remaining assertion see Satz 1 on page 100 of [29], or Theorem 25 on page 74 of [16]. □

Lemma 1.3.2. *Let D be a fundamental discriminant such that $D \neq 1$. Let $K = \mathbb{Q}(\sqrt{D})$, so that K is a quadratic extension of $\mathbb{Q}$. Then* disc$(K) = D$.

Proof. Assume that $D \equiv 1 \pmod 4$. Then D is square-free. From the proof of Proposition 1.3.1 we have disc$(K) = D$. Assume that $D \equiv 0 \pmod 4$. Then $K = \mathbb{Q}(\sqrt{D/4})$, with $D/4$ square-free and $D/4 \equiv 2,3 \pmod 4$. From the proof of Proposition 1.3.1 we again obtain disc$(K) = 4 \cdot (D/4) = D$. □

1.4 Kronecker Symbol

Let Δ be a non-zero integer such that $\Delta \equiv 0, 1$ or $2 \pmod 4$. We define a function,

$$\left(\frac{\Delta}{\cdot}\right) : \mathbb{Z} \longrightarrow \mathbb{C}$$

called the **Kronecker symbol**, in the following way. First, let p be a prime. We define

$$\left(\frac{\Delta}{p}\right) = \begin{cases} \left(\frac{\Delta}{p}\right) \quad \text{(Legendre symbol)} & \text{if } p \text{ is odd}, \\ 0 & \text{if } p = 2 \text{ and } \Delta \text{ is even}, \\ 1 & \text{if } p = 2 \text{ and } \Delta \equiv 1 \pmod 8, \\ -1 & \text{if } p = 2 \text{ and } \Delta \equiv 5 \pmod 8). \end{cases}$$

Note that, since by assumption $\Delta \equiv 0, 1$ or $2 \pmod 4$, the cases $\Delta \equiv 3 \pmod 8$ and $\Delta \equiv 7 \pmod 8$ do not occur. We see that if p is a prime, then $p|\Delta$ if and only if $\left(\frac{\Delta}{p}\right) = 0$. If n is a positive integer, and

$$n = p_1^{e_1} \cdots p_t^{e_t}$$

is the prime factorization of n, where $p_1, \ldots, p_t$ are primes, then we define

$$\left(\frac{\Delta}{n}\right) = \left(\frac{\Delta}{p_1}\right)^{e_1} \cdots \left(\frac{\Delta}{p_t}\right)^{e_t}.$$

This defines $\left(\frac{\Delta}{n}\right)$ for all positive integers n. We also define

$$\left(\frac{\Delta}{-n}\right) = \left(\frac{\Delta}{-1}\right)\left(\frac{\Delta}{n}\right)$$

for all positive integers n, where we define

$$\left(\frac{\Delta}{-1}\right) = \begin{cases} 1 & \text{if } \Delta > 0, \\ -1 & \text{if } \Delta < 0. \end{cases}$$

Finally, we define

$$\left(\frac{\Delta}{0}\right) = \begin{cases} 0 & \text{if } \Delta \neq 1, \\ 1 & \text{if } \Delta = 1. \end{cases}$$

We note that if $\Delta = 1$, then $\left(\frac{\Delta}{a}\right) = \left(\frac{1}{a}\right) = 1$ for $a \in \mathbb{Z}$. Thus, $\left(\frac{1}{\cdot}\right)$ is the unique Dirichlet character modulo 1. It is straightfoward to verify that

$$\left(\frac{\Delta}{ab}\right) = \left(\frac{\Delta}{a}\right)\left(\frac{\Delta}{b}\right)$$

for $a, b \in \mathbb{Z}$. Also, we note that $\left(\frac{\Delta}{a}\right) = 0$ if and only if $(a, \Delta) > 1$.

Lemma 1.4.1. *Let D be a non-zero integer such that $D \equiv 1 \pmod 4$ or $D \equiv 0 \pmod 4$. There exists a unique fundamental discriminant D_{fd} and a unique positive integer m such that*

$$D = m^2 D_{\mathrm{fd}}.$$

Proof. We first prove the existence of m and D_{fd}. We may write $D = 2^e a^2 b$, where e is a positive non-negative integer, a is a positive integer, and b is an odd square-free integer.

Assume that $e = 0$. Then $D \equiv 1 \pmod 4$. Since a is odd, $a^2 \equiv 1 \pmod 4$; therefore, $b \equiv 1 \pmod 4$. It follows that $D = m^2 D_{\mathrm{fd}}$ with $m = a$ and $D_{\mathrm{fd}} = b$ a fundamental discriminant.

The case $e = 1$ is impossible because $D \equiv 1 \pmod 4$ or $D \equiv 0 \pmod 4$.

Assume that $e \geq 2$ and e is odd. Write $e = 2k + 1$ for a positive integer k. Then $D = m^2 D_{\mathrm{fd}}$ with $m = 2^{k-1} a$ and $D_{\mathrm{fd}} = 8b$ a fundamental discriminant.

Assume that $e \geq 2$ and e is even. Write $e = 2k$ for a positive integer k. If $b \equiv 1 \pmod 4$, then $D = m^2 D_{\mathrm{fd}}$ with $m = 2^k a$ and $D_{\mathrm{fd}} = b$ a fundamental discriminant. If $b \equiv 3 \pmod 4$, then $D = m^2 D_{\mathrm{fd}}$ with $m = 2^{k-1} a$ and $D_{\mathrm{fd}} = 4b$ a fundamental discriminant. This completes the proof the existence of m and D_{fd}.

To prove the uniqueness assertion, assume that m and m' are positive integers and D_{fd} and D'_{fd} are fundamental discriminants such that $D = m^2 D_{\mathrm{fd}} = (m')^2 D'_{\mathrm{fd}}$. Assume first that $D_{\mathrm{fd}} = 1$. Then $m^2 = (m')^2 D'_{\mathrm{fd}}$. This implies that D'_{fd} is a square; hence, $D'_{\mathrm{fd}} = 1$. Therefore, $m^2 = (m')^2$, implying that $m = m'$. Now assume that $D_{\mathrm{fd}} \neq 1$. Then also $D'_{\mathrm{fd}} \neq 1$, and D is not a square. Set $K = \mathbb{Q}(\sqrt{D})$. We have $K = \mathbb{Q}(\sqrt{D_{\mathrm{fd}}}) = \mathbb{Q}(\sqrt{D'_{\mathrm{fd}}})$. By Lemma 1.3.2, $\mathrm{disc}(K) = D_{\mathrm{fd}}$ and $\mathrm{disc}(K) = D'_{\mathrm{fd}}$, so that $D_{\mathrm{fd}} = D'_{\mathrm{fd}}$. Since this holds we also conclude that $m = m'$. □

Proposition 1.4.2. *Let Δ be a non-zero integer with $\Delta \equiv 0, 1$ or $2 \pmod 4$. Define*

$$D = \begin{cases} \Delta & \text{if } \Delta \equiv 0 \text{ or } 1 \pmod 4, \\ 4\Delta & \text{if } \Delta \equiv 2 \pmod 4. \end{cases}$$

Write $D = m^2 D_{\mathrm{fd}}$ with m a positive integer, and D_{fd} a fundamental discriminant, as in Lemma 1.4.1. The Kronecker symbol $\left(\frac{\Delta}{\cdot}\right)$ is a Dirichlet character modulo $|D|$, and is the Dirichlet character induced by the mod $|D_{\mathrm{fd}}|$ Dirichlet character $\chi_{D_{\mathrm{fd}}}$.

Proof. Let α be the Dirichlet character modulo $|D|$ induced by $\chi_{D_{\mathrm{fd}}}$. Thus, α is the composition

$$(\mathbb{Z}/|D|\mathbb{Z})^\times \longrightarrow (\mathbb{Z}/|D_{\mathrm{fd}}|\mathbb{Z})^\times \xrightarrow{\chi_{\mathrm{fd}}} \mathbb{C}^\times,$$

extended to $\mathbb{Z}$. Since α and $\left(\frac{\Delta}{\cdot}\right)$ are multiplicative, to prove that $\alpha = \left(\frac{\Delta}{\cdot}\right)$ it will suffice to prove that these two functions agree on all primes, on -1, and on 0. Let p be a prime.

Assume first that p is odd. If $p|D$, then also $p|\Delta$, so that $\alpha(p)$ and $\left(\frac{\Delta}{\cdot}\right)$ evaluated at p are both 0. Assume that $(p, D) = 1$. Then also $(p, \Delta) = 1$. Then

$$\begin{aligned}
\left(\frac{\Delta}{\cdot}\right) \text{ evaluated at } p &= \left(\frac{\Delta}{p}\right) \text{ (Legendre symbol)} \\
&= \begin{cases} \left(\frac{\Delta}{p}\right) & \text{if } \Delta \equiv 0 \text{ or } 1 \pmod 4, \\ \left(\frac{2}{p}\right)^2 \left(\frac{\Delta}{p}\right) & \text{if } \Delta \equiv 2 \pmod 4, \end{cases} \\
&= \begin{cases} \left(\frac{\Delta}{p}\right) & \text{if } \Delta \equiv 0 \text{ or } 1 \pmod 4, \\ \left(\frac{4\Delta}{p}\right) & \text{if } \Delta \equiv 2 \pmod 4, \end{cases} \\
&= \left(\frac{D}{p}\right) \\
&= \left(\frac{m^2 D_{\mathrm{fd}}}{p}\right) \\
&= \left(\frac{D_{\mathrm{fd}}}{p}\right) \\
&= \chi_{D_{\mathrm{fd}}}(p) \\
&= \alpha(p).
\end{aligned}$$

Assume next that $p = 2$. If $2|D$, then also $2|\Delta$, so that $\alpha(2)$ and $\left(\frac{\Delta}{\cdot}\right)$ evaluated at 2 are both 0. Assume that $(2, D) = 1$, so that D is odd. Then $D = \Delta$, and in fact $D \equiv 1 \pmod 4$. This implies that $\Delta \equiv 1$ or $7 \pmod 8$. Also, as $D \equiv 1 \pmod 4$, and $D = m^2 D_{\mathrm{fd}}$, we must have $D_{\mathrm{fd}} \equiv D \pmod 8$ (since $a^2 \equiv 1 \pmod 8$ for any odd integer a). Therefore,

$$\begin{aligned}
\left(\frac{\Delta}{\cdot}\right) \text{ evaluated at } 2 &= \begin{cases} 1 & \text{if } D \equiv 1 \pmod 8, \\ -1 & \text{if } D \equiv 5 \pmod 8, \end{cases} \\
&= \begin{cases} 1 & \text{if } D_{\mathrm{fd}} \equiv 1 \pmod 8\ , \\ -1 & \text{if } D_{\mathrm{fd}} \equiv 5 \pmod 8\ , \end{cases} \\
&= \chi_{D_{\mathrm{fd}}}(2) \\
&= \alpha(2).
\end{aligned}$$

To finish the proof we note that

$$\begin{aligned}
\left(\frac{\Delta}{\cdot}\right) \text{ evaluated at } -1 &= \operatorname{sign}(\Delta) \\
&= \operatorname{sign}(D) \\
&= \operatorname{sign}(D_{\mathrm{fd}}) \\
&= \chi_{D_{\mathrm{fd}}}(-1) \\
&= \alpha(-1).
\end{aligned}$$

Since $\Delta = 1$ if and only if $D_{\mathrm{fd}} = 1$, the evaluation of $\left(\frac{D}{\cdot}\right)$ at 0 is $\chi_{D_{\mathrm{fd}}}(0) = \alpha(0)$. □

Lemma 1.4.3. *Assume that Δ_1 and Δ_2 are non-zero integers that satisfy the congruences $\Delta_1 \equiv 0, 1$ or $2 \pmod 4$ and $\Delta_2 \equiv 0, 1$ or $2 \pmod 4$. Then we have $\Delta_1\Delta_2 \equiv 0, 1$ or $2 \pmod 4$, and*

$$\left(\frac{\Delta_1}{a}\right)\left(\frac{\Delta_2}{a}\right) = \left(\frac{\Delta_1\Delta_2}{a}\right) \tag{1.3}$$

for all integers a.

Proof. It is easy to verify that $\Delta_1\Delta_2 \equiv 0, 1$ or $2 \pmod 4$, and that if $\Delta_1 = 1$ or $\Delta_2 = 1$, then (1.3) holds. Assume that $\Delta_1 \neq 1$ and $\Delta_2 \neq 1$. Since $\left(\frac{\Delta_1}{\cdot}\right)$, $\left(\frac{\Delta_2}{\cdot}\right)$, and $\left(\frac{\Delta_1\Delta_2}{\cdot}\right)$ are multiplicative, it suffices to verify (1.3) for all odd primes, for 2, -1 and 0. These cases follows from the definitions. □

1.5 Quadratic forms

Let f be a positive integer, which will be fixed for the remainder of this section. In this section we regard the elements of $\mathbb{Z}^f$ as column vectors.

Let $A = (a_{i,j}) \in \mathrm{M}(f, \mathbb{Z})$ be a integral symmetric matrix, so that $a_{i,j} = a_{j,i}$ for $i, j \in \{1, \ldots, f\}$. We say that A is **even** if each diagonal entry $a_{i,i}$ for $i \in \{1, \ldots, f\}$ is an even integer.

Lemma 1.5.1. *Let $A \in \mathrm{M}(f, \mathbb{Z})$, and assume that A is symmetric. Then A is even if and only if ${}^{\mathrm{t}}yAy$ is an even integer for all $y \in \mathbb{Z}^f$.*

Proof. Let $y \in \mathbb{Z}^f$, with ${}^{\mathrm{t}}y = (y_1, \dots, y_f)$. Then

$$\begin{aligned} {}^{\mathrm{t}}yAy &= \sum_{i,j=1}^{n} a_{i,j} y_i y_j \\ &= \sum_{i=1}^{f} a_{i,i} y_i^2 + \sum_{1 \le i < j \le f} 2 a_{i,j} y_i y_j. \end{aligned}$$

It is clear that if A is even, then ${}^{\mathrm{t}}yAy$ is an even integer for all $y \in \mathbb{Z}^f$. Assume that ${}^{\mathrm{t}}yAy$ is an even integer for all $y \in \mathbb{Z}^f$. Let $i \in \{1, \dots, f\}$. Let $y_i \in \mathbb{Z}^f$ be defined by

$${}^{\mathrm{t}}y_i = (0, \dots, 0, 1, 0, \dots, 0)$$

where 1 occurs in the i-th position. Then ${}^{\mathrm{t}}y_i A y_i = a_{i,i}$. This is even, as required. □

Suppose that A is an even integral symmetric matrix. To A we associate the polynomial

$$Q(x_1, \dots, x_f) = \frac{1}{2} \sum_{i,j=1}^{f} a_{i,j} x_i x_j,$$

and we refer to $Q(x_1, \dots, x_f)$ as the **quadratic form** determined by A. Evidently,

$$Q(x) = \frac{1}{2}\, {}^{\mathrm{t}}xAx$$

with

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_f \end{bmatrix}.$$

Since $a_{i,i}$ is even for $i \in \{1, \dots, f\}$, the quadratic form $Q(x)$ can also be written as

$$Q(x_1, \dots, x_f) = \sum_{1 \le i \le j \le f} b_{i,j} x_i x_j$$

where

$$b_{i,j} = \begin{cases} a_{i,j} & \text{for } 1 \le i < j \le f, \\ a_{i,i}/2 & \text{for } 1 \le i \le f \end{cases}$$

is an integer. We denote the **determinant** of A by

$$D = D(A) = \det(A).$$

and the **discriminant** of A by

$$\Delta = \Delta(A) = (-1)^k \det(A), \qquad f = \begin{cases} 2k & \text{if } f \text{ is even}, \\ 2k+1 & \text{if } f \text{ is odd}. \end{cases}$$

For example, suppose that $f = 2$. Then every even integral symmetric matrix has the form

$$A = \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}$$

where a, b and c are integers, and the associated quadratic form is:

$$Q(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2.$$

For this example we have

$$D = 4ac - b^2, \qquad \Delta = b^2 - 4ac.$$

Lemma 1.5.2. *Let $A \in \mathrm{M}(f, \mathbb{Z})$ be an even integral symmetric matrix, and let $D = D(A)$ and $\Delta = \Delta(A)$. If f is odd, then $\Delta \equiv D \equiv 0 \pmod 2$. If f is even, then $\Delta \equiv 0, 1 \pmod 4$.*

Proof. Let $A = (a_{i,j})$ with $a_{i,j} \in \mathbb{Z}$ for $i, j \in \{1, \ldots, f\}$. By assumption, $a_{i,j} = a_{j,i}$ and $a_{i,i}$ is even for $i, j \in \{1, \ldots, f\}$.

Assume that f is odd. For $\sigma \in S_f$ (the permutation group of $\{1, \ldots, f\}$, let

$$t(\sigma) = \mathrm{sign}(\sigma) a_{1,\sigma(1)} \cdots a_{f,\sigma(f)} = \mathrm{sign}(\sigma) \prod_{i \in \{1,\ldots,n\}} a_{i,\sigma(i)}$$

We have

$$\begin{aligned} \det(A) &= \sum_{\sigma \in S_f} t(\sigma) \\ &= \sum_{\sigma \in X} t(\sigma) + \sum_{\sigma \in S_f - X} t(\sigma). \end{aligned}$$

Here, X is the subset of $\sigma \in S_f$ such that $\sigma \neq \sigma^{-1}$. Let $\sigma \in S_f$. Then

$$\begin{aligned} t(\sigma^{-1}) &= \mathrm{sign}(\sigma^{-1}) \prod_{i \in \{1,\ldots f\}} a_{i,\sigma^{-1}(i)} \\ &= \mathrm{sign}(\sigma) \prod_{i \in \{1,\ldots f\}} a_{\sigma(i),\sigma^{-1}(\sigma(i))} \\ &= \mathrm{sign}(\sigma) \prod_{i \in \{1,\ldots f\}} a_{\sigma(i),i} \\ &= \mathrm{sign}(\sigma) \prod_{i \in \{1,\ldots f\}} a_{i,\sigma(i)} \end{aligned}$$

$$= t(\sigma).$$

Since the subset X is partitioned into two element subsets of the form $\{\sigma, \sigma^{-1}\}$ for $\sigma \in X$, and since $t(\sigma) = t(\sigma^{-1})$ for $\sigma \in S_f$, it follows that

$$\sum_{\sigma \in X} t(\sigma) \equiv 0 \pmod 2.$$

Let $\sigma \in S_f - X$, so that $\sigma^2 = 1$. Write $\sigma = \sigma_1 \cdots \sigma_t$, where $\sigma_1, \ldots, \sigma_t \in S_f$ are cycles and mutually disjoint. Since $\sigma^2 = 1$, each σ_i for $i \in \{1, \ldots, t\}$ is a two cycle. Since f is odd, there exists $i \in \{1, \ldots, f\}$ such that i does not occur in any of the two cycles $\sigma_1, \ldots, \sigma_t$. It follows that $\sigma(i) = i$. Now $a_{i,\sigma(i)} = a_{i,i}$; by hypothesis, this is an even integer. It follows that $t(\sigma)$ is also an even integer. Hence,

$$\sum_{\sigma \in S_f - X} t(\sigma) \equiv 0 \pmod 2,$$

and we conclude that $\Delta \equiv D \equiv 0 \pmod 2$.

Now assume that f is even, and write $f = 2k$. We will prove that $\Delta \equiv 0, 1 \pmod 4$ by induction on f. Assume that $f = 2$, so that

$$A = \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}$$

where a, b and c are integers. Then $\Delta = b^2 - 4ac \equiv 0, 1 \pmod 4$. Assume now that $f \geq 4$, and that $\Delta(A_1) \equiv 0, 1 \pmod 4$ for all $f_1 \times f_1$ even integral symmetric matrices A_1 with f_1 even and $f > f_1 \geq 2$. Clearly, if all the off-diagonal entries of A are even, then all the entries of A are even, and $\Delta(A) \equiv 0 \pmod 4$. Assume that some off-diagonal entry of A, say $a = a_{i,j}$ is odd with $1 \leq i < j \leq f$. Interchange the first and the i-th row of A, and then the first and the i-th column of A; the result is an even integral symmetric matrix A' with a in the $(1, j)$ position and $\det(A') = \det(A)$. Next, interchange the second and the j-th column of A', and then the second and the j-th row of A'; the result is an even integral symmetric matrix A'' with a in the $(1, 2)$-position and $\det(A'') = \det(A') = \det(A)$. It follows that we may assume that $(i, j) = (1, 2)$. We may write

$$A = \begin{bmatrix} A_1 & B \\ {}^{t}B & A_2 \end{bmatrix},$$

where A_2 is an $(f-2) \times (f-2)$ even integral symmetric matrix,

$$A_1 = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{1,2} & a_{2,2} \end{bmatrix},$$

and B is a $2 \times (f-2)$ matrix with integral entries. Let

$$\mathrm{adj}(A_1) = \begin{bmatrix} a_{2,2} & -a_{1,2} \\ -a_{1,2} & a_{1,1} \end{bmatrix},$$

so that

$$A_1 \cdot \operatorname{adj}(A_1) = \operatorname{adj}(A_1) \cdot A_1 = \det(A_1) \cdot 1_2.$$

Now

$$\begin{bmatrix} 1_2 & \\ -{}^{\mathrm{t}}B \cdot \operatorname{adj}(A_1) & \det(A_1) \cdot 1_{f-2} \end{bmatrix} \begin{bmatrix} A_1 & B \\ {}^{\mathrm{t}}B & A_2 \end{bmatrix} = \begin{bmatrix} A_1 & B \\ & -{}^{\mathrm{t}}B \cdot \operatorname{adj}(A_1) \cdot B + \det(A_1) A_2 \end{bmatrix}. \quad (1.4)$$

Consider the $(f-2) \times (f-2)$ matrix $-{}^{\mathrm{t}}B \cdot \operatorname{adj}(A_1) \cdot B$. This matrix clearly has integral entries. If $y \in \mathbb{Z}^{f-2}$, then $By \in \mathbb{Z}^{f-2}$ and

$${}^{\mathrm{t}}(y)(-{}^{\mathrm{t}}B \cdot \operatorname{adj}(A_1) \cdot B)y = -{}^{\mathrm{t}}(By) \cdot \operatorname{adj}(A_1) \cdot (By);$$

since $\operatorname{adj}(A_1)$ is even, by Lemma 1.5.1 this integer is even. Since the last displayed integer is even for all $y \in \mathbb{Z}^{f-2}$, we can apply Lemma 1.5.1 again to conclude that $-{}^{\mathrm{t}}B \cdot \operatorname{adj}(A_1) \cdot B$ is even. It follows that

$$A_3 = -{}^{\mathrm{t}}B \cdot \operatorname{adj}(A_1) \cdot B + \det(A_1) A_2$$

is an $(f-2) \times (f-2)$ even integral symmetric matrix. Taking determinants of both sides of (1.4), we obtain

$$\begin{aligned} \det(A_1)^{f-2} \cdot \det(A) &= \det(A_1) \cdot \det(A_3) \\ \det(A_1)^{f-2} \cdot (-1)^k \det(A) &= (-1) \det(A_1) \cdot (-1)^{k-1} \det(A_3) \\ \det(A_1)^{f-2} \cdot \Delta(A) &= \Delta(A_1) \cdot \Delta(A_3). \end{aligned}$$

By the induction hypothesis, $\Delta(A_1) \equiv 0, 1 \pmod 4$, and $\Delta(A_3) \equiv 0, 1 \pmod 4$. Hence,

$$\det(A_1)^{f-2} \cdot \Delta(A) \equiv 0, 1 \pmod 4.$$

By hypothesis, $a_{1,2}$ is odd; since $f-2$ is even, this implies that $\det(A_1)^{f-2} \equiv 1 \pmod 4$. We now conclude that $\Delta(A) \equiv 0, 1 \pmod 4$, as desired. □

Let $A \in \mathrm{M}(f, \mathbb{R})$. The **adjoint** of A is the $f \times f$ matrix $\operatorname{adj}(A)$ with entries

$$\operatorname{adj}(A)_{i,j} = (-1)^{i+j} \det\big(A(j|i)\big)$$

for $i, j \in \{1, \ldots, n\}$. Here, for $i, j \in \{1, \ldots, n\}$, $A(j|i)$ is the $(f-1) \times (f-1)$ matrix that is obtained from A by deleting the j-th row and the i-th column. For example, if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then

$$\operatorname{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

We have

$$\mathrm{adj}(A) \cdot A = A \cdot \mathrm{adj}(A) = \det(A) \cdot 1_f.$$

Thus,

$$\begin{aligned} A &= \det(A)\mathrm{adj}(A)^{-1}, \\ \mathrm{adj}(A) &= \det(A) \cdot A^{-1}, \\ A^{-1} &= \det(A)^{-1} \cdot \mathrm{adj}(A), \\ \mathrm{adj}(A)^{-1} &= \det(A)^{-1} \cdot A, \\ \det\big(\mathrm{adj}(A)\big) &= \det(A)^{f-1}. \end{aligned}$$

We let $\mathrm{Sym}(f,\mathbb{R})$ be the set of all symmetric elements of $\mathrm{M}(f,\mathbb{R})$. Let $A \in \mathrm{Sym}(f,\mathbb{R})$. We say that A is **positive-definite** if the following two conditions hold:

1. If $x \in \mathbb{R}^f$, then $Q(x) = \frac{1}{2}\,{}^{\mathrm{t}}xAx \geq 0$;
2. if $x \in \mathbb{R}^f$ and $Q(x) = \frac{1}{2}\,{}^{\mathrm{t}}xAx = 0$, then $x = 0$.

We will also write $A > 0$ to mean that A is positive-definite. We say that A is **positive semi-definite** if the first condition holds; we will write $A \geq 0$ to indicate that A is positive semi-definite. Since A is symmetric with real entries, there exists a matrix $T \in \mathrm{GL}(f,\mathbb{R})$ such that ${}^{\mathrm{t}}TT = T\,{}^{\mathrm{t}}T = 1$ (so that $T^{-1} = {}^{\mathrm{t}}T$) and

$$ {}^{\mathrm{t}}TAT = T^{-1}AT = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_f \end{bmatrix} \tag{1.5}$$

for some $\lambda_1, \dots, \lambda_f \in \mathbb{R}$ (see the corollary on p. 314 of [9]). The symmetric matrix A is positive-definite if and only if $\lambda_1, \dots, \lambda_f$ are all positive, and A is positive semi-definite if and only if $\lambda_1, \dots, \lambda_f$ are all non-negative. It follows that if A is positive-definite, then $\det(A) > 0$, and if A is positive semi-definite, then $\det(A) \geq 0$. Assume that A is positive semi-definite, and that T and $\lambda_1, \dots, \lambda_f$ are as in (1.5); in particular, $\lambda_1, \dots, \lambda_f$ are all non-negative real numbers. Let

$$B = T \begin{bmatrix} \sqrt{\lambda_1} & & & & \\ & \sqrt{\lambda_2} & & & \\ & & \sqrt{\lambda_3} & & \\ & & & \ddots & \\ & & & & \sqrt{\lambda_f} \end{bmatrix} T^{-1}. \tag{1.6}$$

The matrix B is evidently symmetric and positive semi-definite, and we have

$$A = {}^{\mathrm{t}}BB = BB = B^2. \tag{1.7}$$

Also, it is clear that if A is positive-definite, then so is B.

Lemma 1.5.3. *Assume f is even. Let $A \in \mathrm{M}(f,\mathbb{Z})$ be a positive-definite even integral symmetric matrix. The matrix* $\mathrm{adj}(A)$ *is a positive-definite even integral symmetric matrix.*

Proof. We have $\mathrm{adj}(A) = \det(A)\cdot A^{-1}$. Therefore, ${}^{\mathrm{t}}\mathrm{adj}(A) = \det(A)\cdot {}^{\mathrm{t}}(A^{-1}) = \det(A)\cdot({}^{\mathrm{t}}A)^{-1} = \det(A)\cdot A^{-1} = \mathrm{adj}(A)$, so that $\mathrm{adj}(A)$ is symmetric. To see that $\mathrm{adj}(A)$ is positive-definite, let $T \in \mathrm{GL}(f,\mathbb{R})$ and $\lambda_1,\dots,\lambda_f$ be positive real numbers such that (1.5) holds. Then

$$\begin{aligned}{}^{\mathrm{t}}({}^{\mathrm{t}}T)\mathrm{adj}(A)\,{}^{\mathrm{t}}T &= \det(A)\cdot TA^{-1}\,{}^{\mathrm{t}}T\\ &= \begin{bmatrix}\det(A)\lambda_1^{-1} & & & & \\ & \det(A)\lambda_2^{-1} & & & \\ & & \det(A)\lambda_3^{-1} & & \\ & & & \ddots & \\ & & & & \det(A)\lambda_f^{-1}\end{bmatrix}.\end{aligned}$$

This equality implies that $\mathrm{adj}(A)$ is positive-definite. It is clear that $\mathrm{adj}(A)$ has integral entries. To see that $\mathrm{adj}(A)$ is even, let $i \in \{1,\dots,f\}$. Then $\mathrm{adj}(A)_{i,i} = \det\big(A(i|i)\big)$. The matrix $A(i|i)$ is an $(f-1)\times(f-1)$ even integral symmetric matrix. Since $f-1$ is odd, by Lemma 1.5.2 we have $\det\big(A(i|i)\big) \equiv 0 \pmod 2$. Thus, $\mathrm{adj}(A)_{i,i}$ is even. □

Let $A \in \mathrm{M}(f,\mathbb{Z})$ be an even integral symmetric matrix with $\det(A)$ non-zero. The set of all integers N such that NA^{-1} is an even integral symmetric matrix is an ideal of $\mathbb{Z}$. We define the **level** of A, and its associated quadratic form, to be the unique positive generator $N(A)$ of this ideal. Evidently, the level $N(A)$ of A is smallest positive integer N such that NA^{-1} is an even integral symmetric matrix.

Proposition 1.5.4. *Assume f is even. Let $A \in \mathrm{M}(f,\mathbb{Z})$ be a positive-definite even integral symmetric matrix. Define*

$$G = \gcd\left(\left\{\begin{matrix}\frac{\mathrm{adj}(A)_{1,1}}{2} & \mathrm{adj}(A)_{1,2} & \mathrm{adj}(A)_{1,3} & \cdots & \mathrm{adj}(A)_{1,f}\\ \mathrm{adj}(A)_{1,2} & \frac{\mathrm{adj}(A)_{2,2}}{2} & \mathrm{adj}(A)_{2,3} & \cdots & \mathrm{adj}(A)_{2,f}\\ \mathrm{adj}(A)_{1,3} & \mathrm{adj}(A)_{2,3} & \frac{\mathrm{adj}(A)_{3,3}}{2} & \cdots & \mathrm{adj}(A)_{3,f}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \mathrm{adj}(A)_{1,f} & \mathrm{adj}(A)_{2,f} & \mathrm{adj}(A)_{3,f} & \cdots & \frac{\mathrm{adj}(A)_{f,f}}{2}\end{matrix}\right\}\right)$$

Then G divides $\det(A)$, *and the level of A is*

$$N = \frac{\det(A)}{G}.$$

The positive integers N and $\det(A)$ *have the same set of prime divisors.*

Proof. The integer G divides every entry of $\mathrm{adj}(A)$. Therefore, G^f divides $\det\big(\mathrm{adj}(A)\big)$. Since $\det\big(\mathrm{adj}(A)\big) = \det(A)^{f-1}$, G^f divides $\det(A)^{f-1}$. This implies that G divides $\det(A)$. Now by definition, G is the largest integer g such that

$$\frac{1}{g}\mathrm{adj}(A) \quad \text{is even.}$$

Since $\mathrm{adj}(A) = \det(A)A^{-1}$, we therefore have that

$$\frac{\det(A)}{G}A^{-1} \quad \text{is even.}$$

This implies that $\det(A)G^{-1}$ is in the ideal generated by the level N of A, i.e., N divides $\det(A)G^{-1}$; consequently,

$$GN \leq \det(A).$$

On the other hand, NA^{-1} is even. Using $A^{-1} = \det(A)^{-1}\mathrm{adj}(A)$, this is equivalent to

$$\frac{1}{\det(A)N^{-1}}\mathrm{adj}(A) \quad \text{is even.}$$

Since $\det(A)N^{-1}$ is a positive integer (we have already proven that N divides $\det(A)$), the definition of G implies that $G \geq \det(A)N^{-1}$, or equivalently,

$$GN \geq \det(A).$$

We now conclude that $GN = \det(A)$, as desired.

To see that N and $\det(A)$ have the same set of prime divisors, we first note that (since N divides $\det(A)$) every prime divisor of N is a prime divisor of $\det(A)$. Let p be a prime divisor of $\det(A)$. If p does not divide G, then p divides N (because $NG = \det(A)$). Assume that p divides G. Write $\det(A) = p^j d$ and $G = p^k g$ with k and j positive integers and d and g integers such that $(d,p) = (g,p) = 1$. From above, G^f divides $\det(A)^{f-1}$. This implies that $(f-1)j \geq fk$. Therefore,

$$j \geq \frac{f}{f-1}k > k.$$

This means that p divides $N = \det(A)/G$. □

Corollary 1.5.5. *Let f be an even positive integer, let $A \in \mathrm{M}(f,\mathbb{Z})$ be a positive-definite even integral symmetric matrix and let N be the level of A. Then $N = 1$ if and only if* $\det(A) = 1$.

Proof. By Proposition 1.5.4, N and $\det(A)$ have the same set of prime divisors. It follows that $N = 1$ if and only if $\det(A) = 1$. □

Corollary 1.5.6. *Let A be a 2×2 even integral symmetric matrix, so that*

$$A = \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}$$

where a, b and c are integers. Then A is positive-definite if and only if $\det(A) = 4ac - b^2 > 0$*,* $a > 0$*, and* $c > 0$*. Assume that A is positive-definite. The level of A is*

$$N = \frac{4ac - b^2}{\gcd(a, b, c)}.$$

Proof. Assume that A is positive-definite. We have already pointed out that $\det(A) > 0$. Now

$$Q(1,0) = \frac{1}{2}\,{}^{\mathrm{t}}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}2a & b\\ b & 2c\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = a,$$
$$Q(0,1) = \frac{1}{2}\,{}^{\mathrm{t}}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}2a & b\\ b & 2c\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = c.$$

Since A is positive-definite, these numbers are positive. Assume that $\det(A) = 4ac - b^2 > 0$, $a > 0$, and $c > 0$. For $x, y \in \mathbb{R}$ we have

$$\begin{aligned}Q(x,y) &= ax^2 + bxy + cy^2\\ &= \frac{1}{a}(ax + \frac{b}{2}y)^2 + \frac{4ac - b^2}{4a}y^2\\ &= \frac{1}{a}(ax + \frac{b}{2}y)^2 + \frac{\det(A)}{4a}y^2.\end{aligned}$$

Clearly, we have $Q(x,y) \geq 0$ for all $x, y \in \mathbb{R}$. Assume that $x, y \in \mathbb{R}$ are such that $Q(x,y) = 0$. Then since $\det(A) > 0$ and $a > 0$ we must have $ax + \frac{b}{2}y = 0$ and $y = 0$; hence also $x = 0$. It follows that A is positive-definite. The final assertion follows from

$$\mathrm{adj}(A) = \begin{bmatrix}2c & -b\\ -b & 2a\end{bmatrix}$$

and Proposition 1.5.4. □

Corollary 1.5.7. *Let f be an even positive integer, let $A \in \mathrm{M}(f, \mathbb{Z})$ be a positive-definite even integral symmetric matrix and let N be the level of A. Let c be a positive integer. Then the level of the positive-definite even integral symmetric matrix cA is cN.*

Proof. This follows from the formula for level from Proposition 1.5.4. □

Lemma 1.5.8. *Let f be an even positive integer, let $A \in \mathrm{M}(f, \mathbb{Z})$ be a positive-definite even integral symmetric matrix and let N be the level of A. Define the integral quadratic form $Q(x)$ by $Q(x) = \frac{1}{2}\,{}^{\mathrm{t}}xAx$. Let $h \in \mathbb{Z}^f$ be such that $Ah \equiv 0 \pmod{N}$. Then $Q(h) \equiv 0 \pmod{N}$. Also, if $n \in \mathbb{Z}^f$ is such that $n \equiv h \pmod{N}$, then $Q(n) \equiv Q(h) \pmod{N^2}$ and $Q(n) \equiv 0 \pmod{N}$.*

Proof. Since $Ah \equiv 0 \pmod{N}$, there exists $m \in \mathbb{Z}^f$ such that $Ah = Nm$. We have

$$Q(q) = \frac{1}{2}\,{}^{\mathrm{t}}hAh$$

$$= \frac{1}{2}\,{}^{\mathrm{t}}(Ah)A^{-1}(Ah)$$
$$= N \cdot \frac{1}{2}\,{}^{\mathrm{t}}m(NA^{-1})m.$$

By the definition of N, NA^{-1} is an even symmetric integral matrix. Therefore, by Lemma 1.5.1, ${}^{\mathrm{t}}m(NA^{-1})m$ is an even integer. Hence $\frac{1}{2}\,{}^{\mathrm{t}}m(NA^{-1})m$ is an integer, so that $Q(h) \equiv 0 \pmod{N}$. Next, let $n \in \mathbb{Z}^f$ be such that $n \equiv h \pmod{N}$. Let $b \in \mathbb{Z}^f$ be such that $n = h + Nb$. Then

$$\begin{aligned} 2Q(n) &= {}^{\mathrm{t}}(h+Nb)A(h+Nb) \\ &= ({}^{\mathrm{t}}h + N\,{}^{\mathrm{t}}b)A(h+Nb) \\ &= {}^{\mathrm{t}}hAh + 2N\,{}^{\mathrm{t}}bAh + N^2\,{}^{\mathrm{t}}bAb \\ &\equiv {}^{\mathrm{t}}hAh \pmod{2N^2} \\ &\equiv 2Q(h) \pmod{2N^2}. \end{aligned}$$

Here ${}^{\mathrm{t}}bAh \equiv 0 \pmod{N}$ because $Ah \equiv 0 \pmod{N}$ and ${}^{\mathrm{t}}bAb \equiv 0 \pmod{2}$ because A is even. It follows that $Q(n) \equiv Q(h) \pmod{N^2}$. Finally, since $Q(h) \equiv 0 \pmod{N}$ and $Q(n) \equiv Q(h) \pmod{N^2}$, we have $Q(n) \equiv 0 \pmod{N}$. □

1.6 The upper half-plane

Let $\mathrm{GL}(2,\mathbb{R})^+$ be the subgroup of $\sigma \in \mathrm{GL}(2,\mathbb{R})$ such that $\det(\sigma) > 0$. We define and action of $\mathrm{GL}(2,\mathbb{R})^+$ on the upper half-plane $\mathbb{H}_1$ by

$$\sigma \cdot z = \frac{az+b}{cz+d}$$

for $z \in \mathbb{H}_1$ and $\sigma \in \mathrm{GL}(2,\mathbb{R})^+$ such that

$$\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{1.8}$$

We define the cocycle function

$$j : \mathrm{GL}(2,\mathbb{R})^+ \times \mathbb{H}_1 \longrightarrow \mathbb{C}$$

by

$$j(\sigma, z) = cz + d$$

for $z \in \mathbb{H}_1$ and $\sigma \in \mathrm{GL}(2,\mathbb{R})^+$ as in (1.8). We have

$$j(\alpha\beta, z) = j(\alpha, \beta \cdot z)j(\beta, z)$$

for $\alpha, \beta \in \mathrm{GL}(2,\mathbb{R})^+$ and $z \in \mathbb{H}_1$. Let $F : \mathbb{H}_1 \to \mathbb{C}$ be a function, and let ℓ be an integer. Let $\sigma \in \mathrm{GL}(2,\mathbb{R})^+$. We define

$$F|_\ell : \mathbb{H}_1 \longrightarrow \mathbb{C}$$

by the formula

$$\begin{aligned}\big(F|_\ell\sigma\big)(z) &= \det(\sigma)^{\ell/2}(cz+d)^{-\ell}F\Big(\frac{az+b}{cz+d}\Big)\\ &= \det(\sigma)^{\ell/2}j(\sigma,z)^{-\ell}F(\sigma\cdot z)\end{aligned}$$

for $z \in \mathbb{H}_1$. We have

$$\big(F|_\ell\alpha\big)|_\ell\beta = F|_\ell(\alpha\beta)$$

for $\alpha, \beta \in \mathrm{GL}(2,\mathbb{R})^+$.

1.7 Congruence subgroups

Let N be a positive integer. The **principal congruence subgroup** of level N is defined to be

$$\Gamma(N) = \{\begin{bmatrix} a & b \\ c & d\end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : a \equiv d \equiv 1 \pmod N, b \equiv c \equiv 0 \pmod N\}.$$

The **Hecke congruence subgroup** of level N is defined to be

$$\Gamma_0(N) = \{\begin{bmatrix} a & b \\ c & d\end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : c \equiv 0 \pmod N\}.$$

If Γ is a subgroup of $\mathrm{SL}(2,\mathbb{Z})$, then we say that Γ is a **congruence subgroup** of $\mathrm{SL}(2,\mathbb{Z})$ of $\mathrm{SL}(2,\mathbb{Z})$ if there exists a positive integer N such that $\Gamma(N) \subset \Gamma$.

1.8 Modular forms

Let N be a positive integer, and let $R > 0$ be positive number. Let

$$H(N,R) = \{z \in \mathbb{H}_1 : \mathrm{Im}(z) > \frac{N\log(1/R)}{2\pi}\}$$

and

$$D(R) = \{q \in \mathbb{C} : |q| < R\}.$$

The function

$$H(N,R) \longrightarrow D(R)$$

defined by

$$z \mapsto q(z) = e^{2\pi i z/N}$$

is well-defined. We have $q(z+N) = q(z)$ for $z \in H(N,R)$.

Lemma 1.8.1. *Let $f : \mathbb{H}_1 \to \mathbb{C}$ be an analytic function, and let N be a positive integer such that $f(z+N) = f(z)$ for $z \in \mathbb{H}_1$. Assume that there exists a real number such that $0 < R < 1$ and a complex power series*

$$\sum_{n=0}^{\infty} a(n)q^n$$

that converges for $q \in D(R)$ *such that*

$$f(z) = \sum_{n=0}^{\infty} a(n) e^{2\pi i n z/N}$$

for $z \in H(N,R)$. *If* M *is another positive integer such that* $f(z+M) = f(z)$ *for* $z \in \mathbb{H}_1$, *then there exists a real number such that* $0 < T < 1$ *and a complex power series*

$$\sum_{k=0}^{\infty} b(k) q^k$$

that converges for $q \in D(T)$ *such that*

$$\big(F\big|_k \sigma\big)(z) = \sum_{k=0}^{\infty} b(k) e^{2\pi i k z/M}$$

for $z \in H(M,T)$.

Proof. For $z \in H(N,R)$,

$$\begin{aligned} f(z) &= f(z+M) \\ &= \sum_{n=0}^{\infty} a(n) e^{2\pi i n (z+M)/N} \\ \sum_{n=0}^{\infty} a(n) e^{2\pi i n z/N} &= \sum_{n=0}^{\infty} a(n) e^{2\pi i n M/N} \cdot e^{2\pi i n z/N}. \end{aligned}$$

It follows that

$$a(n) = a(n) e^{2\pi i n M/N}$$

for all non-negative integers n. Hence, for every non-negative integer n, if $a(n) \neq 0$, then nM/N is an integer, or equivalently, if nM/N is not an integer, then $a(n) = 0$. Let $z \in H(N,R)$. Then

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a(n) e^{2\pi i n z/N} \\ &= \sum_{n=0}^{\infty} a(n) e^{2\pi i (nM/N) z/M} \\ &= \sum_{k=0}^{\infty} b(k) (e^{2\pi i z/M})^k \end{aligned}$$

where

$$b(k) = \begin{cases} a(kN/M) & \text{if } kN/M \text{ is an integer,} \\ 0 & \text{if } kN/M \text{ is not an integer.} \end{cases}$$

Because the series $\sum_{n=0}^{\infty} a(n) e^{2\pi i n z/N}$ converges for $z \in H(N,R)$, the above equalities imply that the power series $\sum_{k=0}^{\infty} b(k) q^k$ converges for $q \in D(R^{N/M})$. Since $H(M, R^{N/M}) = H(N,R)$, the proof is complete. □

Definition 1.8.2. Let k be a non-negative integer, and let Γ be a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$. Let $F : \mathbb{H}_1 \to \mathbb{C}$ be a function on the upper-half plane $\mathbb{H}_1$. We say that F is a **modular form** of weight k with respect to Γ if the following conditions hold:

1. For all $\alpha \in \Gamma$ we have
$$f\big|_k \alpha = f.$$
2. The function F is analytic on $\mathbb{H}_1$.
3. If $\sigma \in \mathrm{SL}(2,\mathbb{Z})$, then there exists a positive integer N such that $\Gamma(N) \subset \Gamma$, a real number R such that $0 < R < 1$, and a complex power series
$$\sum_{n=0}^{\infty} a(n) q^n$$
that converges for $q \in D(R)$, such that
$$\big(F\big|_k \sigma\big)(z) = \sum_{n=0}^{\infty} a(n) q(z)^n = \sum_{n=0}^{\infty} a(n) e^{2\pi i n z/N}$$
for $z \in H(N,R)$.

The third condition of Definition 1.8.2 is often summarized by saying that F is **holomorphic at the cusps** of Γ. We say that F is a **cusp form** if the three conditions in the definition of a modular form hold, and in addition it is always the case that $a(0) = 0$; this additional condition is summarized by saying that F **vanishes at the cusps** of Γ. The set of modular forms of weight k with respect to Γ is a vector space over $\mathbb{C}$, which we denote by $M_k(\Gamma)$. The set of cusp forms of weight k with respect to Γ is a subspace of $M_k(\Gamma)$, and will be denoted by $S_k(\Gamma)$.

1.9 The symplectic group

Let R be a commutative ring with identity 1, and let n be a positive integer. As usual, we define
$$\mathrm{GL}(2n,R) = \{g \in \mathrm{M}(2n,R) : \det(g) \in R^\times\}.$$
Then $\mathrm{GL}(2n,R)$ is a group under multiplication of matrices; the identity of $\mathrm{GL}(2n,R)$ is the $2n \times 2n$ identity matrix $1 = 1_{2n}$. Let
$$J = \begin{bmatrix} & 1_n \\ -1_n & \end{bmatrix}.$$
We note that
$$J^2 = -1, \qquad J^{-1} = -J.$$
We define
$$\mathrm{Sp}(2n,R) = \{g \in \mathrm{GL}(2n,R) : {}^t g J g = J\}.$$
We refer to $\mathrm{Sp}(2n,R)$ as the **symplectic group of degree n over R**.

Lemma 1.9.1. *If R is a commutative ring with identity and n is a positive integer, then $\mathrm{Sp}(2n,R)$ is a subgroup of $\mathrm{GL}(2n,R)$. If $g \in \mathrm{Sp}(2n,R)$, then ${}^t g \in \mathrm{Sp}(2n,R)$.*

Proof. Evidently, $1 \in \mathrm{Sp}(2n,R)$. Also, it is easy to see that if $g,h \in \mathrm{Sp}(2n,R)$, then $gh \in \mathrm{Sp}(2n,R)$. To complete the proof that $\mathrm{Sp}(2n,R)$ is a subgroup of $\mathrm{GL}(2n,R)$ it will suffice to prove that if $g \in \mathrm{Sp}(2n,R)$, then $g^{-1} \in \mathrm{Sp}(2n,R)$. Let $g \in \mathrm{Sp}(n,R)$. Then ${}^t gJg = J$. This implies that $g^{-1} = J^{-1}\,{}^t gJ = -J\,{}^t gJ$. Now

$$\begin{aligned}
{}^t(g^{-1})Jg^{-1} &= {}^t Jg\,{}^t JJJ\,{}^t gJ \\
&= JgJJJ\,{}^t gJ \\
&= -JgJ\,{}^t gJ \\
&= -JgJ \cdot {}^t gJg \cdot g^{-1} \\
&= -JgJJg^{-1} \\
&= J.
\end{aligned}$$

Next, suppose that $g \in \mathrm{Sp}(2n,R)$. Then

$$\begin{aligned}
gJ\,{}^t g &= gJ\,{}^t gJgg^{-1}J^{-1} \\
&= gJJg^{-1}J^{-1} \\
&= -J^{-1} \\
&= J.
\end{aligned}$$

This implies that $g \in \mathrm{Sp}(2n,R)$. □

Lemma 1.9.2. *Let R be a commutative ring with identity and let n be a positive integer. Let*

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{GL}(2n,R).$$

Then $g \in \mathrm{Sp}(2n,R)$ if and only if

$${}^t AC = {}^t CA, \quad {}^t BD = {}^t DB, \quad {}^t AD - {}^t CB = 1.$$

If $g \in \mathrm{Sp}(2n,R)$, then

$$g^{-1} = \begin{bmatrix} {}^t D & -{}^t B \\ -{}^t C & {}^t A \end{bmatrix},$$

and

$$A\,{}^t B = B\,{}^t A, \quad C\,{}^t D = D\,{}^t C, \quad A\,{}^t D - B\,{}^t C = 1.$$

Proof. The first assertion follows by direct computations from the definition of $\mathrm{Sp}(2n,R)$. To prove the second assertion, assume that $g \in \mathrm{Sp}(2n,R)$. Then

$$\begin{bmatrix} {}^t D & -{}^t B \\ -{}^t C & {}^t A \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} {}^t DA - {}^t BC & {}^t DB - {}^t BD \\ {}^t AC - {}^t CA & {}^t AD - {}^t CB \end{bmatrix} = 1$$

by the first assertion. It follows that g^{-1} has the indicated form. But we also have

$$1 = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} {}^{\mathrm{t}}D & -{}^{\mathrm{t}}B \\ -{}^{\mathrm{t}}C & {}^{\mathrm{t}}A \end{bmatrix} = \begin{bmatrix} A\,{}^{\mathrm{t}}D - B\,{}^{\mathrm{t}}C & B\,{}^{\mathrm{t}}A - A\,{}^{\mathrm{t}}B \\ C\,{}^{\mathrm{t}}D - D\,{}^{\mathrm{t}}C & D\,{}^{\mathrm{t}}A - C\,{}^{\mathrm{t}}B. \end{bmatrix}$$

This implies the remaining claims. □

Lemma 1.9.3. *Let R be a commutative ring with identity. Then* $\mathrm{Sp}(2,R) = \mathrm{SL}(2,R)$.

Proof. Let $g \in \mathrm{GL}(2,R)$, and write

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

for some $a, b, c, d \in R$. A calculations shows that

$${}^{\mathrm{t}}gJg = \begin{bmatrix} & ad-bc \\ -(ad-bc) & \end{bmatrix} = \det(g) \cdot J.$$

It follows that $g \in \mathrm{Sp}(2,R)$ if and only if $\det(g) = 1$, i.e., $g \in \mathrm{SL}(2,R)$. □

Lemma 1.9.4. *Let R be a commutative ring with identity, and let n be a positive integer. The following matrices are contained in* $\mathrm{Sp}(2n,R)$*:*

$$J = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, \quad \begin{bmatrix} -1 & \\ & -1 \end{bmatrix},$$
$$\begin{bmatrix} A & \\ & {}^{\mathrm{t}}A^{-1} \end{bmatrix}, \quad A \in \mathrm{GL}(n,R),$$
$$\begin{bmatrix} 1 & X \\ & 1 \end{bmatrix}, \quad X \in \mathrm{M}(n,R),\ {}^{\mathrm{t}}X = X,$$
$$\begin{bmatrix} 1 & \\ Y & 1 \end{bmatrix}, \quad Y \in \mathrm{M}(n,R),\ {}^{\mathrm{t}}Y = Y.$$

Proof. These assertions follow by direct computations. □

Lemma 1.9.5. *Let R be a commutative ring with identity, and let n be a positive integer. The sets*

$$P = \{\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(2n,R) : C = 0\},$$
$$M = \{\begin{bmatrix} A & \\ & {}^{\mathrm{t}}A^{-1} \end{bmatrix} : A \in \mathrm{GL}(n,R)\},$$
$$U = \{\begin{bmatrix} 1 & X \\ & 1 \end{bmatrix} : X \in \mathrm{M}(n,R),\ {}^{\mathrm{t}}X = X\}$$

are subgroups of $\mathrm{Sp}(2n,R)$*. The subgroup M normalizes U, and $P = MU = UM$.*

Proof. These assertions follow by direct computations. □

Let R be a commutative ring with identity. Assume further that R is a domain. We say that R is **Euclidean domain** if there exists a function $|\cdot|: R \to \mathbb{Z}$ satisfying the following three properties:

1. If $a \in R$, then $|a| \geq 0$.
2. If $a \in R$, then $|a| = 0$ if and only if $a = 0$.
3. If $a, b \in R$ and $b \neq 0$, then there exist $x, y \in R$ such that $a = bx + y$ with $|y| < |b|$.

Any field F is an Euclidean domain with the definition $|a| = 1$ for $a \in F$ with $a \neq 0$ and $|0| = 0$. Also, $\mathbb{Z}$ is an Euclidean domain with the usual absolute value.

Theorem 1.9.6. *Let R be an Euclidean domain, and let n be a positive integer. The group $\mathrm{Sp}(2n, R)$ is generated by the elements*

$$J = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, \quad \begin{bmatrix} 1 & X \\ & 1 \end{bmatrix}$$

for $X \in \mathrm{M}(n, R)$ with ${}^{\mathrm{t}}X = X$.

Proof. See Satz A 5.4 on page 326 of [5]. □

Corollary 1.9.7. *Let R be an Euclidean domain, and let n be a positive integer. If $g \in \mathrm{Sp}(2n, R)$, then $\det(g) = 1$.*

Proof. This follows from Theorem 1.9.6. □

Theorem 1.9.8. *Let F be a field, and let n be a positive integer. Assume that the pair $(2n, F)$ is not $(2, \mathbb{Z}/2\mathbb{Z})$, $(2, \mathbb{Z}/3\mathbb{Z})$ or $(4, \mathbb{Z}/2\mathbb{Z})$. Then the only normal subgroups of $\mathrm{Sp}(2n, F)$ are $\{1\}$, $\{1, -1\}$, and $\mathrm{Sp}(2n, F)$.*

Proof. See Theorem 5.1 of [3]. □

1.10 The Siegel upper half-space

Let n be a positive integer. We define $\mathbb{H}_n$ to be the subset of $\mathrm{M}(n, \mathbb{C})$ consisting of the matrices $Z = X + iY$ with $X, Y \in \mathrm{M}(n, \mathbb{R})$ such that ${}^{\mathrm{t}}X = X$, ${}^{\mathrm{t}}Y = Y$, and Y is positive-definite. We refer to $\mathbb{H}_n$ as the **Siegel upper half-space of degree** n.

Lemma 1.10.1. *Let n be a positive integer. The set $\mathrm{Sym}(n, \mathbb{R})^+$ is open in $\mathrm{Sym}(n, \mathbb{R})$.*

Proof. For $1 \leq k \leq n$ and $V \in \mathrm{Sym}(n,\mathbb{R})$, let $V(k \times k) = (V_{ij})_{1\leq i,j\leq k}$. An element $V \in \mathrm{Sym}(n,\mathbb{R})$ is positive-definite if and only if $\det V(k \times k) > 0$ for $1 \leq k \leq n$. Consider the function

$$f : \mathrm{Sym}(n,\mathbb{R}) \longrightarrow \mathbb{R}^n, \quad f(V) = (\det V(1 \times 1), \dots, \det V(n \times n)).$$

The function f is continuous, and therefore $f^{-1}((\mathbb{R}_{>0})^n)$ is an open subset of $\mathrm{Sym}(n,\mathbb{R})$; since $f^{-1}((\mathbb{R}_{>0})^n)$ is exactly $\mathrm{Sym}(n,\mathbb{R})^+$, the proof is complete. □

Proposition 1.10.2. *Let n be a positive integer. The set $\mathbb{H}_n$ is an open subset of* $\mathrm{Sym}(n,\mathbb{C})$.

Proof. There is a natural homeomorphism $\mathrm{Sym}(n,\mathbb{C}) \cong \mathrm{Sym}(n,\mathbb{R})\times\mathrm{Sym}(n,\mathbb{R})$. Under this homeomorphism, $\mathbb{H}_n \cong \mathrm{Sym}(n,\mathbb{R}) \times \mathrm{Sym}(n,\mathbb{R})^+$. By Lemma 1.10.1, the set $\mathrm{Sym}(n,\mathbb{R})^+$ is open in $\mathrm{Sym}(n,\mathbb{R})$. It follows that $\mathbb{H}_n$ is an open subset of $\mathrm{Sym}(n,\mathbb{C})$. □

Proposition 1.10.3. *Let n be a positive integer. Let $Z_1, Z_2 \in \mathbb{H}_n$. Then $(1-t)Z_1+tZ_2 \in \mathbb{H}_n$ for all $t \in [0,1]$. Therefore, $\mathbb{H}_n$ is convex, and in particular, connected.*

Proof. Write $Z_1 = U_1 + iV_1$ and $Z_2 = U_2 + iV_2$. Then $(1 - t)Z_1 + tZ_2 = (1 - t)U_1 + tU_2 + i((1 - t)V_1 + tV_2)$ for $t \in [0,1]$. Since $(1 - t)U_1 + tU_2 \in \mathrm{Sym}(n,\mathbb{R})$ for $t \in [0,1]$, to prove the proposition it will suffice to prove that $f(t) = (1 - t)V_1 + tV_2 \in \mathrm{Sym}(n,\mathbb{R})^+$ for $t \in [0,1]$. Write $V_1 = W^2$ where $W \in \mathrm{Sym}(n,\mathbb{R})^+$ (see (1.7)). Then $W^{-1}f(t)W^{-1} = (1 - t) \cdot 1_n + tW^{-1}V_2W^{-1}$ for $t \in [0,1]$. We have $W^{-1}V_2W^{-1} \in \mathrm{Sym}(n,\mathbb{R})^+$, and for each $t \in [0,1]$, $W^{-1}f(t)W^{-1} \in \mathrm{Sym}(n,\mathbb{R})^+$ if and only if $f(t) \in \mathrm{Sym}(n,\mathbb{R})$. It follows that we may assume that $V_1 = 1$. Let $t \in [0,1]$; we need to prove that $A = f(t)$ is positive-definite. It is clear that A is positive semi-definite. If $B \in \mathrm{M}(n,\mathbb{R})$, and $k \in \{1,\dots,n\}$, then we define $B(k) = (B_{ij})_{1\leq i,j\leq k}$. Since A is positive semi-definite, by Sylvester's Criterion for positive semi-definite matrices, we have $\det(A(k)) \geq 0$ for $k \in \{1,\dots,n\}$; by Sylvester's Criterion for positive-definite matrices, we need to prove that $\det(A(k)) > 0$ for $k \in \{1,\dots,n\}$. Assume that there exists $k \in \{1,\dots,n\}$ such that $\det(A(k)) = 0$. Then

$$\det\big((1 - t)1_k + V_2(k)\big) = 0,$$

so that

$$\det\big((t - 1)1_k - V_2(k)\big) = 0.$$

It follows that $t - 1$ is an eigenvalue for $V_2(k)$; this implies that $t - 1$ is an eigenvalue for V_2. This is a contradiction since all the eigenvalues of V_2 are positive, and $t - 1 \leq 0$. □

Corollary 1.10.4. *Let n be a positive integer. The topological space $\mathbb{H}_n$ is simply connected.*

Lemma 1.10.5. *Let k be positive integer. Let $f : \mathbb{H}_k \to \mathbb{C}$ be an analytic function. If $f(iU) = 0$ for all U in an open subset S of* $\mathrm{Sym}(k,\mathbb{R})^+$, *then $f = 0$.*

Proof. By Proposition 1.10.3, the open subset $\mathbb{H}_k$ of $\mathrm{Sym}(k,\mathbb{C})$ is connected. By Proposition 1 on page 3 of [19] it suffices to prove that f vanishes on a non-empty open subset of $\mathbb{H}_k$. Let U be any element of S. Since f is analytic at iU and $\mathbb{H}_k$ is an open subset of $\mathrm{Sym}(k,\mathbb{C})$, there exists $\epsilon > 0$ such that

$$D = \{Z \in \mathrm{Sym}(n,\mathbb{C}) : |Z_{ij} - iU_{ij}| < \epsilon, 1 \le i \le j \le k\} \subset \mathbb{H}_k,$$

and a power series

$$\sum_{\alpha \in \mathbb{Z}^k_{\ge 0}} c_\alpha (Z - iU)^\alpha$$

that converges absolutely and uniformly on compact subsets of D, such that this power series converges to $f(Z)$ for $Z \in D$. Evidently, $iU \in D$. Define

$$D' = \{Y \in \mathrm{Sym}(n,\mathbb{R}) : |Y_{ij} - U_{ij}| < \epsilon, 1 \le i \le j \le k\}.$$

Then $U \in D'$. We may assume that $D' \subset S$. If $Y \in D'$, then $iY \in D$. Define $h : D' \to \mathbb{C}$ by $h(Y) = f(iY)$. We have

$$h(Y) = \sum_{\alpha \in \mathbb{Z}^k_{\ge 0}} c_\alpha (iY - iU)^\alpha = \sum_{\alpha \in \mathbb{Z}^k_{\ge 0}} i^{|\alpha|} c_\alpha (Y - U)^\alpha$$

for $Y \in D'$. The function h is C^∞, and we have

$$i^{|\alpha|} c_\alpha = \frac{1}{\alpha!} (D^\alpha h)(U).$$

Since by assumption $f(iY) = 0$ for $Y \in S$, we have $h = 0$. This implies that $c_\alpha = 0$ for $\alpha \in \mathbb{Z}^k_{\ge 0}$, which in turn implies that f vanishes on the open subset $D \subset \mathbb{H}_k$. □

Lemma 1.10.6. *Let n be a positive integer. Let*

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(2n,\mathbb{R})$$

and $Z \in \mathbb{H}_n$. Then $CZ + D$ is invertible, and

$$(AZ + B)(CZ + D)^{-1} \in \mathbb{H}_n.$$

Proof. We follow the argument from [13]. Write $Z = X + iY$ with $X, Y \in \mathrm{M}(n,\mathbb{R})$. Define

$$P = AZ + B, \qquad Q = CZ + D.$$

We will first prove that Q is invertible. Assume that $v \in \mathbb{C}^n$ is such that $Qv = 0$; we need to prove that $v = 0$. We then have:

$$\begin{aligned} {}^{\mathrm{t}}P\overline{Q} - {}^{\mathrm{t}}Q\overline{P} &= (Z\,{}^{\mathrm{t}}A + {}^{\mathrm{t}}B)(C\overline{Z} + D) - (Z\,{}^{\mathrm{t}}C + {}^{\mathrm{t}}D)(A\overline{Z} + B) \\ &= Z\,{}^{\mathrm{t}}AC\overline{Z} + Z\,{}^{\mathrm{t}}AD + {}^{\mathrm{t}}BC\overline{Z} + {}^{\mathrm{t}}BD \end{aligned}$$

$$
\begin{aligned}
&- Z\,{}^{t}CA\overline{Z} - Z\,{}^{t}CB - {}^{t}DA\overline{Z} - {}^{t}DB \\
&= Z - \overline{Z} \quad \text{(cf. Lemma 1.9.2)} \\
&= 2iY. \qquad (1.9)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
{}^{t}v\big({}^{t}P\overline{Q} - {}^{t}Q\overline{P}\big)\overline{v} &= 2i\,{}^{t}vY\overline{v} \\
{}^{t}v\,{}^{t}P\overline{Q}\overline{v} - {}^{t}v\,{}^{t}Q\overline{P}\overline{v} &= 2i\,{}^{t}vY\overline{v} \\
{}^{t}v\,{}^{t}P\overline{Qv} - {}^{t}(Qv)\overline{P}\overline{v} &= 2i\,{}^{t}vY\overline{v} \\
0 &= 2i\,{}^{t}vY\overline{v} \\
0 &= {}^{t}vY\overline{v}.
\end{aligned}
$$

Write $v = v_1 + iv_2$ with $v_1, v_2 \in \mathbb{R}^n$. Then

$$
0 = {}^{t}vY\overline{v} = {}^{t}v_1Yv_1 + {}^{t}v_2Yv_2.
$$

Since Y is positive-definite, the real numbers ${}^{t}v_1Yv_1$ and ${}^{t}v_2Yv_2$ are both non-negative; since the sum of these two numbers is zero, both are zero. Again, since Y is positive-definite, this implies that $v_1 = v_2 = 0$ so that $v = 0$. Hence, Q is invertible. Now we prove that PQ^{-1} is symmetric. Evidently, PQ^{-1} is symmetric if and only if ${}^{t}PQ = {}^{t}QP$. Now

$$
\begin{aligned}
{}^{t}PQ - {}^{t}QP &= {}^{t}(AZ+B)(CZ+D) - {}^{t}(CZ+D)(AZ+B) \\
&= ({}^{t}Z\,{}^{t}A + {}^{t}B)(CZ+D) - ({}^{t}Z\,{}^{t}C + {}^{t}D)(AZ+B) \\
&= {}^{t}Z\,{}^{t}ACZ + {}^{t}Z\,{}^{t}AD + {}^{t}BCZ + {}^{t}BD \\
&\quad - {}^{t}Z\,{}^{t}CAZ - {}^{t}Z\,{}^{t}CB - {}^{t}DAZ - {}^{t}DB \\
&= 0 \quad \text{(cf Lemma 1.9.2)}
\end{aligned}
$$

as desired. It follows that PQ^{-1} is symmetric. Write $PQ^{-1} = X' + iY'$ where $X', Y' \in \mathrm{M}(n, \mathbb{R})$ with ${}^{t}X' = X'$ and ${}^{t}Y' = Y'$. To complete the proof of the lemma we need to show that Y' is positive-definite. Now

$$
\begin{aligned}
Y' &= \frac{1}{2i}\big((X'+iY') - \overline{(X'+iY')}\big) \\
&= \frac{1}{2i}\big(PQ^{-1} - \overline{PQ^{-1}}\big) \\
&= \frac{1}{2i}\big({}^{t}(PQ^{-1}) - \overline{PQ^{-1}}\big) \\
&= \frac{1}{2i}\big({}^{t}Q^{-1}\,{}^{t}P - \overline{PQ^{-1}}\big) \\
&= \frac{1}{2i}\,{}^{t}Q^{-1}\big({}^{t}P\overline{Q} - {}^{t}Q\overline{P}\big)\overline{Q^{-1}} \\
&= \frac{1}{2i}\,{}^{t}Q^{-1}\big(2iY\big)\overline{Q^{-1}} \quad \text{(cf. (1.9))} \\
&= {}^{t}Q^{-1}Y\overline{Q^{-1}}.
\end{aligned}
$$

Using that Y is positive-definite, it is easy to verify that $Y' = {}^{t}Q^{-1}Y\overline{Q^{-1}}$ is positive-definite. $\square$

Lemma 1.10.7. *Let n be a positive integer. For $g = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{Sp}(2n, \mathbb{R})$ and $Z \in \mathbb{H}_n$ we define*

$$g \cdot Z = (AZ + B)(CZ + D)^{-1}, \qquad j(g, Z) = \det(CZ + D).$$

We have

$$(gh) \cdot Z = g \cdot (h \cdot Z),$$
$$j(gh, Z) = j(g, h \cdot Z) j(h, Z)$$

for $g, h \in \mathrm{Sp}(2n, \mathbb{R})$ and $Z \in \mathbb{H}_n$.

Proposition 1.10.8. *Let n be a positive integer, and let*

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

There exists an analytic function

$$s(g, \cdot) : \mathbb{H}_n \longrightarrow \mathbb{C}$$

such that

$$s(g, Z)^2 = \det(CZ + D)$$

for $Z \in \mathbb{H}_n$. Moreover, there exists an eighth root of unity ξ such that

$$s(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, iU) = \xi \det(U)^{1/2}$$

for all $U \in \mathrm{Sym}(n, \mathbb{R})^+$. Here, $\det(U)^{1/2}$ is the positive square root of the positive number $\det(U)$ for $U \in \mathrm{Sym}(n, \mathbb{R})^+$.

Proof. We follow an idea from [5], page 19. Define a function

$$\alpha : [0, 1] \times \mathbb{H}_n \longrightarrow \mathbb{C}$$

by

$$\begin{aligned} \alpha(t, Z) &= \det\Big((1 - t)(C(i1_n) + D) + t(CZ + D)\Big) \\ &= \det\Big(C\big((1 - t)(i1_n) + tZ\big) + D)\Big) \end{aligned}$$

for $t \in [0, 1]$ and $Z \in \mathbb{H}_n$. Here, given $Z \in \mathbb{H}_n$, the points $W(t) = (1-t)(i1_n)+tZ$ for $t \in [0, 1]$ are the points on the line between iI_n and Z; by Proposition 1.10.3, all these points are in $\mathbb{H}_n$, and by Lemma 1.10.6, $\det(CW(t) + D)$ is non-zero for $t \in [0, 1]$. Thus, α actually takes values in $\mathbb{C} - \{0\}$. Evidently, for fixed $Z \in \mathbb{H}_n$, the $\alpha(\cdot, Z)$ is a polynomial in t, and hence $\alpha(\cdot, Z) : [0, 1] \to \mathbb{C} - \{0\}$

is a piecewise C^1 curve (see [17], page 75. Also, for fixed $t \in [0,1]$, $\alpha(t,\cdot)$ is a function on $\mathbb{H}_n$ that is a polynomial in each entry of $Z \in \mathbb{H}_n$, and is hence analytic in each variable. Define

$$H : \mathbb{H}_n \longrightarrow \mathbb{C}$$

by the contour integral (see [17], page 76)

$$H(Z) = \int\limits_{\alpha(\cdot,Z)} \frac{1}{w}\, dw,$$

or more concretely,

$$H(Z) = \int\limits_0^1 \frac{\alpha'(t,Z)}{\alpha(t,Z)}\, dt,$$

for $Z \in \mathbb{H}_n$. Here, the derivative is taken with respect to $t \in [0,1]$ for fixed $Z \in \mathbb{H}_n$. We claim that $e^{H(Z)} = \det(-iZ)$ for $Z \in \mathbb{H}_n$. To see this, fix $Z \in \mathbb{H}_n$. Since $|\alpha(\cdot,Z)|$ is continuous, $[0,1]$ is compact, and $|\alpha(t,Z)| > 0$ for $t \in [0,1]$, the number $\epsilon = \inf(\{|\alpha(t,Z)| : t \in [0,1]\}$ is positive (see Theorem 5 on page 88 of [18]). The function $\alpha(\cdot,Z) : [0,1] \to \mathbb{C}$ is uniformly continuous (see Theorem 7 on page 92 of [18]). Hence, there exists a positive integer n such that if $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| \le 1/n$, then $|\alpha(t_1,Z) - \alpha(t_2,Z)| < \epsilon/2$. Let $k \in \{0,1,2,\ldots,n-1\}$. If $t \in [k/n,(k+1)/n]$, then $\alpha(t,Z)$ lies in the disc $D_k = \{w \in \mathbb{C} : |\alpha(k/n,Z) - w| < \epsilon/2\}$. By the definition of ϵ, the disc D_k does not contain 0. Therefore, there exists $\theta_k \in [0,2\pi)$ such that none of the points on the ray $R(\theta_k) = \{re^{i\theta_k} : r \in [0,\infty)\}$ lie in D_k. For $\theta \in [0,2\pi)$, let $\log_\theta : \mathbb{C} - R(\theta) \to \mathbb{C}$ be the branch of the logarithm function given by

$$\log_\theta(z) = \log(|z|) + i\mathrm{arg}_\theta(z),$$

where $z \in \mathbb{C} - R(\theta)$ and $\theta < \mathrm{arg}_\theta(z) < \theta + 2\pi i$. The function $\log_\theta$ is analytic in its domain, and we have

$$\frac{d}{dz}(\log_\theta)(z) = \frac{1}{z}$$

for $z \in \mathbb{C} - R(\theta)$. Now using Theorem 4 on page 83 of [17],

$$\begin{aligned} H(Z) &= \int\limits_{\alpha(\cdot,Z)} \frac{1}{z}\, dz \\ &= \sum_{k=0}^{n-1} \int\limits_{k/n}^{(k+1)/n} \frac{\alpha'(t,Z)}{\alpha(t,Z)}\, dt \\ &= \sum_{k=0}^{n-1} \log_{\theta_k}(\alpha((k+1)/n,Z)) - \log_{\theta_k}(\alpha(k/n,Z)). \end{aligned}$$

For each $k \in \{0, \ldots, n-1\}$, $\log_{\theta_k}(\alpha((k+1)/n, Z)) = \log_{\theta_{k+1}}(\alpha((k+1)/n, Z) + 2\pi i m$ for some integer m. It follows that

$$H(Z) = \log_{\theta_{n-1}}(\alpha(1, Z)) - \log_{\theta_0}(\alpha(0, Z)) + 2\pi i N$$

for some integer N. Therefore,

$$\begin{aligned} e^{H(Z)} &= e^{\log_{\theta_{n-1}}(\alpha(1,Z)) - \log_{\theta_0}(\alpha(0,Z)) + 2\pi i N} \\ &= \alpha(1, Z)\alpha(0, Z)^{-1} \\ &= \det(CZ + D)\det(C(i1_n) + D)^{-1}. \end{aligned}$$

Next, we claim that $H : \mathbb{H}_n \to \mathbb{C}$ is an analytic function on $\mathbb{H}_n$. To see this, we note that the function sending $(t, Z) \in [0, 1] \times \mathbb{H}_n$ to

$$\frac{\alpha'(t, Z)}{\alpha(t, Z)}$$

is continuous, and for fixed $t \in [0, 1]$, is analytic on $\mathbb{H}_n$. We thus may differentiate under the integral sign in the definition of H (see 2. on page 324 of [18]), and use the Cauchy-Riemann equations criterion (see Theorem 19 on page 48 of [17]) to see that H is analytic on $\mathbb{H}_n$. Fix $w \in \mathbb{C}^\times$ such that $w^2 = \det(C(i1_n) + D)$. We now define $s(g, \cdot) : \mathbb{H}_n \to \mathbb{C}$ by

$$s(g, Z) = we^{H(Z)/2}.$$

Then for $Z \in \mathbb{H}_n$ we have

$$\begin{aligned} s(g, Z)^2 &= w^2 e^{H(Z)} \\ &= \det(C(i1_n) + D)\det(CZ + D)\det(C(i1_n) + D)^{-1} \\ &= \det(CZ + D). \end{aligned}$$

To prove the uniqueness statement, we first note that

$$s(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, iU)^2 = \det((-1)iU) = (-i)^n \det(U)$$

for $U \in \mathrm{Sym}(n, \mathbb{R})^+$. Fix $\zeta \in \mathbb{C}^\times$ such that $\zeta^2 = (-i)^n$. Then ζ is an eighth root of unity. It follows that for every $U \in \mathrm{Sym}(n, \mathbb{R})^+$ there exists $\epsilon(U) \in \{\pm 1\}$ such that

$$s(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, iU) = \epsilon(U)\zeta \det(U)^{1/2}$$

for $U \in \mathrm{Sym}(n, \mathbb{R})^+$. Consider the function $\mathrm{Sym}(n, \mathbb{R})^+ \to \mathbb{R}$ defined by $U \mapsto s(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, iU)/\det(U)^{1/2}$ for $U \in \mathrm{Sym}(n, \mathbb{R})^+$. This function is continuous and defined on the connected set $\mathrm{Sym}(n, \mathbb{R})^+$. Since this function takes values in the eighth roots of unity, it follows from the intermediate value theorem (see

Theorem 6 on page 90 of [18]) that this function is constant. Hence, there exists an eighth root of unity ξ such that

$$s(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, iU) = \xi \det(U)^{1/2}$$

for all $U \in \mathrm{Sym}(n, \mathbb{R})^+$. □

Corollary 1.10.9. *Let n be a positive integer. Let $s : \mathrm{Sp}(2n, \mathbb{R}) \times \mathbb{H}_n \to \mathbb{C}$ be the function from Proposition 1.10.8. If $g, h \in \mathrm{Sp}(2n, \mathbb{R})$, then there exists $\varepsilon \in \{\pm 1\}$ such that*

$$s(gh, Z) = \varepsilon s(g, h \cdot Z) s(h, Z)$$

for all $Z \in \mathbb{H}_n$.

Proof. Let $g, h \in \mathrm{Sp}(2n, \mathbb{R})$. If $Z \in \mathbb{H}_n$, then

$$\begin{aligned} s(gh, Z)^2 &= j(gh, Z) \\ &= j(g, h \cdot Z) j(h, Z) \qquad \text{(see Lemma 1.10.7)} \\ &= s(g, h \cdot Z)^2 s(h, Z)^2 \\ &= \big(s(g, h \cdot Z) s(h, Z)\big)^2. \end{aligned}$$

It follows that for each $Z \in \mathbb{H}_n$ there exists $\varepsilon(Z) \in \{\pm 1\}$ such that $s(gh, Z) = \varepsilon(Z) s(g, h \cdot Z) s(h, Z)$. The function on $\mathbb{H}_n$ that sends Z to $\varepsilon(Z)$ is continuous and takes values in $\{\pm 1\}$. Since $\mathbb{H}_n$ is connected (see Proposition 1.10.3), the intermediate value theorem (see Theorem 6 on page 90 of [18]) implies now that this function is constant. □

1.11 The theta group

Let k be a positive integer, and let $M \in \mathrm{M}(k, \mathbb{C})$. We define an element of $\mathrm{M}(k, 1, \mathbb{C})$ by

$$\mathrm{diag}(M) = \begin{bmatrix} m_{11} \\ \vdots \\ m_{kk} \end{bmatrix}.$$

Lemma 1.11.1. *Let k be a positive integer, Assume that $M \in \mathrm{M}(k, \mathbb{Z})$ and $X \in \mathrm{Sym}(k, \mathbb{Z})$. Then*

$$\mathrm{diag}(MX\,{}^{\mathrm{t}}M) \equiv M \mathrm{diag}(X) \pmod 2.$$

Proof. If A is a $k \times k$ matrix, and $i, j \in \{1, \ldots, k\}$, then we let A_{ij} be the (i, j)-th entry of A. Let $i \in \{1, \ldots, k\}$. Then the i-th entry of $\mathrm{diag}(MX\,{}^{\mathrm{t}}M)$ is:

$$\sum_{\ell=1}^{k} M_{i\ell} (X\,{}^{\mathrm{t}}M)_{\ell i} = \sum_{\ell=1}^{k} M_{i\ell} \sum_{j=1}^{k} X_{\ell j} ({}^{\mathrm{t}}M)_{ji}$$

$$
\begin{aligned}
&= \sum_{\ell=1}^{k}\sum_{j=1}^{k} M_{i\ell}M_{ij}X_{\ell j}\\
&= \sum_{\substack{\ell,j\in\{1,\ldots,k\}\\ \ell=j}} M_{i\ell}M_{ij}X_{\ell j} + \sum_{\substack{\ell,j\in\{1,\ldots,k\}\\ \ell\neq j}} M_{i\ell}M_{ij}X_{\ell j}\\
&= \sum_{j\in\{1,\ldots,k\}} M_{ij}^2X_{jj} + \sum_{\substack{\ell,j\in\{1,\ldots,k\}\\ \ell<j}} (M_{i\ell}M_{ij}X_{\ell j} + M_{ij}M_{i\ell}X_{j\ell})\\
&= \sum_{j\in\{1,\ldots,k\}} M_{ij}^2X_{jj} + \sum_{\substack{\ell,j\in\{1,\ldots,k\}\\ \ell<j}} 2M_{i\ell}M_{ij}X_{\ell j}\\
&\equiv \sum_{j\in\{1,\ldots,k\}} M_{ij}^2X_{jj} \pmod 2\\
&\equiv \sum_{j\in\{1,\ldots,k\}} M_{ij}X_{jj} \pmod 2.
\end{aligned}
$$

Since $\sum_{j=1}^{k} M_{ij}X_{jj}$ is the i-th entry of $M\mathrm{diag}(X)$, the proof is complete. □

For the next proposition, we follow Lemma 7.6 from p. 457 of [7].

Proposition 1.11.2. *Let n be a positive integer. Define a function*

$$\mathrm{Sp}(2n,\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})^{2n} \to (\mathbb{Z}/2\mathbb{Z})^{2n}$$

by

$$g\{m\} = {}^{\mathrm{t}}g^{-1}m + \begin{bmatrix}\mathrm{diag}(C\,{}^{\mathrm{t}}D)\\ \mathrm{diag}(A\,{}^{\mathrm{t}}B)\end{bmatrix},$$

for $g = \left[\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right] \in \mathrm{Sp}(2n,\mathbb{Z})$ and $m \in (\mathbb{Z}/2\mathbb{Z})^{2n}$. Then this function is an action, i.e.,

$$g\{h\{m\}\} = (gh)\{m\}$$

for $g, h \in \mathrm{Sp}(2n,\mathbb{Z})$ and $m \in (\mathbb{Z}/2\mathbb{Z})^{2n}$.

Proof. Let $g, h \in \mathrm{Sp}(2n,\mathbb{Z})$ with

$$g = \begin{bmatrix}A & B\\ C & D\end{bmatrix} \in \mathrm{Sp}(2n,\mathbb{Z}),$$

and let $m \in (\mathbb{Z}/2\mathbb{Z})^{2n}$. To prove that $g\{h\{m\}\} = (gh)\{m\}$ we may assume that h is a generator for $\mathrm{Sp}(2n,\mathbb{Z})$ as described in Theorem 1.9.6. Assume first that h has the form

$$h = \begin{bmatrix}1 & X\\ & 1\end{bmatrix}$$

for some $X \in \mathrm{Sym}(n,\mathbb{Z})$. Then

$$(gh)\{m\} \equiv \begin{bmatrix}A & AX+B\\ C & CX+D\end{bmatrix}\{m\} \pmod 2$$

$$
\begin{aligned}
&\equiv {}^{\mathrm{t}}(gh)^{-1}m + \begin{bmatrix} \operatorname{diag}(C\,{}^{\mathrm{t}}(CX+D)) \\ \operatorname{diag}(A\,{}^{\mathrm{t}}(AX+B)) \end{bmatrix} \pmod 2 \\
&\equiv {}^{\mathrm{t}}(gh)^{-1}m + \begin{bmatrix} \operatorname{diag}(CX\,{}^{\mathrm{t}}C + C\,{}^{\mathrm{t}}D) \\ \operatorname{diag}(AX\,{}^{\mathrm{t}}A + A\,{}^{\mathrm{t}}B) \end{bmatrix} \pmod 2 \\
&\equiv {}^{\mathrm{t}}(gh)^{-1}m + \begin{bmatrix} \operatorname{diag}(CX\,{}^{\mathrm{t}}C) + \operatorname{diag}(C\,{}^{\mathrm{t}}D) \\ \operatorname{diag}(AX\,{}^{\mathrm{t}}A) + \operatorname{diag}(A\,{}^{\mathrm{t}}B) \end{bmatrix} \pmod 2,
\end{aligned}
$$

And

$$
\begin{aligned}
g\{h\{m\}\} &\equiv g\{{}^{\mathrm{t}}h^{-1}m + \begin{bmatrix} \\ \operatorname{diag}(X) \end{bmatrix}\} \pmod 2 \\
&\equiv {}^{\mathrm{t}}g^{-1}\,{}^{\mathrm{t}}h^{-1}m + {}^{\mathrm{t}}g^{-1}\begin{bmatrix} \\ \operatorname{diag}(X) \end{bmatrix} + \begin{bmatrix} \operatorname{diag}(C\,{}^{\mathrm{t}}D) \\ \operatorname{diag}(A\,{}^{\mathrm{t}}B) \end{bmatrix} \pmod 2 \\
&\equiv {}^{\mathrm{t}}(gh)^{-1}m + \begin{bmatrix} D & -C \\ -B & A \end{bmatrix}\begin{bmatrix} \\ \operatorname{diag}(X) \end{bmatrix} + \begin{bmatrix} \operatorname{diag}(C\,{}^{\mathrm{t}}D) \\ \operatorname{diag}(A\,{}^{\mathrm{t}}B) \end{bmatrix} \pmod 2 \\
&\equiv {}^{\mathrm{t}}(gh)^{-1}m + \begin{bmatrix} -C\cdot\operatorname{diag}(X) + \operatorname{diag}(C\,{}^{\mathrm{t}}D) \\ A\cdot\operatorname{diag}(X) + \operatorname{diag}(A\,{}^{\mathrm{t}}B) \end{bmatrix} \pmod 2.
\end{aligned}
$$

The equality $g\{h\{m\}\} = (gh)\{m\}$ follows now from Lemma 1.11.1. Next, assume that

$$
h = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}.
$$

Then

$$
\begin{aligned}
(g\begin{bmatrix} & 1 \\ -1 & \end{bmatrix})\{m\} &\equiv {}^{\mathrm{t}}g^{-1}\,{}^{\mathrm{t}}\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}^{-1}m + \begin{bmatrix} \operatorname{diag}(-D\,{}^{\mathrm{t}}C) \\ \operatorname{diag}(-B\,{}^{\mathrm{t}}A) \end{bmatrix} \pmod 2 \\
&\equiv {}^{\mathrm{t}}g^{-1}\,{}^{\mathrm{t}}\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}^{-1}m + \begin{bmatrix} \operatorname{diag}(D\,{}^{\mathrm{t}}C) \\ \operatorname{diag}(B\,{}^{\mathrm{t}}A) \end{bmatrix} \pmod 2.
\end{aligned}
$$

And

$$
\begin{aligned}
g\{h\{m\}\} &\equiv g\{{}^{\mathrm{t}}\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}^{-1}m\} \pmod 2 \\
&\equiv {}^{\mathrm{t}}g^{-1}\,{}^{\mathrm{t}}\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}^{-1}m + \begin{bmatrix} \operatorname{diag}(C\,{}^{\mathrm{t}}D) \\ \operatorname{diag}(A\,{}^{\mathrm{t}}B) \end{bmatrix} \pmod 2.
\end{aligned}
$$

Because $g \in \mathrm{Sp}(2n,\mathbb{Z})$, the matrices $C\,{}^{\mathrm{t}}D$ and $A\,{}^{\mathrm{t}}B$ are symmetric; this now implies that $(gh)\{m\} = g\{h\{m\}\}$. □

Let n be a positive integer. By Proposition 1.11.2, the group $\mathrm{Sp}(2n,\mathbb{Z})$ acts on $(\mathbb{Z}/2\mathbb{Z})^{2n}$. We define the **theta group** Γ_θ to be the stabilizer of the point 0 in $(\mathbb{Z}/2\mathbb{Z})^{2n}$. When we need to indicate that Γ_θ is contained in $\mathrm{Sp}(2n,\mathbb{Z})$ we will write $\Gamma_{\theta,2n}$ for Γ_θ. The definition of this action implies that the theta group is the subset of all $[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}] \in \mathrm{Sp}(2n,\mathbb{Z})$ such that $\operatorname{diag}(A\,{}^{\mathrm{t}}B) \equiv 0 \pmod 2$ and $\operatorname{diag}(C\,{}^{\mathrm{t}}D) \equiv 0 \pmod 2$. Let $g = [\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}] \in \mathrm{Sp}(2n,\mathbb{Z})$. Then

$$
g^{-1} = \begin{bmatrix} {}^{\mathrm{t}}D & -{}^{\mathrm{t}}B \\ -{}^{\mathrm{t}}C & {}^{\mathrm{t}}A \end{bmatrix}.
$$

Since Γ_θ is a group, we have $g \in \Gamma_\theta$ if and only if $g^{-1} \in \Gamma_\theta$. Thus, for $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{Sp}(2n, \mathbb{Z})$,

$$\begin{aligned} \begin{matrix} \mathrm{diag}(A\,{}^{\mathrm{t}}B) \equiv 0 \pmod 2 \\ \mathrm{diag}(C\,{}^{\mathrm{t}}D) \equiv 0 \pmod 2 \end{matrix} &\iff g \in \Gamma_\theta \\ &\iff g^{-1} \in \Gamma_\theta \iff \begin{matrix} \mathrm{diag}({}^{\mathrm{t}}BD) \equiv 0 \pmod 2 \\ \mathrm{diag}({}^{\mathrm{t}}CA) \equiv 0 \pmod 2 \end{matrix}. \end{aligned}$$

1.12 Elementary divisors

Theorem 1.12.1 (Theorem on elementary divisors)**.** *Let n be a positive integer. Let $M \in \mathrm{M}(n, \mathbb{Z})$. There exist a non-negative integer k, positive integers $d_1, \ldots, d_k$ and $g_1, g_2 \in \mathrm{SL}(n, \mathbb{Z})$ such that $k \le n$,*

$$g_1 M g_2 = \begin{bmatrix} d_1 &&&&&&& \\ & d_2 &&&&&& \\ && d_3 &&&&& \\ &&& \ddots &&&& \\ &&&& d_k &&& \\ &&&&& 0 && \\ &&&&&& \ddots & \\ &&&&&&& 0 \end{bmatrix}$$

and

$$d_1 \big| d_2, \quad d_2 \big| d_3, \quad \ldots, \quad d_{k-1} | d_k.$$

If M is non-zero, then the greatest common divisor of the entries of M is d_1.

Proof. For the first assertion see Proposition 2.11 on p. 339 of [10], or p. 8 of [4]. Assume that M is non-zero. If $X \in \mathrm{M}(n, \mathbb{Z})$ is non-zero, then let $I(X)$ be the ideal of $\mathbb{Z}$ generated by X. If $X \in \mathrm{M}(n, \mathbb{Z})$ is non-zero, then the greatest common divisor of the entries of X is the positive generator of $I(X)$. Since $g_1, g_2 \in \mathrm{SL}(n, \mathbb{Z})$ we have $I(M) = I(g_1 M g_2) = (d_1)$; thus, the greatest common divisor of the entries of M is d_1. $\square$

Chapter 2

Classical theta series on $\mathbb{H}_1$

2.1 Definition and convergence

Lemma 2.1.1. *Let f be a positive integer. Let $A \in \mathrm{M}(f, \mathbb{R})$ be a positive-definite symmetric matrix, and for $x \in \mathbb{R}^f$ let*

$$Q(x) = \frac{1}{2}\,{}^{\mathrm{t}}xAx.$$

For $z \in \mathbb{H}_1$, define

$$\theta(A, z) = \sum_{m \in \mathbb{Z}^f} e^{\pi i z\,{}^{\mathrm{t}}mAm} = \sum_{m \in \mathbb{Z}^f} e^{2\pi i z Q(m)}$$

For every $\delta > 0$, this series converges absolutely and uniformly on the set

$$\{z \in \mathbb{H}_1 : \mathrm{Im}(z) \geq \delta\}.$$

The function $\theta(A, \cdot)$ is an analytic function on $\mathbb{H}_1$.

Proof. Since A is positive-definite, the function defined by $x \mapsto \sqrt{Q(x)}$ defines a norm on $\mathbb{R}^f$. All norms on $\mathbb{R}^f$ equivalent; in particular, this norm is equivalent to the standard norm $\|\cdot\|$ on $\mathbb{R}^f$. Hence, there exists $\epsilon > 0$ such that

$$\varepsilon\|x\| \leq \sqrt{Q(x)},$$

or equivalently,

$$\varepsilon^2\|x\|^2 = \varepsilon^2(x_1^2 + \cdots x_f^2) \leq Q(x)$$

for $x = {}^{\mathrm{t}}(x_1, \ldots, x_f) \in \mathbb{R}^f$.

Now let $\delta > 0$, and let $z \in \mathbb{H}_1$ be such that $\mathrm{Im}(z) \geq \delta$. Let $m = {}^{\mathrm{t}}(m_1, \ldots, m_f) \in \mathbb{Z}^f$. Then

$$|e^{2\pi i z Q(m)}| = e^{-2\pi \mathrm{Im}(z) Q(m)}$$

$$\begin{aligned}
&\le e^{-2\pi\delta Q(m)}\\
&\le e^{-2\pi\delta\varepsilon^2\|m\|^2}\\
&= q^{\|m\|^2}\\
&= q^{m_1^2+\cdots+m_f^2}.
\end{aligned}$$

where $q = e^{-2\pi\delta\varepsilon^2}$. Since $0 < q < 1$, the series

$$\sum_{n\in\mathbb{Z}} q^{n^2}$$

converges absolutely. This implies that the series

$$(\sum_{n\in\mathbb{Z}} q^{n^2})^f = \sum_{m\in\mathbb{Z}^f} q^{m_1^2+\cdots+m_f^2} = \sum_{m\in\mathbb{Z}^f} q^{\|m\|^2}$$

converges absolutely. It follows from the Weierstrass M-test that our series

$$\sum_{m\in\mathbb{Z}^f} e^{2\pi i z Q(m)}$$

converges absolutely and uniformly on $\{z \in \mathbb{H}_1 : \mathrm{Im}(z) \ge \delta\}$ (see, for example, [17], p. 160). Since for each $m \in \mathbb{Z}^f$ the function on $\mathbb{H}_1$ defined by $z \mapsto e^{2\pi i z Q(m)}$ is an analytic function, and since our series converges absolutely and uniformly on every closed disk in $\mathbb{H}_1$, it follows that $\theta(A, \cdot)$ is analytic on $\mathbb{H}_1$ (see [17], p. 162). □

Proposition 2.1.2. *Let f be a positive integer. Let ε be a real number such that $0 < \varepsilon < 1$. Let K_1 be a compact subset of $\mathbb{H}_1$, and let K_2 be a compact subset of $\mathbb{C}^f$. Then there exists a positive real number $R > 0$ such that*

$$\mathrm{Im}\big(z \cdot {}^{\mathrm{t}}(w+g)(w+g)\big) \ge \varepsilon\, \mathrm{Im}\big(z \cdot {}^{\mathrm{t}}gg\big),$$

or equivalently

$$-\mathrm{Im}\big(z \cdot {}^{\mathrm{t}}(w+g)(w+g)\big) \le -\varepsilon\, \mathrm{Im}\big(z \cdot {}^{\mathrm{t}}gg\big),$$

for $z \in K_1$, $w \in K_2$ and $g \in \mathbb{R}^f$ such that $\|g\| \ge R$.

Proof. Let $M > 0$ be a positive real number such that

$$M \ge |\mathrm{Re}(z)|, |\mathrm{Im}(z)|, \|\mathrm{Re}(w)\|, \|\mathrm{Im}(w)\|$$

for $z \in K_1$ and $w \in K_2$. Let $\delta > 0$ be such that

$$\mathrm{Im}(z) \ge \delta > 0$$

for $z \in K_1$. Let $R > 0$ be such that if $x \in \mathbb{R}$ and $x \ge R$, then

$$0 \le (1-\varepsilon)\delta x^2 - 4M^2 x - 4M^3,$$

or equivalently,

$$4M^2(x+M) \le (1-\varepsilon)\delta x^2.$$

Now let $z \in K_1$, $w \in K_2$, and let $g \in \mathbb{R}^f$ with $\|g\| \ge R$. Write $z = \sigma + it$ for some $\sigma, t \in \mathbb{R}$ with $t > 0$. Also, write $w = a + bi$ with $a, b \in \mathbb{R}^f$. Then calculations show that

$$\begin{aligned} 2 \cdot \mathrm{Im}\big(z\,{}^{\mathrm{t}}wg\big) &= 2t\,{}^{\mathrm{t}}ag + 2\sigma\,{}^{\mathrm{t}}bg, \\ \mathrm{Im}\big(z\,{}^{\mathrm{t}}ww\big) &= \sigma({}^{\mathrm{t}}aa - {}^{\mathrm{t}}bb) - 2t\,{}^{\mathrm{t}}ab. \end{aligned}$$

It follows that

$$\begin{aligned} &- 2 \cdot \mathrm{Im}\big(z\,{}^{\mathrm{t}}wg\big) - \mathrm{Im}\big(z\,{}^{\mathrm{t}}ww\big) \\ &\le |2 \cdot \mathrm{Im}\big(z\,{}^{\mathrm{t}}wg\big)| + |\mathrm{Im}\big(z\,{}^{\mathrm{t}}ww\big)| \\ &\le 2t|\,{}^{\mathrm{t}}ag| + 2|\sigma||\,{}^{\mathrm{t}}bg| + |\sigma||\,{}^{\mathrm{t}}aa| + |\sigma||\,{}^{\mathrm{t}}bb| + 2t|\,{}^{\mathrm{t}}ab| \\ &\le 2t\|a\|\|g\| + 2|\sigma|\|b\|\|g\| + |\sigma|\|a\|^2 + |\sigma|\|b\|^2 + 2t\|a\|\|b\| \\ &\le 2M^2\|g\| + 2M^2\|g\| + M^3 + M^3 + 2M^3 \\ &= 4M^2\|g\| + 4M^3 \\ &= 4M^2(\|g\| + M) \\ &\le (1-\varepsilon)\delta\|g\|^2 \\ &\le (1-\varepsilon)t\|g\|^2 \\ &= (1-\varepsilon)\mathrm{Im}\big(z \cdot {}^{\mathrm{t}}gg\big). \end{aligned}$$

Therefore,

$$\begin{aligned} -2 \cdot \mathrm{Im}\big(z\,{}^{\mathrm{t}}wg\big) - \mathrm{Im}\big(z\,{}^{\mathrm{t}}ww\big) &\le (1-\varepsilon)\mathrm{Im}\big(z \cdot {}^{\mathrm{t}}gg\big) \\ \varepsilon\mathrm{Im}\big(z \cdot {}^{\mathrm{t}}gg\big) &\le \mathrm{Im}\big(z \cdot {}^{\mathrm{t}}gg\big) + 2 \cdot \mathrm{Im}\big(z\,{}^{\mathrm{t}}wg\big) + \mathrm{Im}\big(z\,{}^{\mathrm{t}}ww\big) \\ \varepsilon\mathrm{Im}\big(z \cdot {}^{\mathrm{t}}gg\big) &\le \mathrm{Im}\big(z \cdot {}^{\mathrm{t}}(w+g)(w+g)\big). \end{aligned}$$

This is the desired inequality. □

Corollary 2.1.3. *Let f be a positive integer. Let $A \in \mathrm{M}(f, \mathbb{R})$ be a positive-definite symmetric matrix. Let ε be real number such that $0 < \varepsilon < 1$. Let K_1 be a compact subset of $\mathbb{H}_1$, and let K_2 be a compact subset of $\mathbb{C}^f$. For $x \in \mathbb{C}^f$, define*

$$Q(x) = \frac{1}{2}\,{}^{\mathrm{t}}xAx.$$

Then there exists a positive real number $R > 0$ such that

$$\mathrm{Im}\big(z \cdot Q(w+g)\big) \ge \varepsilon\,\mathrm{Im}\big(z \cdot Q(g)\big),$$

or equivalently,

$$-\mathrm{Im}\big(z \cdot Q(w+g)\big) \le -\varepsilon\,\mathrm{Im}\big(z \cdot Q(g)\big),$$

for $z \in K_1$, $w \in K_2$, and all $g \in \mathbb{R}^f$ such that $\|g\| \ge R$.

Proof. Since A is a positive-definite symmetric matrix, there exists a positive-definite symmetric matrix $B \in \mathrm{M}(f, \mathbb{R})$ such that $A = {}^{\mathrm{t}}BB = BB$ (see (1.7)). The set $B(K_2)$ is a compact subset of $\mathbb{C}^f$. By Proposition 2.1.2 there exists a positive real number $T > 0$ such that

$$\mathrm{Im}\big(z \cdot {}^{\mathrm{t}}(w' + g')(w' + g')\big) \geq \varepsilon \, \mathrm{Im}\big(z \cdot {}^{\mathrm{t}}g'g'\big)$$

for $z \in K_1$, $w' \in B(K_2)$, and $g' \in \mathbb{R}^f$ with $\|g'\| \geq T$. We may regard the matrix B^{-1} as a operator from $\mathbb{R}^f$ to $\mathbb{R}^f$; as such, B^{-1} is bounded. Hence,

$$\|B^{-1}(g)\| \leq \|B^{-1}\| \|g\|$$

for $g \in \mathbb{R}^f$. Define $R = \|B^{-1}\|T$. Let $z \in K_1$, $w \in K_2$ and $g \in \mathbb{R}^f$ with $\|g\| \geq R$. Then $w' = Bw \in B(K_2)$, and:

$$\begin{aligned} \|B^{-1}\big(B(g)\big)\| &\leq \|B^{-1}\| \|B(g)\| \\ \|g\| &\leq \|B^{-1}\| \|B(g)\| \\ R &\leq \|B^{-1}\| \|B(g)\| \\ \|B^{-1}\|^{-1} R &\leq \|B(g)\| \\ T &\leq \|B(g)\|. \end{aligned}$$

Therefore, with $g' = B(g)$,

$$\begin{aligned} \mathrm{Im}\big(z \cdot {}^{\mathrm{t}}(w' + g')(w' + g')\big) &\geq \varepsilon \, \mathrm{Im}\big(z \cdot {}^{\mathrm{t}}g'g'\big) \\ \mathrm{Im}\big(z \cdot {}^{\mathrm{t}}(Bw + Bg)(Bw + Bg)\big) &\geq \varepsilon \, \mathrm{Im}\big(z \cdot {}^{\mathrm{t}}(Bg)Bg\big) \\ \mathrm{Im}\big(z \cdot {}^{\mathrm{t}}(w + g)\, {}^{\mathrm{t}}BB(w + g))\big) &\geq \varepsilon \, \mathrm{Im}\big(z \cdot {}^{\mathrm{t}}g \, {}^{\mathrm{t}}BBg\big) \\ \mathrm{Im}\big(z \cdot {}^{\mathrm{t}}(w + g)A(w + g))\big) &\geq \varepsilon \, \mathrm{Im}\big(z \cdot {}^{\mathrm{t}}gAg\big) \\ \mathrm{Im}\big(z \cdot Q(w + g))\big) &\geq \varepsilon \, \mathrm{Im}\big(z \cdot Q(g)\big) \end{aligned}$$

This completes the proof. □

Proposition 2.1.4. *Let f be a positive integer. Let $A \in \mathrm{M}(f, \mathbb{R})$ be a positive-definite symmetric matrix, and for $x \in \mathbb{R}^f$ let*

$$Q(x) = \frac{1}{2} \, {}^{\mathrm{t}}xAx.$$

For $z \in \mathbb{H}_1$ and $w = {}^{\mathrm{t}}(w_1, \ldots, w_f) \in \mathbb{C}^f$, define

$$\theta(A, z, w) = \sum_{m \in \mathbb{Z}^f} e^{\pi i z \, {}^{\mathrm{t}}(m+w)A(m+w)} = \sum_{m \in \mathbb{Z}^f} e^{2\pi i z Q(m+w)}.$$

Let D be a closed disk in $\mathbb{H}_1$, and let $D_1, \ldots, D_f$ be closed disks in $\mathbb{C}^f$. Then $\theta(A, z, w_1, \ldots, w_f)$ converges absolutely and uniformly on $D \times D_1 \times \cdots \times D_f$. The function $\theta(A, z, w_1, \ldots, w_f)$ on $\mathbb{H}_1 \times \mathbb{C}^f$ is analytic in each variable.

Proof. We apply Corollary 2.1.3 with $\varepsilon = 1/2$, $K_1 = D$ and $K_2 = D_1 \times \cdots \times D_f$. By this corollary, there exists a finite set X of $\mathbb{Z}^f$ such that for $m \in \mathbb{Z}^f - X$, $z \in K_1$ and $w \in K_2$ we have:

$$\begin{aligned} |e^{2\pi i z Q(m+w)}| &= e^{\mathrm{Re}\big(2\pi i z Q(m+w)\big)} \\ &= e^{-2\pi \mathrm{Im}\big(zQ(m+w)\big)} \\ &\le e^{-2\pi\cdot(1/2)\cdot\mathrm{Im}\big(zQ(m)\big)} \\ &= e^{-2\pi Q(m)\mathrm{Im}(z/2)} \\ &\le e^{-2\pi\delta Q(m)} \\ &= |e^{2\pi i(\delta i)Q(m)}|. \end{aligned}$$

Here, $\delta > 0$ is such that $\delta \le \mathrm{Im}(z/2)$ for $z \in D$. By Lemma 2.1.1 the series

$$\sum_{m\in\mathbb{Z}^f} |e^{2\pi i(\delta i)Q(m)}|$$

converges. The Weierstrass M-test (see [17], p. 160) now implies that the series

$$\theta(A, z, w) = \sum_{m\in\mathbb{Z}^f} e^{2\pi i z Q(m+w)}$$

converges absolutely and uniformly on $D \times D_1 \times \cdots \times D_f$. Since for each $m \in \mathbb{Z}^f$ the function on $\mathbb{H}_1 \times \mathbb{C}^f$ defined by $(z, w) \mapsto e^{2\pi i z Q(m+w)}$ is an analytic function in each variable $z, w_1, \ldots, w_f$, and since our series converges absolutely and uniformly on all products of closed disks, it follows that $\theta(A, z, w_1, \ldots, w_f)$ is analytic in each variable (see [17], p. 162). □

2.2 The Poisson summation formula

Let f be a positive integer. Let $g : \mathbb{R}^f \to \mathbb{C}$ be a function, and write $g = u + iv$, where $u, v : \mathbb{R}^f \to \mathbb{R}$ are functions. We say that g is **smooth** if u and v are both infinitely differentiable. Assume that g is smooth. Let $(\alpha_1, \ldots, \alpha_f) \in \mathbb{Z}^f_{>0}$. We define

$$D^\alpha g = \Big(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_f}}{\partial x_f^{\alpha_f}}\Big) g.$$

We say that f is a **Schwartz function** if

$$\sup_{x\in\mathbb{R}^f} |P(x)(D^\alpha)(x)|$$

is finite for all $P(X) = P(X_1, \ldots, X_f) \in \mathbb{C}[X_1, \ldots, X_f]$ and $\alpha \in \mathbb{Z}^f_{>0}$. The set $\mathcal{S}(\mathbb{R}^f)$ of all Schwartz functions is a complex vector space, called the **Schwartz**

space on $\mathbb{R}^f$. If $g \in \mathcal{S}(\mathbb{R}^f)$, then we define the **Fourier transform** of g to be the function $\mathcal{F}g : \mathbb{R}^f \to \mathbb{C}$ defined by

$$(\mathcal{F}g)(x) = \int\limits_{\mathbb{R}^f} g(y) e^{-2\pi i\, {}^{\mathrm{t}}xy}\, dy$$

for $x \in \mathbb{R}^f$. If $g \in \mathcal{S}(\mathbb{R}^f)$, then the integral defining $\mathcal{F}g$ converges absolutely for every $x \in \mathbb{R}^f$. In fact, if $g \in \mathcal{S}(\mathbb{R}^f)$, then $\mathcal{F}g \in \mathcal{S}(\mathbb{R}^f)$, and a number of other properties hold; see, for example, chapter 7 of [23], or chapter 13 of [15].

Lemma 2.2.1. *Let f be a positive integer. Let $A \in \mathrm{M}(f, \mathbb{R})$ be a positive-definite symmetric matrix, and for $x \in \mathbb{R}^f$ let*

$$Q(x) = \frac{1}{2}\, {}^{\mathrm{t}}xAx.$$

Let $w \in \mathbb{C}^f$. The function $g : \mathbb{R}^f \to \mathbb{C}$ defined by

$$g(x) = e^{-2\pi Q(x+w)} = e^{-\pi\, {}^{\mathrm{t}}(x+w)A(x+w)}$$

for $x \in \mathbb{R}^f$ is in the Schwartz space $\mathcal{S}(\mathbb{R}^f)$.

Proof. We begin with some simplifications. Also, there exists a positive-definte symmetric matrix $B \in \mathrm{GL}(f, \mathbb{R})$ such that $A = {}^{\mathrm{t}}BB = BB$ (see (1.7)). The function g is in $\mathcal{S}(\mathbb{R}^f)$ if and only if $g \circ B^{-1}$ in in $\mathcal{S}(\mathbb{R}^f)$. Now

$$\begin{aligned} g(B^{-1}x) &= e^{-\pi\, {}^{\mathrm{t}}(B^{-1}x+w)A(B^{-1}x+w)} \\ &= e^{-\pi\, {}^{\mathrm{t}}(B^{-1}x+w)\, {}^{\mathrm{t}}BB(B^{-1}x+w)} \\ &= e^{-\pi\, {}^{\mathrm{t}}(x+Bw)(x+Bw)}. \end{aligned}$$

It follows that we may assume that $A = 1$. Next, let $w = u + iv$ where $u, v \in \mathbb{R}^f$. Since g is in $\mathcal{S}(\mathbb{R}^f)$ if and only if the function defined by $x \mapsto g(x-u)$ for $x \in \mathbb{R}^f$ is in $\mathcal{S}(\mathbb{R}^f)$, we may also assume that $u = 0$. Now

$$\begin{aligned} g(x) &= e^{-\pi\, {}^{\mathrm{t}}(x+iv)(x+iv)} \\ &= e^{-\pi\, {}^{\mathrm{t}}xx - 2\pi i\, {}^{\mathrm{t}}xv + \pi\, {}^{\mathrm{t}}vv} \\ &= e^{\pi\, {}^{\mathrm{t}}vv} e^{-\pi\, {}^{\mathrm{t}}xx - 2\pi i\, {}^{\mathrm{t}}xv}. \end{aligned}$$

Since $e^{\pi\, {}^{\mathrm{t}}vv}$ is a constant, it suffices to prove that the function $h : \mathbb{R}^f \to \mathbb{C}$ defined by

$$h(x) = e^{-\pi\, {}^{\mathrm{t}}xx - 2\pi i\, {}^{\mathrm{t}}xv}$$

for $x \in \mathbb{R}^f$ is contained in $\mathcal{S}(\mathbb{R}^f)$. Let $\alpha = (\alpha_1, \dots, \alpha_f) \in \mathbb{Z}^f_{\geq 0}$. Then there exists a polynomial $Q_\alpha(X_1, \dots, X_f) \in \mathbb{C}[X_1, \dots, X_f]$ such that

$$(D^\alpha h)(x) = Q_\alpha(x) e^{-\pi\, {}^{\mathrm{t}}xx - 2\pi i\, {}^{\mathrm{t}}xv}$$

for $x \in \mathbb{R}^f$. Hence, if $P(X_1, \dots, X_f) \in \mathbb{C}[X_1, \dots, X_f]$, then

$$\begin{aligned} |P(x)(D^\alpha h)(x)| &= |P(x)Q_\alpha(x)e^{-\pi\, {}^{t}xx - 2\pi i\, {}^{t}xv}| \\ &= |P(x)Q_\alpha(x)e^{-\pi\, {}^{t}xx}| \end{aligned}$$

for $x \in \mathbb{R}^f$. This equality implies that it now suffices to prove that the function defined by $x \mapsto e^{-\pi\, {}^{t}xx}$ for $x \in \mathbb{R}^f$ is contained in $\mathcal{S}(\mathbb{R}^f)$. This is a well-known fact that can be proven using L'Hôpital's rule. □

Lemma 2.2.2. *Let f be a positive integer. If $w \in \mathbb{C}^f$, then*

$$\int_{\mathbb{R}^f} e^{-\pi\, {}^{t}(y+w)(y+w)}\, dy = \int_{\mathbb{R}^f} e^{-\pi\, {}^{t}yy}\, dy.$$

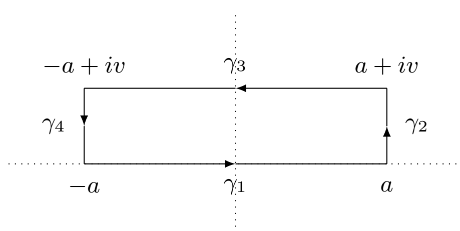
Proof. By Fubini's theorem

$$\begin{aligned} \int_{\mathbb{R}^f} e^{-\pi\, {}^{t}(y+w)(y+w)}\, dy &= \int_{\mathbb{R}^f} e^{-\pi(y_1+w_1)^2 - \dots - \pi(y_f+w_f)^2}\, dy \\ &= \int_{\mathbb{R}^f} e^{-\pi(y_1+w_1)^2} \cdots e^{-\pi(y_f+w_f)^2}\, dy \\ &= (\int_{\mathbb{R}} e^{-\pi(y_1+w_1)^2}\, dy_1) \cdots (\int_{\mathbb{R}} e^{-\pi(y_f+w_f)^2}\, dy_f). \end{aligned}$$

It thus suffices to prove the lemma when $f = 1$. Write $w = u + iv$ with $u, v \in \mathbb{R}$. Then

$$\int_{\mathbb{R}} e^{-\pi(y+u+iv)^2}\, dy = \int_{\mathbb{R}} e^{-\pi(y+iv)^2}\, dy.$$

To complete the proof we will use Cauchy's theorem. Assume, say, $v > 0$. Let $a > 0$, and let $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ be the closed piecewise smooth curve as below:



By Cauchy's theorem (see chapter 2 of [17]) applied to the analytic function $z \mapsto e^{-\pi z^2}$ we have

$$0 = \int_{\gamma} e^{-\pi z^2}\, dz = \int_{\gamma_1} e^{-\pi z^2}\, dz + \int_{\gamma_2} e^{-\pi z^2}\, dz + \int_{\gamma_3} e^{-\pi z^2}\, dz + \int_{\gamma_4} e^{-\pi z^2}\, dz.$$

Using the definitions of these contour integrals, this is:

$$0 = \int_{-a}^{a} e^{-\pi y^2}\,dy + \int\limits_{\gamma_2} e^{-\pi z^2}\,dz - \int_{-a}^{a} e^{-\pi(y+iv)^2}\,dy + \int\limits_{\gamma_4} e^{-\pi z^2}\,dz,$$

or equivalently,

$$\int_{-a}^{a} e^{-\pi(y+iv)^2}\,dy = \int_{-a}^{a} e^{-\pi y^2}\,dy + \int\limits_{\gamma_2} e^{-\pi z^2}\,dz + + \int\limits_{\gamma_4} e^{-\pi z^2}\,dz. \tag{2.1}$$

On the curves γ_2 and γ_4 the function $z \mapsto e^{-\pi z^2}$ is bounded by $e^{-\pi a^2 + \pi v^2}$. Therefore (see Theorem 3 on page 81 of [17]),

$$|\int\limits_{\gamma_2} e^{-\pi z^2}\,dz| \le v e^{-\pi a^2+\pi v^2}, \qquad |\int\limits_{\gamma_3} e^{-\pi z^2}\,dz| \le v e^{-\pi a^2+\pi v^2}.$$

These bounds imply that

$$\lim_{a\to\infty} \int\limits_{\gamma_2} e^{-\pi z^2}\,dz = \lim_{a\to\infty} \int\limits_{\gamma_4} e^{-\pi z^2}\,dz = 0.$$

Letting $a \to \infty$ in (2.1), we thus obtain

$$\int_{-\infty}^{\infty} e^{-\pi(y+iv)^2}\,dy = \int_{-\infty}^{\infty} e^{-\pi y^2}\,dy.$$

This is the desired result. If $v < 0$, then there is a similar proof. □

Lemma 2.2.3. *Let f be a positive integer. Let $A \in \mathrm{M}(f, \mathbb{R})$ be a positive-definite symmetric matrix, and for $x \in \mathbb{R}^f$ let*

$$Q(x) = \frac{1}{2}\,{}^{\mathrm{t}}xAx.$$

Let $w \in \mathbb{C}^f$. Define $g : \mathbb{R}^f \to \mathbb{C}$ by

$$g(x) = e^{-2\pi Q(x+w)} = e^{-\pi\,{}^{\mathrm{t}}(x+w)A(x+w)}$$

for $x \in \mathbb{R}^f$. Then

$$\big(\mathcal{F}g\big)(x) = \det(A)^{-1/2} e^{2\pi i\,{}^{\mathrm{t}}xw} e^{-\pi\,{}^{\mathrm{t}}xA^{-1}x}$$

for $x \in \mathbb{R}^f$.

Proof. There exists positive-definite symmetric matrix $B \in \mathrm{GL}(f, \mathbb{R})$ such that $A = {}^{\mathrm{t}}BB = BB$ (see (1.7)). Let $x \in \mathbb{R}^f$. Then:

$$\big(\mathcal{F}g\big)(x) = \int\limits_{\mathbb{R}^f} \exp(-2\pi Q(y+w)) \exp(-2\pi i\,{}^{\mathrm{t}}xy)\,dy$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^f} \exp\Big(-\pi\big(2Q(y+w) + 2i\,{}^{\mathrm{t}}xy\big)\Big)\,dy \\
&= \int_{\mathbb{R}^f} \exp\Big(-\pi\big({}^{\mathrm{t}}(y+w)A(y+w) + 2i\,{}^{\mathrm{t}}xy\big)\Big)\,dy \\
&= \int_{\mathbb{R}^f} \exp\Big(-\pi\big({}^{\mathrm{t}}(y+w)A(y+w) + 2i\,{}^{\mathrm{t}}yx\big)\Big)\,dy \\
&= \int_{\mathbb{R}^f} \exp\Big(-\pi\big({}^{\mathrm{t}}(y+w)\,{}^{\mathrm{t}}BB(y+w) + 2i\,{}^{\mathrm{t}}(By)\,{}^{\mathrm{t}}B^{-1}x\big)\Big)\,dy \\
&= \int_{\mathbb{R}^f} \exp\Big(-\pi\big({}^{\mathrm{t}}(By+Bw)(By+Bw) + 2i\,{}^{\mathrm{t}}(By)\,{}^{\mathrm{t}}B^{-1}x\big)\Big)\,dy
\end{aligned}
$$

$$
\big(\mathcal{F}g\big)(x) = \det(B)^{-1} \int_{\mathbb{R}^f} \exp\Big(-\pi\big({}^{\mathrm{t}}(y+Bw)(y+Bw) + 2i\,{}^{\mathrm{t}}y\,{}^{\mathrm{t}}B^{-1}x\big)\Big)\,dy.
$$

In the last step we used the formula for a linear change of variables (see Theorem 2.20, (e) on page 50 and section 2.23 of [24]; note also that $\det(A)$ and $\det(B)$ are positive, as A and B are positive-definite symmetric matrices). Now $\det(B)^2 = \det(A)$, so that $\det(A)^{1/2} = \det(B)$. Hence,

$$
\begin{aligned}
&\big(\mathcal{F}g\big)(x) \\
&= \det(A)^{-1/2} \int_{\mathbb{R}^f} \exp\Big(-\pi\big({}^{\mathrm{t}}yy + 2\,{}^{\mathrm{t}}yBw + {}^{\mathrm{t}}(Bw)Bw + 2i\,{}^{\mathrm{t}}y\,{}^{\mathrm{t}}B^{-1}x\big)\Big)\,dy \\
&= \det(A)^{-1/2} \exp(-\pi\,{}^{\mathrm{t}}wAw) \int_{\mathbb{R}^f} \exp\Big(-\pi\big({}^{\mathrm{t}}yy + 2\,{}^{\mathrm{t}}yBw + 2i\,{}^{\mathrm{t}}y\,{}^{\mathrm{t}}B^{-1}x\big)\Big)\,dy \\
&= \det(A)^{-1/2} \exp(-\pi\,{}^{\mathrm{t}}wAw) \int_{\mathbb{R}^f} \exp\Big(-\pi\big({}^{\mathrm{t}}yy + 2\,{}^{\mathrm{t}}y(Bw + i\,{}^{\mathrm{t}}B^{-1}x)\big)\Big)\,dy \\
&= \det(A)^{-1/2} \exp(-\pi\,{}^{\mathrm{t}}wAw) \exp\big(\pi\,{}^{\mathrm{t}}(Bw + i\,{}^{\mathrm{t}}B^{-1}x)(Bw + i\,{}^{\mathrm{t}}B^{-1}x)\big) \\
&\qquad \times \int_{\mathbb{R}^f} \exp\Big(-\pi\big({}^{\mathrm{t}}yy + 2\,{}^{\mathrm{t}}y(Bw + i\,{}^{\mathrm{t}}B^{-1}x) \\
&\qquad\qquad + {}^{\mathrm{t}}(Bw + i\,{}^{\mathrm{t}}B^{-1}x)(Bw + i\,{}^{\mathrm{t}}B^{-1}x)\big)\Big)\,dy \\
&= \det(A)^{-1/2} \exp\big(-\pi\,{}^{\mathrm{t}}wAw\big) \exp\big(\pi\,{}^{\mathrm{t}}wAw + 2\pi i\,{}^{\mathrm{t}}xw - \pi\,{}^{\mathrm{t}}xA^{-1}x\big) \\
&\qquad \times \int_{\mathbb{R}^f} \exp\Big(-\pi\,{}^{\mathrm{t}}(y + Bw + i\,{}^{\mathrm{t}}B^{-1}x)(y + Bw + i\,{}^{\mathrm{t}}B^{-1}x)\Big)\,dy.
\end{aligned}
$$

Applying now Lemma 2.2.2, we obtain:

$$
\big(\mathcal{F}g\big)(x) = \det(A)^{-1/2} \exp\big(2\pi i\,{}^{\mathrm{t}}xw - \pi\,{}^{\mathrm{t}}xA^{-1}x\big) \int_{\mathbb{R}^f} \exp\big(-\pi\,{}^{\mathrm{t}}yy\big)\,dy
$$

$$\big(\mathcal{F}g\big)(x) = \det(A)^{-1/2} \exp\big(2\pi i\,{}^{\mathrm{t}}xw - \pi\,{}^{\mathrm{t}}xA^{-1}x\big).$$

Here, we have used the well-known classical fact that

$$\int_{\mathbb{R}^f} \exp\big(-\pi\,{}^{\mathrm{t}}yy\big)\,dy = 1.$$

This completes the calculation. □

Theorem 2.2.4 (Poisson summation formula)**.** *Let f be a positive integer. Let $g \in \mathcal{S}(\mathbb{R}^f)$. Then*

$$\sum_{m\in\mathbb{Z}^f} g(m) = \sum_{m\in\mathbb{Z}^f} (\mathcal{F}g)(m),$$

where both series converge absolutely.

Proof. See page 249 of [15]. □

Lemma 2.2.5. *Let f be a positive integer. Let $A \in \mathrm{M}(f,\mathbb{R})$ be a positive-definite symmetric matrix. Let ε be real number such that $0 < \varepsilon < 1$. Let K_1 be a compact subset of $\mathbb{H}_1$, and let K_2 be a compact subset of $\mathbb{C}^f$. For $x \in \mathbb{C}^f$, define*

$$Q(x) = \frac{1}{2}\,{}^{\mathrm{t}}xAx.$$

Then there exists a positive real number $R > 0$ such that

$$-\mathrm{Im}\big((-1/z)\,{}^{\mathrm{t}}gA^{-1}g + 2\,{}^{\mathrm{t}}gw\big) \le -\varepsilon\,\mathrm{Im}\big((-1/z)\cdot{}^{\mathrm{t}}gA^{-1}g\big),$$

for $z \in K_1$, $w \in K_2$, and all $g \in \mathbb{R}^f$ such that $\|g\| \ge R$.

Proof. This proof is similar to the proof of Proposition 2.1.2. First of all, there exists a positive-definite symmetric matrix $B \in \mathrm{GL}(f,\mathbb{R})$ such that $A = {}^{\mathrm{t}}BB$ (see (1.7)). If $m \in \mathbb{R}^f$, then we note that

$$\begin{aligned}
{}^{\mathrm{t}}gA^{-1}g &= |\,{}^{\mathrm{t}}gA^{-1}g| \\
&= |\,{}^{\mathrm{t}}gB^{-1}\,{}^{\mathrm{t}}B^{-1}g| \\
&= |\,{}^{\mathrm{t}}({}^{\mathrm{t}}B^{-1}g)\cdot({}^{\mathrm{t}}B^{-1}g)| \\
&= \|\,{}^{\mathrm{t}}B^{-1}g\|^2 \\
&= \Big(\frac{1}{\|\,{}^{\mathrm{t}}B\|}\cdot\|\,{}^{\mathrm{t}}B\|\|\,{}^{\mathrm{t}}B^{-1}g\|\Big)^2 \\
&\ge \Big(\frac{1}{\|\,{}^{\mathrm{t}}B\|}\cdot\|g\|\Big)^2 \\
&= \frac{1}{\|\,{}^{\mathrm{t}}B\|^2}\cdot\|g\|^2.
\end{aligned}$$

Next, let $M > 0$ be such that

$$|\mathrm{Im}(-1/z)|, |\mathrm{Im}(w)| \le M$$

for $z \in K_1$ and $w \in K_2$; note that the set consisting of $-1/z$ for $z \in K_1$ is also a compact subset of $\mathbb{H}_1$. Let $\delta > 0$ be such that

$$\mathrm{Im}(-1/z) \geq \delta > 0.$$

Let $R > 0$ be such that if $x \geq R$, then

$$\delta(1-\varepsilon) \cdot \frac{1}{\| {}^{\mathrm{t}}B\|^2} \cdot x^2 \geq 2Mx.$$

Now $z \in K_1$, $w \in K_2$, and $g \in \mathbb{R}^f$ with $\|g\| \geq R$. Write $-1/z = \sigma + it$ for $\sigma, t \in \mathbb{R}$ and $w = a + bi$ for $a, b \in \mathbb{R}^f$. We have

$$\begin{aligned} -\mathrm{Im}(2\,{}^{\mathrm{t}}gw) &= -2\,{}^{\mathrm{t}}gb \\ &\leq 2|\,{}^{\mathrm{t}}gb| \\ &\leq 2M\|g\|. \end{aligned}$$

On the other hand,

$$\begin{aligned} (1-\varepsilon) \cdot \mathrm{Im}\big((-1/z)\,{}^{\mathrm{t}}gA^{-1}g\big) &= t \cdot {}^{\mathrm{t}}gA^{-1}g \\ &\geq \delta(1-\varepsilon) \cdot \frac{1}{\| {}^{\mathrm{t}}B\|^2} \cdot \|g\|^2 \end{aligned}$$

It follows that

$$\begin{aligned} -\mathrm{Im}(2\,{}^{\mathrm{t}}gw) &\leq (1-\varepsilon) \cdot \mathrm{Im}\big((-1/z)\,{}^{\mathrm{t}}gA^{-1}g\big) \\ -\mathrm{Im}\big((-1/z)\,{}^{\mathrm{t}}gA^{-1}g + 2\,{}^{\mathrm{t}}gw\big) &\leq -\varepsilon \cdot \mathrm{Im}\big((-1/z)\,{}^{\mathrm{t}}gA^{-1}g\big). \end{aligned}$$

This is the desired result. □

Theorem 2.2.6. *Let f be a positive integer. Assume that f is even, and set*

$$k = \frac{f}{2}.$$

Let $A \in \mathrm{M}(f, \mathbb{R})$ be a positive-definite symmetric matrix, and for $x \in \mathbb{R}^f$ let

$$Q_A(x) = \frac{1}{2}\,{}^{\mathrm{t}}xAx, \qquad Q_{A^{-1}}(x) = \frac{1}{2}\,{}^{\mathrm{t}}xA^{-1}x.$$

The series

$$\sum_{m \in \mathbb{Z}^f} e^{\pi i(-1/z)\,{}^{\mathrm{t}}mA^{-1}m + 2\pi i\,{}^{\mathrm{t}}mw}$$

converges absolutely and uniformly for $(z, w) \in D \times D_1 \times \cdots \times D_f$, where D is any closed disk in $\mathbb{H}_1$, and $D_1, \ldots, D_f$ are any closed disks in $\mathbb{C}^f$. The function that sends $(z, w) \in \mathbb{H}_1 \times \mathbb{C}^f$ to this series is analytic in each variable. We have

$$\theta(A, z, w) = \frac{i^k}{z^k \sqrt{\det(A)}} \sum_{m \in \mathbb{Z}^f} e^{\pi i(-1/z)\,{}^{\mathrm{t}}mA^{-1}m + 2\pi i\,{}^{\mathrm{t}}mw}$$

for $z \in \mathbb{H}_1$ and $w \in \mathbb{C}^f$.

Proof. We apply Lemma 2.2.5 with $\varepsilon = 1/2$, $K_1 = D$, and $K_2 = D_1 \times \cdots \times D_f$. By this corollary, there exists a finite set X of $\mathbb{Z}^f$ such that for $m \in \mathbb{Z}^f - X$, $z \in K_1$ and $w \in K_2$ we have:

$$\begin{aligned}
|e^{\pi i(-1/z)\,{}^{\mathrm{t}}mA^{-1}m+2\pi i\,{}^{\mathrm{t}}mw}| &= e^{-\pi \mathrm{Im}\left((-1/z)\,{}^{\mathrm{t}}mA^{-1}m+2\,{}^{\mathrm{t}}mw\right)}\\
&= e^{-\pi\cdot(1/2)\cdot\mathrm{Im}\left((-1/z)\cdot{}^{\mathrm{t}}mA^{-1}m\right)}\\
&\le e^{-\pi\cdot\mathrm{Im}\left((-1/z)\cdot Q_{A^{-1}}(m)\right)}\\
&= e^{-2\pi Q_{A^{-1}}(m)\cdot\mathrm{Im}(-1/(2z))}\\
&\le e^{-2\pi\delta Q_{A^{-1}}(m)}\\
&= |e^{2\pi i(\delta i)Q_{A^{-1}}(m)}|.
\end{aligned}$$

Here, $\delta > 0$ is such that $\delta \le \mathrm{Im}(-1/(2z))$ for $z \in D$. By Lemma 2.1.1 the series

$$\sum_{m\in\mathbb{Z}^f} |e^{2\pi i(\delta i)Q_{A^{-1}}(m)}|$$

converges. The Weierstrass M-test (see [17], p. 160) now implies that the series

$$\sum_{m\in\mathbb{Z}^f} e^{\pi i(-1/z)\,{}^{\mathrm{t}}mA^{-1}m+2\pi i\,{}^{\mathrm{t}}mw}$$

converges absolutely and uniformly on $D\times D_1\times\cdots\times D_f$. Since for each $m \in \mathbb{Z}^f$ the function on $\mathbb{H}_1 \times \mathbb{C}^f$ defined by $(z, w) \mapsto e^{\pi i(-1/z)\,{}^{\mathrm{t}}mA^{-1}m+2\pi i\,{}^{\mathrm{t}}mw}$ is an analytic function in each variable $z, w_1, \ldots, w_f$, and since our series converges absolutely and uniformly on all products of closed disks, it follows that this series is analytic in each variable (see [17], p. 162).

Now fix $w \in \mathbb{C}^f$. Define $g : \mathbb{R}^f \to \mathbb{C}$ by

$$g(x) = e^{-2\pi Q_A(x+w)} = e^{-\pi\,{}^{\mathrm{t}}(x+w)A(x+w)}$$

for $x \in \mathbb{R}^f$. Then by Lemma 2.2.3,

$$\big(\mathcal{F}g\big)(x) = \det(A)^{-1/2}e^{-\pi\,{}^{\mathrm{t}}xA^{-1}x+2\pi i\,{}^{\mathrm{t}}xw}$$

for $x \in \mathbb{R}^f$. By Theorem 2.2.4, the Poisson summation formula, we have:

$$\begin{aligned}
\sum_{m\in\mathbb{Z}^f} e^{-2\pi Q_A(m+w)} &= \sum_{m\in\mathbb{Z}^f} \det(A)^{-1/2}e^{-\pi\,{}^{\mathrm{t}}xA^{-1}x+2\pi i\,{}^{\mathrm{t}}xw}\\
\sum_{m\in\mathbb{Z}^f} e^{2\pi i\cdot i\cdot Q_A(m+w)} &= \det(A)^{-1/2}\sum_{m\in\mathbb{Z}^f} e^{\pi i\cdot(-1/i)\,{}^{\mathrm{t}}xA^{-1}x+2\pi i\,{}^{\mathrm{t}}xw}.
\end{aligned}$$

Let $t > 0$. Replacing A by tA, we obtain similarly,

$$\sum_{m\in\mathbb{Z}^f} e^{2\pi i\cdot it\cdot Q_A(m+w)} = \frac{1}{\det(tA)^{1/2}}\sum_{m\in\mathbb{Z}^f} e^{\pi i\cdot(-1/(it))\,{}^{\mathrm{t}}xA^{-1}x+2\pi i\,{}^{\mathrm{t}}xw}$$

$$= \frac{i^k}{(it)^k\sqrt{\det(A)}} \sum_{m\in\mathbb{Z}^f} e^{\pi i\cdot(-1/(it))\,{}^{\mathrm{t}}xA^{-1}x+2\pi i\,{}^{\mathrm{t}}xw}$$

$$\sum_{m\in\mathbb{Z}^f} e^{2\pi i\cdot z\cdot Q_A(m+w)} = \frac{i^k}{z^k\sqrt{\det(A)}} \sum_{m\in\mathbb{Z}^f} e^{\pi i\cdot(-1/z)\,{}^{\mathrm{t}}xA^{-1}x+2\pi i\,{}^{\mathrm{t}}xw}$$

$$\theta(A,z,w) = \frac{i^k}{z^k\sqrt{\det(A)}} \sum_{m\in\mathbb{Z}^f} e^{\pi i\cdot(-1/z)\,{}^{\mathrm{t}}xA^{-1}x+2\pi i\,{}^{\mathrm{t}}xw},$$

for $z \in \mathbb{H}_1$ of the form $z = it$ for $t > 0$. Since both sides of the last equation are analytic functions in z for $z \in \mathbb{H}_1$, the Identity Principle (see p. 307 of [17]) implies that this equality holds for all $z \in \mathbb{H}_1$. □

2.3 Differential operators

Let f be a positive integer. Let $H(\mathbb{C}^f)$ be the $\mathbb{C}$-algebra of all functions

$$F : \mathbb{C}^f \to \mathbb{C}$$

that are analytic in each variable. Let $\ell = {}^{\mathrm{t}}(\ell_1,\dots,\ell_f) \in \mathbb{C}^f$. We define a $\mathbb{C}$ linear map

$$L_\ell : H(\mathbb{C}^f) \longrightarrow H(\mathbb{C}^f)$$

by

$$L_\ell(F) = \sum_{i=1}^{f} \ell_i \frac{\partial F}{\partial w_i}.$$

Lemma 2.3.1. *Let f be a positive integer, and let $\ell \in \mathbb{C}^f$. Then*

$$L_\ell\big(F_1\cdot F_2\big) = L_\ell(F_1)\cdot F_2 + F_1\cdot L_\ell(F_2)$$

for $F_1, F_2 \in H(\mathbb{C}^f)$. Also,

$$L_\ell(e^F) = L_\ell(F)\cdot e^F$$

for $F \in H(\mathbb{C}^f)$.

Proof. Let $F_1, F_2 \in H(\mathbb{C}^f)$. We have

$$\begin{aligned} L_\ell\big(F_1\cdot F_2\big) &= \sum_{i=1}^{f} \ell_i \frac{\partial}{\partial w_i}\big(F_1\cdot F_2\big) \\ &= \sum_{i=1}^{f} \ell_i \big(\frac{\partial F_1}{\partial w_i}\cdot F_2 + F_1\cdot\frac{\partial F_2}{\partial w_i}\big) \\ &= \sum_{i=1}^{f} \ell_i \frac{\partial F_1}{\partial w_i}\cdot F_2 + \sum_{i=1}^{f} \ell_i F_1\cdot\frac{\partial F_2}{\partial w_i} \end{aligned}$$

$$= \Big(\sum_{i=1}^{f} \ell_i \frac{\partial F_1}{\partial w_i}\Big) \cdot F_2 + F_1 \cdot \Big(\sum_{i=1}^{f} \ell_i \frac{\partial F_2}{\partial w_i}\Big)$$
$$= L_\ell(F_1) \cdot F_2 + F_1 \cdot L_\ell(F_2).$$

Let $F \in H(\mathbb{C}^f)$. Then:

$$\begin{aligned} L_\ell(e^F) &= \sum_{i=1}^{f} \ell_i \frac{\partial}{\partial w_i}(e^F) \\ &= \sum_{i=1}^{f} \ell_i \frac{\partial F}{\partial w_i} \cdot e^F \\ &= \Big(\sum_{i=1}^{f} \ell_i \frac{\partial F}{\partial w_i}\Big) \cdot e^F \\ &= L_\ell(F) \cdot e^F. \end{aligned}$$

This completes the proof. □

Lemma 2.3.2. *Let f be a positive integer and let $A \in \mathrm{M}(f, \mathbb{R})$ be a positive-definite symmetric matrix. Assume that $\ell \in \mathbb{C}^f$ is such that*

$${}^{\mathrm{t}}\ell A \ell = 0.$$

Let $m \in \mathbb{C}^f$ be fixed, and let r be a non-negative integer. Then:

$$\begin{aligned} L_\ell\big({}^{\mathrm{t}}(m+w)A(m+w)\big) &= 2\,{}^{\mathrm{t}}\ell A(m+w), \\ L_\ell\Big(\big({}^{\mathrm{t}}\ell A(m+w)\big)^r\Big) &= 0, \\ L_\ell\big({}^{\mathrm{t}}mw\big) &= {}^{\mathrm{t}}\ell m. \end{aligned}$$

Here, all functions are variables in $w \in \mathbb{C}^f$.

Proof. We have

$$\begin{aligned} &L_\ell\big({}^{\mathrm{t}}(m+w)A(m+w)\big) \\ &= L_\ell\Big(\sum_{i,j=1}^{f} a_{ij}(m_i+w_i)(m_j+w_j)\Big) \\ &= \sum_{i,j=1}^{f} a_{ij} L_\ell\big((m_i+w_i)(m_j+w_j)\big) \\ &= \sum_{i,j=1}^{f} a_{ij}\Big(L_\ell\big((m_i+w_i)\big)(m_j+w_j) + (m_i+w_i)L_\ell\big((m_j+w_j)\big)\Big) \\ &= \sum_{i,j=1}^{f} a_{ij}\big(\ell_i(m_j+w_j) + \ell_j(m_i+w_i)\big) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i,j=1}^{f} a_{ij}\ell_i(m_j+w_j) + \sum_{i,j=1}^{f} a_{ij}\ell_j(m_i+w_i)\\
&= {}^{\mathrm{t}}\ell A(m+w) + {}^{\mathrm{t}}(m+w)A\ell\\
&= 2\,{}^{\mathrm{t}}\ell A(m+w).
\end{aligned}$$

We prove the second assertion by induction on r. The assertion is clear if $r=0$. For $r=1$, we have:

$$\begin{aligned}
L_\ell\big({}^{\mathrm{t}}lA(m+w)\big) &= L_\ell\big(\sum_{i,j=1}^{f} a_{ij}\ell_i(m_j+w_j)\big)\\
&= \sum_{i,j=1}^{f} a_{ij}\ell_i L_\ell(m_j+w_j)\\
&= \sum_{i,j=1}^{f} a_{ij}\ell_i\ell_j\\
&= {}^{\mathrm{t}}\ell A\ell\\
&= 0.
\end{aligned}$$

Assume now that $r \geq 2$ and that the claim holds for the non-negative integers $0,1,\dots,r-1$. Then

$$\begin{aligned}
&L_\ell\Big(\big({}^{\mathrm{t}}\ell A(m+w)\big)^r\Big)\\
&= L_\ell\Big({}^{\mathrm{t}}\ell A(m+w)\cdot\big({}^{\mathrm{t}}\ell A(m+w)\big)^{r-1}\Big)\\
&= L_\ell\big({}^{\mathrm{t}}\ell A(m+w)\big)\cdot\big({}^{\mathrm{t}}\ell A(m+w)\big)^{r-1} + {}^{\mathrm{t}}\ell A(m+w)\cdot L_\ell\Big(\big({}^{\mathrm{t}}\ell A(m+w)\big)^{r-1}\Big)\\
&= 0\cdot\big({}^{\mathrm{t}}\ell A(m+w)\big)^{r-1} + {}^{\mathrm{t}}\ell A(m+w)\cdot 0\\
&= 0.
\end{aligned}$$

The final assertion of the lemma is straightforward. □

Proposition 2.3.3. *Let f be a positive even integer, and let $A \in \mathrm{M}(f,\mathbb{R})$ be a positive-definite symmetric matrix. Define*

$$k = \frac{f}{2}.$$

Let $\ell \in \mathbb{C}^f$ be such that

$${}^{\mathrm{t}}\ell A\ell = 0.$$

For every non-negative integer r the series

$$\sum_{m\in\mathbb{Z}^f} \big({}^{\mathrm{t}}\ell A(m+w)\big)^r e^{\pi i z\,{}^{\mathrm{t}}(m+w)A(m+w)}$$

and

$$\sum_{m\in\mathbb{Z}^f} \big({}^{\mathrm{t}}\ell m\big)^r e^{\pi i(-1/z)\,{}^{\mathrm{t}}mA^{-1}m+2\pi i\,{}^{\mathrm{t}}mw}$$

converge absolutely and uniformly for $(z,w)\in D\times D_1\times\cdots\times D_f$, where D is any closed disk in $\mathbb{H}_1$, and $D_1,\dots,D_f$ are any closed disks in $\mathbb{C}^f$. Both series define functions on $\mathbb{H}_1\times\mathbb{C}^f$ that are analytic in each variable. Moreover,

$$\sum_{m\in\mathbb{Z}^f} \big({}^{\mathrm{t}}\ell A(m+w)\big)^r e^{\pi iz\,{}^{\mathrm{t}}(m+w)A(m+w)} = \frac{i^k}{z^{k+r}\sqrt{\det(A)}}\sum_{m\in\mathbb{Z}^f} \big({}^{\mathrm{t}}\ell m\big)^r e^{\pi i(-1/z)\,{}^{\mathrm{t}}mA^{-1}m+2\pi i\,{}^{\mathrm{t}}mw}.$$

Proof. We prove this result by induction on r. The case $r=0$ is Theorem 2.2.6. Assume the claims hold for r; we will prove that they hold for $r+1$. Let

$$S_1(z,w)=\sum_{m\in\mathbb{Z}^f} \big({}^{\mathrm{t}}\ell A(m+w)\big)^r e^{\pi iz\,{}^{\mathrm{t}}(m+w)A(m+w)}$$

for $s\in\mathbb{H}_1$ and $w\in\mathbb{C}^f$. Let D be any closed disk in $\mathbb{H}_1$, and let $D_1,\dots,D_f$ be any closed disks in $\mathbb{C}^f$. Since the above series converge absolutely and uniformly on $D\times D_1\times\cdots\times D_f$ to S_1, and since the terms of this series are analytic functions in each of the variables $z,w_1,\dots,w_f$, the series

$$\sum_{m\in\mathbb{Z}^f} L_\ell\Big(\big({}^{\mathrm{t}}\ell A(m+w)\big)^r e^{\pi iz\,{}^{\mathrm{t}}(m+w)A(m+w)}\Big)$$

converges absolutely and uniformly on $D\times D_1\times\cdots\times D_f$ to the analytic function $L_\ell S_1$ (see p. 162 of [17]). We have for $z\in\mathbb{H}_1$ and $w\in\mathbb{C}^f$, using Lemma 2.3.1 and Lemma 2.3.2,

$$\begin{aligned}
&(L_\ell S_1)(z,w)\\
&=\sum_{m\in\mathbb{Z}^f} L_\ell\Big(\big({}^{\mathrm{t}}\ell A(m+w)\big)^r e^{\pi iz\,{}^{\mathrm{t}}(m+w)A(m+w)}\Big)\\
&=\sum_{m\in\mathbb{Z}^f} L_\ell\Big(\big({}^{\mathrm{t}}\ell A(m+w)\big)^r\Big) e^{\pi iz\,{}^{\mathrm{t}}(m+w)A(m+w)}\\
&\qquad+\big({}^{\mathrm{t}}\ell A(m+w)\big)^r L_\ell\big(e^{\pi iz\,{}^{\mathrm{t}}(m+w)A(m+w)}\big)\\
&=\sum_{m\in\mathbb{Z}^f} \big({}^{\mathrm{t}}\ell A(m+w)\big)^r\cdot L_\ell\big(\pi iz\,{}^{\mathrm{t}}(m+w)A(m+w)\big)\cdot e^{\pi iz\,{}^{\mathrm{t}}(m+w)A(m+w)}\\
&=2\pi iz\sum_{m\in\mathbb{Z}^f} \big({}^{\mathrm{t}}\ell A(m+w)\big)^{r+1} e^{\pi iz\,{}^{\mathrm{t}}(m+w)A(m+w)}.
\end{aligned}$$

Next, for $z\in\mathbb{H}_1$ and $w\in\mathbb{C}^f$, let

$$S_2(z,w)=\frac{i^k}{z^{k+r}\sqrt{\det(A)}}\sum_{m\in\mathbb{Z}^f} \big({}^{\mathrm{t}}\ell m\big)^r e^{\pi i(-1/z)\,{}^{\mathrm{t}}mA^{-1}m+2\pi i\,{}^{\mathrm{t}}mw}.$$

Comments similar to those above apply to S_2 and the series defining S_2. For S_2 we have for $z \in \mathbb{H}_1$ and $w \in \mathbb{C}^f$, using Lemma 2.3.1 and Lemma 2.3.2,

$$\begin{aligned}
&(L_\ell S_2)(z,w)\\
&= \frac{i^k}{z^{k+r}\sqrt{\det(A)}} \sum_{m\in\mathbb{Z}^f} L_\ell\Big(\big({}^{\mathrm{t}}\ell m\big)^r e^{\pi i(-1/z)\,{}^{\mathrm{t}}mA^{-1}m + 2\pi i\,{}^{\mathrm{t}}mw}\Big)\\
&= \frac{i^k}{z^{k+r}\sqrt{\det(A)}} \sum_{m\in\mathbb{Z}^f} \big({}^{\mathrm{t}}\ell m\big)^r L_\ell\Big(e^{\pi i(-1/z)\,{}^{\mathrm{t}}mA^{-1}m + 2\pi i\,{}^{\mathrm{t}}mw}\Big)\\
&= \frac{i^k}{z^{k+r}\sqrt{\det(A)}} \sum_{m\in\mathbb{Z}^f} \big({}^{\mathrm{t}}\ell m\big)^r L_\ell\big(\pi i(-1/z)\,{}^{\mathrm{t}}mA^{-1}m + 2\pi i\,{}^{\mathrm{t}}mw\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \times e^{\pi i(-1/z)\,{}^{\mathrm{t}}mA^{-1}m + 2\pi i\,{}^{\mathrm{t}}mw}\\
&= 2\pi i\cdot \frac{i^k}{z^{k+r}\sqrt{\det(A)}} \sum_{m\in\mathbb{Z}^f} \big({}^{\mathrm{t}}\ell m\big)^r \cdot {}^{\mathrm{t}}\ell m \cdot e^{\pi i(-1/z)\,{}^{\mathrm{t}}mA^{-1}m + 2\pi i\,{}^{\mathrm{t}}mw}\\
&= 2\pi i\cdot \frac{i^k}{z^{k+r}\sqrt{\det(A)}} \sum_{m\in\mathbb{Z}^f} \big({}^{\mathrm{t}}\ell m\big)^{r+1} \cdot e^{\pi i(-1/z)\,{}^{\mathrm{t}}mA^{-1}m + 2\pi i\,{}^{\mathrm{t}}mw}.
\end{aligned}$$

Since $(L_\ell S_1)(z,w) = (L_\ell S_2)(z,w)$, we have for $(z,w) \in \mathbb{H}_1 \times \mathbb{C}^f$,

$$\begin{aligned}
&2\pi i z \sum_{m\in\mathbb{Z}^f} \big({}^{\mathrm{t}}\ell A(m+w)\big)^{r+1} e^{\pi i z\,{}^{\mathrm{t}}(m+w)A(m+w)}\\
&\qquad = 2\pi i\cdot \frac{i^k}{z^{k+r}\sqrt{\det(A)}} \sum_{m\in\mathbb{Z}^f} \big({}^{\mathrm{t}}\ell m\big)^{r+1} \cdot e^{\pi i(-1/z)\,{}^{\mathrm{t}}mA^{-1}m + 2\pi i\,{}^{\mathrm{t}}mw},
\end{aligned}$$

or equivalently,

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}^f} \big({}^{\mathrm{t}}\ell A(m+w)\big)^{r+1} e^{\pi i z\,{}^{\mathrm{t}}(m+w)A(m+w)}\\
&\qquad = \frac{i^k}{z^{k+r+1}\sqrt{\det(A)}} \sum_{m\in\mathbb{Z}^f} \big({}^{\mathrm{t}}\ell m\big)^{r+1} \cdot e^{\pi i(-1/z)\,{}^{\mathrm{t}}mA^{-1}m + 2\pi i\,{}^{\mathrm{t}}mw}.
\end{aligned}$$

By induction, the proof is complete. □

Let f be a positive even integer, and let $A \in \mathrm{M}(f,\mathbb{R})$ be a positive-definite symmetric matrix. For r a non-negative integer, we let $\mathcal{H}_r(A)$ be the $\mathbb{C}$ vector space spanned by the polynomials in $w_1,\dots,w_f$ given by

$$({}^{\mathrm{t}}\ell A w)^r$$

where $w = {}^{\mathrm{t}}(w_1,\dots,w_f)$ and $\ell \in \mathbb{C}^f$ with ${}^{\mathrm{t}}\ell A\ell = 0$. The elements of $\mathcal{H}_r(A)$ are homogeneous polynomials of degree r, and are called **spherical functions** with respect to A.

2.4 A space of theta series

Lemma 2.4.1. *Let f be a positive even integer, and define $k = f/2$. Let $A \in \mathrm{M}(f,\mathbb{Z})$ be an even symmetric positive-definite matrix, and let N be the level of A. Define the quadratic form $Q(x)$ in f variables by*

$$Q(x) = \frac{1}{2}\,{}^{\mathrm{t}}xAx.$$

Let r be a non-negative integer, and let $P \in \mathcal{H}_r(A)$. Let $h \in \mathbb{Z}^f$ be such that

$$Ah \equiv 0 \pmod{N}.$$

For $z \in \mathbb{H}_1$ define

$$\theta(A,P,h,z) = \sum_{\substack{n\in\mathbb{Z}^f\\ n\equiv h \pmod{N}}} P(n)e^{2\pi i z\frac{Q(n)}{N^2}}.$$

This series converges absolutely and uniformly on closed disks in $\mathbb{H}_1$ to an analytic function. If $h, h' \in \mathbb{Z}^f$ are such that $Ah \equiv 0 \pmod{N}$, $Ah' \equiv 0 \pmod{N}$, and $h \equiv h' \pmod{N}$, then

$$\theta(A,P,h,z) = \theta(A,P,h',z), \tag{2.2}$$
$$\theta(A,P,h,z) = (-1)^r\theta(A,P,-h,z), \tag{2.3}$$

for $z \in \mathbb{H}_1$. For $h \in \mathbb{Z}^f$ with $Ah \equiv 0 \pmod{N}$ and $P \in \mathcal{H}_r(A)$ we have

$$\theta(A,P,h,z)\,\big|_{k+r}\begin{bmatrix} & 1\\ -1 & \end{bmatrix} = \frac{i^k}{\sqrt{\det(A)}}\sum_{\substack{g \pmod{N}\\ Ag\equiv 0 \pmod{N}}} e^{2\pi i\frac{{}^{\mathrm{t}}gAh}{N^2}}\cdot\theta(A,P,g,z) \tag{2.4}$$

and

$$\theta(A,P,h,z)\,\big|_{k+r}\begin{bmatrix} 1 & b\\ & 1\end{bmatrix} = e^{2\pi i b\frac{Q(h)}{N^2}}\theta(A,P,h,z) \tag{2.5}$$

for $z \in \mathbb{H}_1$. Let $P \in \mathcal{H}_r(A)$, and let $V(A,P)$ be the $\mathbb{C}$ vector space spanned by the functions $\theta(A,P,h,\cdot)$ for $h \in \mathbb{Z}^f$ with $Ah \equiv 0 \pmod{N}$. The $\mathbb{C}$ vector space $V(A,P)$ is a right $\mathrm{SL}(2,\mathbb{Z})$ module under the $|_{k+r}$ action.

Proof. The assertions (2.2) and (2.3) follow from the involved definitions.

To prove (2.4) and (2.5), let $h \in \mathbb{Z}^f$ with $Ah \equiv 0 \pmod{N}$ and $P \in \mathcal{H}_r(A)$. Using the definition of $\mathcal{H}_r(A)$, it is clear that may assume that the polynomial P is of the form

$$P(w) = ({}^{\mathrm{t}}\ell Aw)^r.$$

for some $\ell \in \mathbb{C}^f$ such that ${}^{\mathrm{t}}\ell A\ell = 0$. We recall from Proposition 2.3.3 that

$$\sum_{m\in\mathbb{Z}^f} \big({}^{\mathrm{t}}\ell A(m+w)\big)^r e^{\pi i z\,{}^{\mathrm{t}}(m+w)A(m+w)}$$
$$= \frac{i^k}{z^{k+r}\sqrt{\det(A)}} \sum_{m\in\mathbb{Z}^f} \big({}^{\mathrm{t}}\ell m\big)^r e^{\pi i(-1/z)\,{}^{\mathrm{t}}mA^{-1}m + 2\pi i\,{}^{\mathrm{t}}mw}.$$

for $z\in\mathbb{H}_1$ and $w\in\mathbb{C}^f$. Replacing w with h/N, we obtain:

$$\sum_{m\in\mathbb{Z}^f} \big({}^{\mathrm{t}}\ell A(m+\frac{h}{N})\big)^r e^{\pi i z\,{}^{\mathrm{t}}(m+\frac{h}{N})A(m+\frac{h}{N})}$$
$$= \frac{i^k}{z^{k+r}\sqrt{\det(A)}} \sum_{m\in\mathbb{Z}^f} \big({}^{\mathrm{t}}\ell m\big)^r e^{\pi i(-1/z)\,{}^{\mathrm{t}}mA^{-1}m + 2\pi i\frac{{}^{\mathrm{t}}mh}{N}}.$$

Let $m\in\mathbb{Z}^f$. Then

$$\begin{aligned} m+\frac{h}{N} &= \frac{h+mN}{N} \\ &= \frac{n}{N}, \end{aligned}$$

where $n=h+mN$. The map

$$\mathbb{Z}^f \xrightarrow{\sim} \{n\in\mathbb{Z}^f : n\equiv h \pmod N\}$$

defined by $m\mapsto n=h+mN$ is a bijection, the inverse of which is given by $n\mapsto(n-h)/N$. It follows that

$$N^{-r} \sum_{\substack{n\in\mathbb{Z}^f\\ n\equiv h \pmod N}} \big({}^{\mathrm{t}}\ell An\big)^r e^{\pi i z\frac{{}^{\mathrm{t}}nAn}{N^2}}$$
$$= \frac{i^k}{z^{k+r}\sqrt{\det(A)}} \sum_{m\in\mathbb{Z}^f} \big({}^{\mathrm{t}}\ell m\big)^r e^{\pi i(-1/z)\,{}^{\mathrm{t}}mA^{-1}m + 2\pi i\frac{{}^{\mathrm{t}}mh}{N}}.$$

Next, consider the map

$$\mathbb{Z}^f \xrightarrow{\sim} \{g\in\mathbb{Z}^f : Ag\equiv 0 \pmod N\}$$

defined by $m\mapsto g=NA^{-1}m$; note that $NA^{-1}m\in\mathbb{Z}_f$ for $m\in\mathbb{Z}^f$ because NA^{-1} is integral by the definition of the level N. This map is a bijection, with inverse defined by $g\mapsto m=N^{-1}Ag$. Hence,

$$N^{-r} \sum_{\substack{n\in\mathbb{Z}^f\\ n\equiv h \pmod N}} \big({}^{\mathrm{t}}\ell An\big)^r e^{\pi i z\frac{{}^{\mathrm{t}}nAn}{N^2}}$$
$$= N^{-r}\frac{i^k}{z^{k+r}\sqrt{\det(A)}} \sum_{\substack{g\in\mathbb{Z}^f\\ Ag\equiv 0 \pmod N}} \big({}^{\mathrm{t}}\ell Ag\big)^r e^{\pi i(-1/z)\frac{{}^{\mathrm{t}}gAg}{N^2} + 2\pi i\frac{{}^{\mathrm{t}}gAh}{N^2}}.$$

Canceling the common factor N^{-r}, we get:

$$\sum_{\substack{n\in\mathbb{Z}^f \\ n\equiv h \pmod N}} \left({}^{\mathrm{t}}\ell A n\right)^r e^{\pi i z \frac{{}^{\mathrm{t}}nAn}{N^2}}$$

$$= \frac{i^k}{z^{k+r}\sqrt{\det(A)}} \sum_{\substack{g\in\mathbb{Z}^f \\ Ag\equiv 0 \pmod N}} \left({}^{\mathrm{t}}\ell A g\right)^r e^{\pi i(-1/z)\frac{{}^{\mathrm{t}}gAg}{N^2}+2\pi i\frac{{}^{\mathrm{t}}gAh}{N^2}}.$$

The set of $g\in\mathbb{Z}^f$ such that $Ag\equiv 0 \pmod N$ is a subgroup of $\mathbb{Z}^f$; this subgroup in turn contains the subgroup $N\mathbb{Z}^f$. We may therefore sum in stages on the right-hand side. Let $F(g)$ be the summand on the right-hand side for $g\in\mathbb{Z}^f$ with $Ag\equiv 0 \pmod N$. The form of this summation in stages is then:

$$\begin{aligned}\sum_{\substack{g\in\mathbb{Z}^f \\ Ag\equiv 0 \pmod N}} F(n) &= \sum_{\substack{g\in\mathbb{Z}^f/NZ^f \\ Ag\equiv 0 \pmod N}} \sum_{m\in N\mathbb{Z}^f} F(g+m) \\ &= \sum_{\substack{g \pmod N \\ Ag\equiv 0 \pmod N}} \sum_{\substack{n_1\in\mathbb{Z}^f \\ n_1\equiv g \pmod N}} F(n_1).\end{aligned}$$

Applying this observation, we have:

$$\sum_{\substack{n\in\mathbb{Z}^f \\ n\equiv h \pmod N}} \left({}^{\mathrm{t}}\ell A n\right)^r e^{\pi i z \frac{{}^{\mathrm{t}}nAn}{N^2}} = \frac{i^k}{z^{k+r}\sqrt{\det(A)}} \sum_{\substack{g \pmod N \\ Ag\equiv 0 \pmod N}}$$

$$\sum_{\substack{n_1\in\mathbb{Z}^f \\ n_1\equiv g \pmod N}} \left({}^{\mathrm{t}}\ell A n_1\right)^r e^{\pi i(-1/z)\frac{{}^{\mathrm{t}}n_1An_1}{N^2}+2\pi i\frac{{}^{\mathrm{t}}n_1Ah}{N^2}}.$$

Let $g\in\mathbb{Z}^f$ with $Ag\equiv 0 \pmod N$ and let $n_1\in\mathbb{Z}^f$ with $n_1\equiv g \pmod N$. Write $n_1 = g+Nm$ for some $m\in\mathbb{Z}^f$. Then

$$\begin{aligned}e^{2\pi i\frac{{}^{\mathrm{t}}n_1Ah}{N^2}} &= e^{2\pi i\frac{{}^{\mathrm{t}}gAh}{N^2}} e^{2\pi i\frac{N\,{}^{\mathrm{t}}mAh}{N^2}} \\ &= e^{2\pi i\frac{{}^{\mathrm{t}}gAh}{N^2}} e^{2\pi i\frac{{}^{\mathrm{t}}mAh}{N}} \\ &= e^{2\pi i\frac{{}^{\mathrm{t}}gAh}{N^2}}.\end{aligned}$$

In the last step we used that $Ah\equiv 0 \pmod N$, so that $\frac{{}^{\mathrm{t}}mAh}{N}$ is an integer. We therefore have:

$$\sum_{\substack{n\in\mathbb{Z}^f \\ n\equiv h \pmod N}} \left({}^{\mathrm{t}}\ell A n\right)^r e^{\pi i z \frac{{}^{\mathrm{t}}nAn}{N^2}}$$

$$= \frac{i^k}{z^{k+r}\sqrt{\det(A)}} \sum_{\substack{g \pmod N \\ Ag\equiv 0 \pmod N}} e^{2\pi i \frac{{}^t gAh}{N^2}} \sum_{\substack{n_1\in\mathbb{Z}^f \\ n_1\equiv g \pmod N}} \left({}^t\ell A n_1\right)^r e^{\pi i(-1/z)\frac{{}^t n_1 A n_1}{N^2}}.$$

Interchanging z and $-1/z$, we obtain:

$$\sum_{\substack{n\in\mathbb{Z}^f \\ n\equiv h \pmod N}} \left({}^t\ell A n\right)^r e^{\pi i(-1/z)\frac{{}^t nAn}{N^2}}$$
$$= \frac{(-1)^{k+r} i^k z^{k+r}}{\sqrt{\det(A)}} \sum_{\substack{g \pmod N \\ Ag\equiv 0 \pmod N}} e^{2\pi i \frac{{}^t gAh}{N^2}} \sum_{\substack{n_1\in\mathbb{Z}^f \\ n_1\equiv g \pmod N}} \left({}^t\ell A n_1\right)^r e^{\pi i z\frac{{}^t n_1 A n_1}{N^2}}.$$

This implies that

$$\theta(A,P,h,\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\cdot z)$$
$$= \frac{(-i)^{k+2r} z^{k+r}}{\sqrt{\det(A)}} \sum_{\substack{g \pmod N \\ Ag\equiv 0 \pmod N}} e^{2\pi i \frac{{}^t gAh}{N^2}} \theta(A,P,g,z), \quad (2.6)$$

which is equivalent to (2.4).

Next, let $b \in \mathbb{Z}$. We have

$$\theta(A,P,h,z)\big|_{k+r}\begin{bmatrix} 1 & b \\ & 1 \end{bmatrix}$$
$$= \theta(A,P,h,z+b)$$
$$= \sum_{\substack{n\in\mathbb{Z}^f \\ n\equiv h \pmod N}} P(n) e^{2\pi i(z+b)\frac{Q(n)}{N^2}}$$
$$= \sum_{\substack{n\in\mathbb{Z}^f \\ n\equiv h \pmod N}} P(n) e^{2\pi i b\frac{Q(n)}{N^2}} e^{2\pi i z\frac{Q(n)}{N^2}}$$
$$= e^{2\pi i b\frac{Q(h)}{N^2}} \sum_{\substack{n\in\mathbb{Z}^f \\ n\equiv h \pmod N}} P(n) e^{2\pi i z\frac{Q(n)}{N^2}} \quad \text{(cf. Lemma 1.5.8)}$$
$$= e^{2\pi i b\frac{Q(h)}{N^2}} \theta(A,P,h,z).$$

This is (2.5).

Finally, the vector space $V(A,P)$ is mapped into itself by $\mathrm{SL}(2,\mathbb{Z})$ via the $|_{k+r}$ right action because $\mathrm{SL}(2,\mathbb{Z})$ is generated by the matrices

$$\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$$

and because (2.4) and (2.5) hold. □

2.5 The case $N = 1$

Proposition 2.5.1. *Let f be a positive even integer, and define $k = f/2$. Let $A \in \mathrm{M}(f, \mathbb{Z})$ be a even symmetric positive-definite matrix, and let N be the level of A. By Corollary 1.5.5 $N = 1$ if and only if $\det(A) = 1$; assume that $N = 1$ so that also $\det(A) = 1$. Then f is divisible by 8. Let r be a non-negative integer, and let $P \in \mathcal{H}_r(A)$. The $\mathbb{C}$ vector space $V(A, P)$ has dimension at most one, and is spanned by the theta series*

$$\theta(A, P, 0, z) = \sum_{n \in \mathbb{Z}^f} P(n) e^{2\pi i z Q(n)}.$$

We have

$$\theta(A, P, 0, z)\big|_{k+r}\alpha = \theta(A, P, 0, z) \tag{2.7}$$

for all $\alpha \in \mathrm{SL}(2, \mathbb{Z})$, and $\theta(A, P, 0, z)$ is a modular form of weight $k + r$ with respect to $\mathrm{SL}(2, \mathbb{Z})$.

Proof. Let $h \in \mathbb{Z}^f$; since $N = 1$, we have $Ah \equiv 0 \pmod N$. Now

$$\begin{aligned} \theta(A, P, h, z) &= \sum_{\substack{n \in \mathbb{Z}^f \\ n \equiv h \pmod 1}} P(n) e^{2\pi i z Q(n)} \\ &= \sum_{\substack{n \in \mathbb{Z}^f \\ n \equiv 0 \pmod 1}} P(n) e^{2\pi i z Q(n)} \\ &= \theta(A, P, 0, z). \end{aligned}$$

It follows that $V(A, P)$ is at most one-dimensional, and is spanned by the function $\theta(A, P, 0, z)$. By Lemma 2.4.1, we have

$$\theta(A, P, 0, z)\big|_{k+r} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} = i^k \theta(A, P, 0, z), \tag{2.8}$$

$$\theta(A, P, 0, z)\big|_{k+r} \begin{bmatrix} 1 & b \\ & 1 \end{bmatrix} = \theta(A, P, 0, z) \tag{2.9}$$

for $b \in \mathbb{Z}$. Since $\mathrm{SL}(2, \mathbb{Z})$ is generated by the elements

$$\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$$

it follows that there exists a function $t : \mathrm{SL}(2, \mathbb{Z}) \to \mathbb{C}^\times$ such that

$$\theta(A, P, 0, z)\big|_{k+r}\alpha = t(\alpha) \cdot \theta(A, P, 0, z) \tag{2.10}$$

for $\alpha \in \mathrm{SL}(2, \mathbb{Z})$ and for all non-negative integers r and $P \in \mathrm{SL}(2, \mathbb{Z})$. We claim that $t(\alpha) = 1$ for all $\alpha \in \mathrm{SL}(2, \mathbb{Z})$. Assume that $r = 0$ and let $P \in \mathcal{H}_0(A)$ be the polynomial such that $P(X_1, \ldots, X_f) = 1$. Then the function $\theta(A, P, 0, z)$ is not identically zero. Since $\theta(A, P, 0, z)$ is not identically zero, and since $|_k$ is a right action, equation (2.10) implies that t is a homomorphism. Also, by (2.8) and (2.9) we have

$$t(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}) = i^k, \quad t(\begin{bmatrix} 1 & b \\ & 1 \end{bmatrix}) = 1$$

for $b \in \mathbb{Z}$. Now

$$\begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} = \begin{bmatrix} -1 & \\ & -1 \end{bmatrix}.$$

Applying these matrices to $\theta(A, P, 0, z)$ we obtain:

$$\begin{aligned} \theta(A, P, 0, z)\big|_k \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} &= \theta(A, P, 0, z)\big|_k \begin{bmatrix} -1 & \\ & -1 \end{bmatrix} \\ i^{2k}\theta(A, P, 0, z) &= (-1)^k \theta(A, P, 0, z). \end{aligned}$$

Since $\theta(A, P, 0, z)$ is not identically zero, we have $i^{2k} = (-1)^k$. We also have the matrix identity

$$\begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} 1 & -b \\ & 1 \end{bmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} = \begin{bmatrix} -1 & \\ & -1 \end{bmatrix} \begin{bmatrix} 1 & \\ b & 1 \end{bmatrix}$$

for $b \in \mathbb{Z}$. Applying these matrices to $\theta(A, P, 0, z)$, we find that:

$$i^{2k}\theta(A, P, 0, z) = (-1)^k \theta(A, P, 0, z)\big|_k \begin{bmatrix} 1 & \\ b & 1 \end{bmatrix}$$

for $b \in \mathbb{Z}$. Since $i^{2k} = (-1)^k$, this implies that

$$\theta(A, P, 0, z)\big|_{k+r} \begin{bmatrix} 1 & \\ b & 1 \end{bmatrix} = \theta(A, P, 0, z)$$

for $b \in \mathbb{Z}$. Therefore, t is trivial on all matrices of the form

$$\begin{bmatrix} 1 & b \\ & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & \\ b & 1 \end{bmatrix}$$

for $b \in \mathbb{Z}$. Since these matrices generate $\mathrm{SL}(2, \mathbb{Z})$ it follows that the homomorphism t is trivial. This proves (2.7) for all $\alpha \in \mathrm{SL}(2, \mathbb{Z})$, for all non-negative integers r and $P \in \mathcal{H}_r(A)$. Also, since t is trivial, we must have $i^k = 1$. Write $k = 4a + b$ where a and b are non-negative integers with $b \in \{0, 1, 2, 3\}$. Then $1 = i^k = (i^4)^a i^b = i^b$. This equality implies that $4|k$, so that $8|f$.

Given what we have already proven, to complete the proof that $\theta(A, P, 0, z)$ is a modular form of weight $k + r$ for $\mathrm{SL}(2, \mathbb{Z})$, it will suffice to prove that $\theta(A, P, 0, z)$ is holomorphic at the cusps of $\mathrm{SL}(2, \mathbb{Z})$, i.e., that the third condition of the definition of a modular form holds (see section 1.7). Clearly, the smallest positive integer N such that $\Gamma(N) \subset \mathrm{SL}(2, \mathbb{Z})$ is $N = 1$. Let $\sigma \in \mathrm{SL}(2, \mathbb{Z})$. We have already proven that $\theta(A, P, 0, z)|_{k+r}\sigma = \theta(A, P, 0, z)$. Thus, to complete the proof we need to prove the existence of a positive number R and a complex power series

$$\sum_{m=0}^{\infty} a(m)q^m$$

that converges in $D(R) = \{q \in \mathbb{C} : |q| < R\}$ such that

$$\theta(A, P, 0, z) = \sum_{m=0}^{\infty} a(m)e^{2\pi imz}$$

for $z \in H(1, R) = \{z \in \mathbb{H}_1 : \mathrm{Im}(z) > -\frac{\log(R)}{2\pi}\}$ (note that $H(1, R)$ is mapped into $D(R)$ under the map defined by $z \mapsto e^{2\pi iz}$). Consider the power series

$$\sum_{n\in\mathbb{Z}^f} P(n)q^{Q(n)} \tag{2.11}$$

in the complex variable q. Let q be any element of $\mathbb{C}$ with $|q| < 1$. Since $q = e^{2\pi iz}$ for some $z \in \mathbb{H}_1$, and since

$$\sum_{n\in\mathbb{Z}^f} P(n)e^{2\pi izQ(n)} = \sum_{n\in\mathbb{Z}^f} P(n)q^{Q(n)}$$

converges absolutely by Lemma 2.4.1, it follows that the power series (2.11) converges absolutely at q. Hence, the radius of convergence of the power series (2.11) is greater than 0, and in fact at least 1 (see Theorem 8 on p. 172 of [17]). Since by the definition of $\theta(A, P, 0, z)$ we have

$$\theta(A, P, 0, z) = \sum_{n\in\mathbb{Z}^f} P(n)e^{2\pi izQ(n)},$$

for $z \in \mathbb{H}_1$, the proof is complete. □

2.6 Example: a quadratic form of level one

If the level N of A is 1, so that the $\theta(A, P, h, z)$ are modular forms with respect to $\mathrm{SL}(2, \mathbb{Z})$, then necessarily $8|f$ by Proposition 2.5.1. Assume that $f = 8$. Up to equivalence, there is the only positive-definite even integral symmetric matrix A in $\mathrm{M}(8, \mathbb{Z})$ with $\det(A) = 1$. This matrix arises in the following way. Consider the root system E_8 inside $\mathbb{R}^8$. To describe this root system with 240 elements, let $e_1, \ldots, e_8$ be the standard basis for $\mathbb{R}^8$. The root system E_8 consists of the 112 vectors

$$\delta_1 e_i + \delta_2 e_k \quad \text{where } 1 \le i, k \le 8,\ i \ne k, \text{ and } \delta_1, \delta_2 \in \{\pm 1\}$$

and the 128 vectors

$$\frac{1}{2}(\epsilon_1 e_1 + \cdots + \epsilon_8 e_8) \quad \text{where } \epsilon_1, \ldots, \epsilon_8 \in \{\pm 1\} \text{ and } \quad \epsilon_1 \cdots \epsilon_8 = 1.$$

Every element of E_8 has length $\sqrt{2}$. As a base for this root system we can take the 8 vectors

$$\begin{aligned}
\alpha_1 &= \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8),\\
\alpha_2 &= e_1 + e_2,\\
\alpha_3 &= -e_1 + e_2,\\
\alpha_4 &= -e_2 + e_3,\\
\alpha_5 &= -e_3 + e_4,\\
\alpha_6 &= -e_4 + e_5,\\
\alpha_7 &= -e_5 + e_6,\\
\alpha_8 &= -e_6 + e_7.
\end{aligned}$$

Every element of E_8 can be written as a $\mathbb{Z}$ linear combination of $\alpha_1, \dots, \alpha_8$ such that all the coefficients are either all non-negative or all non-positive. Let A be the Cartan matrix of E_8 with respect to the above base; this turns out to be $A = ((\alpha_i, \alpha_j))_{1 \le i,j \le 8}$. Here, $(\cdot, \cdot)$ is the usual inner product on $\mathbb{R}^8$. Explicitly, we have:

$$A = \begin{bmatrix}
2 & & -1 & & & & & \\
 & 2 & & -1 & & & & \\
-1 & & 2 & -1 & & & & \\
 & -1 & -1 & 2 & -1 & & & \\
 & & & -1 & 2 & -1 & & \\
 & & & & -1 & 2 & -1 & \\
 & & & & & -1 & 2 & -1 \\
 & & & & & & -1 & 2
\end{bmatrix}.$$

Clearly, A is the matrix of $(\cdot, \cdot)$ with respect to the ordered basis $\alpha_1, \dots, \alpha_8$ for $\mathbb{R}^8$; hence, A is positive-definite. Evidently A is an even integral symmetric matrix, and a computation shows that $\det(A) = 1$. Since $\det(A) = 1$, the level of A is $N = 1$. The quadratic form Q is given by:

$$\begin{aligned}
Q(x_1, x_2, x_3, \dots, x_8) &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2\\
&\quad - x_1x_3 - x_2x_4 - x_3x_4 - x_4x_5 - x_5x_6 - x_6x_7 - x_7x_8.
\end{aligned}$$

Let $r = 0$, and let $1 \in \mathcal{H}_0(A)$ be the constant polynomial. The theta series

$$\theta(A, z) = \theta(A, 1, 0, z) = \sum_{m \in \mathbb{Z}^8} e^{2\pi i Q(m)}$$

is a non-zero modular form for $\mathrm{SL}(2, \mathbb{Z})$ of weight $8/2 = 4$. We may also write

$$\theta(A, z) = \sum_{n=0}^{\infty} r(n) e^{2\pi i n}$$

where

$$r(n) = \#\{m \in \mathbb{Z}^8 : Q(m) = n\}.$$

It is known that the dimension of the space of modular forms for $\mathrm{SL}(2,\mathbb{Z})$ of weight 4 is one (see Proposition 2.26 on p. 46 of [27]). Moreover, this space contains the Eisenstein series

$$E(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) e^{2\pi i n z}$$

where

$$\sigma_3(n) = \sum_{a|n, a>0} a^3$$

for positive integers n. Since $r(0) = 1$, we have $\theta(A, z) = E(z)$. Thus,

$$r(n) = 240 \cdot \sigma_3(n)$$

for all positive integers n. Evidently, $240 \cdot \sigma_3(1) = 240$. Thus, there are 240 solutions $m \in \mathbb{Z}^8$ to the equation $Q(m) = 1$. These 240 solutions are exactly the coordinates of the elements of E_8 when the elements of E_8 are written in our chosen base (note that the coordinates are automatically in $\mathbb{Z}$, as this is property of a base for a root system). □

2.7 The case $N > 1$

The action of $\mathrm{SL}(2,\mathbb{Z})$

Lemma 2.7.1. *Let f be a positive even integer, and define $k = f/2$. Let $A \in \mathrm{M}(f,\mathbb{Z})$ be an even symmetric positive-definite matrix, and let N be the level of A. Let c be a positive integer; by Corollary 1.5.7, the level of cA is cN. Let r be a non-negative integer. We have $\mathcal{H}_r(cA) = \mathcal{H}_r(A)$. Let $h \in \mathbb{Z}^f$ be such that $Ah \equiv 0 \pmod{N}$ and let $P \in \mathcal{H}_r(A)$. If $g \in \mathbb{Z}_f$ is such that $g \equiv h \pmod{N}$, then $(cA)g \equiv 0 \pmod{cN}$ so that $\theta(cA, P, g, \cdot)$ is defined, and*

$$\theta(A, P, h, z) = \sum_{\substack{g \pmod{cN} \\ g \equiv h \pmod{N}}} \theta(cA, P, g, cz)$$

for $z \in \mathbb{H}_1$.

Proof. If $\ell \in \mathbb{C}^f$, then ${}^{\mathrm{t}}\ell A \ell = 0$ if and only if ${}^{\mathrm{t}}\ell (cA) \ell = 0$; this observation, and the involved definitions, imply that $\mathcal{H}_r(cA) = \mathcal{H}_r(A)$. Next, let $z \in \mathbb{H}_1$. Then:

$$\begin{aligned} \theta(A, P, h, z) &= \sum_{\substack{n \in \mathbb{Z}^f \\ n \equiv h \pmod{N}}} P(n) e^{2\pi i z \frac{Q(n)}{N^2}} \\ &= \sum_{\substack{g \in \mathbb{Z}^f / cN\mathbb{Z}^f \\ g \equiv h \pmod{N}}} \sum_{n_1 \in cN\mathbb{Z}^f} P(g + n_1) e^{2\pi i z \frac{Q(g+n_1)}{N^2}}. \end{aligned}$$

Let $g \in \mathbb{Z}^f$ with $g \equiv h \pmod N$. There is a bijection

$$cN\mathbb{Z}^f \xrightarrow{\sim} \{m \in \mathbb{Z}^f : m \equiv g \pmod{cN}\}$$

given by $n_1 \mapsto m = g + n_1$. Hence,

$$\begin{aligned}
\theta(A,P,h,z) &= \sum_{\substack{g \pmod{cN} \\ g\equiv h \pmod N}} \sum_{\substack{m\in\mathbb{Z}^f \\ m\equiv g \pmod{cN}}} P(m)e^{2\pi i z\frac{Q(m)}{N^2}} \\
&= \sum_{\substack{g \pmod{cN} \\ g\equiv h \pmod N}} \sum_{\substack{m\in\mathbb{Z}^f \\ m\equiv g \pmod{cN}}} P(m)e^{\pi i z\frac{{}^{\mathrm{t}}mAm}{N^2}} \\
&= \sum_{\substack{g \pmod{cN} \\ g\equiv h \pmod N}} \sum_{\substack{m\in\mathbb{Z}^f \\ m\equiv g \pmod{cN}}} P(m)e^{\pi i c z\frac{{}^{\mathrm{t}}mcAm}{(cN)^2}} \\
&= \sum_{\substack{g \pmod{cN} \\ g\equiv h \pmod N}} \theta(cA,P,g,cz).
\end{aligned}$$

This completes the proof. □

Lemma 2.7.2. *Let f be a positive even integer. Let $A \in \mathrm{M}(f,\mathbb{Z})$ be an even symmetric positive-definite matrix, and let N be the level of A. Let*

$$\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z}),$$

and assume that $c \neq 0$. Let

$$Y(A) = \{m \in \mathbb{Z}^f : Am \equiv 0 \pmod N\}.$$

Define a function

$$s_\alpha : Y(A) \times Y(A) \longrightarrow \mathbb{C}$$

by

$$s_\alpha(g_1,g_2) = \sum_{\substack{g \pmod{cN} \\ g\equiv g_2 \pmod N}} e^{2\pi i\left(\frac{aQ(g)+{}^{\mathrm{t}}g_1Ag+dQ(g_1)}{cN^2}\right)}.$$

The function s_α is well-defined. If $g_1, g_1', g_2, g_2' \in Y(A)$ and $g_1 \equiv g_1' \pmod N$ and $g_2 \equiv g_2' \pmod N$, then $s_\alpha(g_1,g_2) = s_\alpha(g_1',g_2')$. Moreover,

$$s_\alpha(g_1,g_2) = e^{-2\pi i\left(\frac{b\,{}^{\mathrm{t}}g_2Ag_1+bdQ(g_1)}{N^2}\right)} s_\alpha(0, g_2+dg_1) \tag{2.12}$$

for $g_1, g_2 \in Y(A)$.

Proof. To prove that s_α is well-defined, let $g_1, g_2 \in Y(A)$, and $g, g' \in \mathbb{Z}^f$ with $g \equiv g' \pmod{cN}$ and $g \equiv g' \equiv g_2 \pmod N$. Write $g' = g + cNm$ for some $m \in \mathbb{Z}^f$. Then

$$e^{2\pi i\left(\frac{aQ(g')+{}^{\mathrm{t}}g_1Ag'+dQ(g_1)}{cN^2}\right)} = e^{2\pi i\left(\frac{aQ(g+cNm)+{}^{\mathrm{t}}g_1A(g+cNm)+dQ(g_1)}{cN^2}\right)}$$

$$= e^{2\pi i\left(\frac{aQ(g)+acN\,{}^{\mathrm{t}}gAm+ac^2N^2Q(m)+{}^{\mathrm{t}}g_1Ag+cN\,{}^{\mathrm{t}}g_1Am+dQ(g_1)}{cN^2}\right)}$$
$$= e^{2\pi i\left(\frac{aQ(g)+{}^{\mathrm{t}}g_1Ag+dQ(g_1)+acN\,{}^{\mathrm{t}}(Ag)m+ac^2N^2Q(m)+cN\,{}^{\mathrm{t}}(Ag_1)m}{cN^2}\right)}$$
$$= e^{2\pi i\left(\frac{aQ(g)+{}^{\mathrm{t}}g_1Ag+dQ(g_1)}{cN^2}\right)},$$

where in the last step we used that $Ag \equiv Ag_1 \equiv 0 \pmod N$. It follows that s_α is well-defined.

Next we prove (2.12). Let $g_1, g_2 \in Y(A)$. Then

$$\begin{aligned}
&e^{-2\pi i\left(\frac{b\,{}^{\mathrm{t}}g_2Ag_1+bdQ(g_1)}{N^2}\right)} s_\alpha(0, g_2+dg_1)\\
&= \sum_{\substack{g \pmod{cN} \\ g\equiv g_2+dg_1 \pmod N}} e^{-2\pi i\left(\frac{b\,{}^{\mathrm{t}}g_2Ag_1+bdQ(g_1)}{N^2}\right)} e^{2\pi i\left(\frac{aQ(g)}{cN^2}\right)}\\
&= \sum_{\substack{g \pmod{cN} \\ g\equiv g_2+dg_1 \pmod N}} e^{2\pi i\left(\frac{aQ(g)-bc\,{}^{\mathrm{t}}g_2Ag_1-bcdQ(g_1)}{cN^2}\right)}\\
&= \sum_{\substack{g \pmod{cN} \\ g\equiv g_2 \pmod N}} e^{2\pi i\left(\frac{aQ(g+dg_1)-bc\,{}^{\mathrm{t}}g_2Ag_1-bcdQ(g_1)}{cN^2}\right)}\\
&= \sum_{\substack{g \pmod{cN} \\ g\equiv g_2 \pmod N}} e^{2\pi i\left(\frac{aQ(g)+ad\,{}^{\mathrm{t}}g_1Ag+ad^2Q(g_1)-bc\,{}^{\mathrm{t}}g_2Ag_1-bcdQ(g_1)}{cN^2}\right)}\\
&= \sum_{\substack{g \pmod{cN} \\ g\equiv g_2 \pmod N}} e^{2\pi i\left(\frac{aQ(g)+{}^{\mathrm{t}}g_1A(adg-bcg_2)+dQ(g_1)}{cN^2}\right)}.
\end{aligned}$$

Let $g \in \mathbb{Z}_f$ with $g \equiv g_2 \pmod N$. Write $g_2 = g + Nm$ for some $m \in \mathbb{Z}^f$. Then

$$\begin{aligned}
e^{2\pi i\left(\frac{{}^{\mathrm{t}}g_1A(adg-bcg_2)}{cN^2}\right)} &= e^{2\pi i\left(\frac{{}^{\mathrm{t}}g_1A((ad-bc)g-bcNm)}{cN^2}\right)}\\
&= e^{2\pi i\left(\frac{{}^{\mathrm{t}}g_1A(g-bcNm)}{cN^2}\right)}\\
&= e^{2\pi i\left(\frac{{}^{\mathrm{t}}g_1Ag}{cN^2}\right)} e^{2\pi i\left(\frac{-bcN\,{}^{\mathrm{t}}(Ag_1)m}{cN^2}\right)}\\
&= e^{2\pi i\left(\frac{{}^{\mathrm{t}}g_1Ag}{cN^2}\right)} e^{2\pi i\left(\frac{-b\,{}^{\mathrm{t}}(Ag_1)m}{N}\right)}\\
&= e^{2\pi i\left(\frac{{}^{\mathrm{t}}g_1Ag}{cN^2}\right)},
\end{aligned}$$

where the last step follows because $Ag_1 \equiv 0 \pmod N$. We therefore have:

$$e^{-2\pi i\left(\frac{b\,{}^{\mathrm{t}}g_2Ag_1+bdQ(g_1)}{N^2}\right)} s_\alpha(0, g_2+dg_1) = \sum_{\substack{g \pmod{cN} \\ g\equiv g_2 \pmod N}} e^{2\pi i\left(\frac{aQ(g)+{}^{\mathrm{t}}g_1Ag+dQ(g_1)}{cN^2}\right)}$$
$$e^{-2\pi i\left(\frac{b\,{}^{\mathrm{t}}g_2Ag_1+bdQ(g_1)}{N^2}\right)} s_\alpha(0, g_2+dg_1) = s_\alpha(g_1, g_2).$$

This completes the proof of (2.12).

Finally, let $g_1, g_1', g_2, g_2' \in Y(A)$ with $g_1 \equiv g_1' \pmod N$ and $g_2 \equiv g_2' \pmod N$. It is evident from the definition of s_α that $s_\alpha(g_1,g_2) = s_\alpha(g_1,g_2')$. Write $g_1' = g_1 + Nm$ for some $m \in \mathbb{Z}^f$. Then

$$\begin{aligned}
s_\alpha(g_1', g_2) &= e^{-2\pi i\left(\frac{b\,{}^{\mathrm{t}}g_2 A g_1' + bdQ(g_1')}{N^2}\right)} s_\alpha(0, g_2 + dg_1')\\
&= e^{-2\pi i\left(\frac{b\,{}^{\mathrm{t}}g_2 A(g_1+Nm) + bdQ(g_1+Nm)}{N^2}\right)} s_\alpha(0, g_2 + d(g_1+Nm))\\
&= e^{-2\pi i\left(\frac{b\,{}^{\mathrm{t}}g_2 A g_1 + bdQ(g_1) + bdN\,{}^{\mathrm{t}}(Ag_1)m + bdN^2Q(m) + bN\,{}^{\mathrm{t}}(Ag_2)m}{N^2}\right)}\\
&\qquad \times s_\alpha(0, g_2 + dg_1 + dNm)\\
&= e^{-2\pi i\left(\frac{b\,{}^{\mathrm{t}}g_2 A g_1 + bdQ(g_1)}{N^2}\right)} s_\alpha(0, g_2 + dg_1)\\
&= s_\alpha(g_1, g_2).
\end{aligned}$$

Here we used that $Ag_1 \equiv Ag_2 \equiv 0 \pmod N$. This completes the proof. □

Lemma 2.7.3. *Let f be a positive even integer, and define $k = f/2$. Let $A \in \mathrm{M}(f,\mathbb{Z})$ be an even symmetric positive-definite matrix, and let N be the level of A. Define the quadratic form $Q(x)$ in f variables by*

$$Q(x) = \frac{1}{2}\,{}^{\mathrm{t}}xAx.$$

Let r be a non-negative integer, and let $P \in \mathcal{H}_r(A)$. Let $h \in \mathbb{Z}^f$ be such that

$$Ah \equiv 0 \pmod N.$$

Let

$$\alpha = \begin{bmatrix} a & b\\ c & d\end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z}),$$

and assume that c is a positive integer. Then

$$\begin{aligned}
&\theta(A,P,h,z)\big|_{k+r}\begin{bmatrix} a & b\\ c & d\end{bmatrix}\\
&\qquad = \frac{1}{i^{k+2r}c^k\sqrt{\det(A)}} \sum_{\substack{g \pmod N\\ Ag\equiv 0 \pmod N}} s_\alpha(g,h)\cdot\theta(A,P,g,z), \qquad (2.13)
\end{aligned}$$

where s_α is defined in Lemma 2.7.2.

Proof. We have

$$\begin{aligned}
&\theta(A,P,h,z)\big|_{k+r}\begin{bmatrix} a & b\\ c & d\end{bmatrix}\\
&= j(\alpha,z)^{-k-r}\theta\Big(A,P,h,\ \frac{az+b}{cz+d}\ \Big)
\end{aligned}$$

$$= j(\alpha,z)^{-k-r} \sum_{\substack{g \pmod{cN} \\ g\equiv h \pmod N}} \theta\Big(cA, P, g,\ c\cdot\frac{az+b}{cz+d}\Big)$$

$$= j(\alpha,z)^{-k-r} \sum_{\substack{g \pmod{cN} \\ g\equiv h \pmod N}} \theta\Big(cA, P, g,\ -\frac{1}{cz+d}+a\Big)$$

$$= j(\alpha,z)^{-k-r} \sum_{\substack{g \pmod{cN} \\ g\equiv h \pmod N}} e^{2\pi i a\frac{Q_{cA}(g)}{(cN)^2}}\theta\Big(cA, P, g,\ -\frac{1}{cz+d}\Big)$$

$$= j(\alpha,z)^{-k-r} \sum_{\substack{g \pmod{cN} \\ g\equiv h \pmod N}} e^{2\pi i a\frac{Q(g)}{cN^2}}\theta\Big(cA, P, g,\ -\frac{1}{cz+d}\Big)$$

$$= (-1)^{k+r} \sum_{\substack{g \pmod{cN} \\ g\equiv h \pmod N}} e^{2\pi i a\frac{Q(g)}{cN^2}}\big(\theta(cA,P,g,\cdot)\big|_{k+r}\begin{bmatrix} & 1\\ -1 & \end{bmatrix}\big)(cz+d)$$

$$= \frac{i^k(-1)^{k+r}}{\sqrt{\det(cA)}} \sum_{\substack{g \pmod{cN} \\ g\equiv h \pmod N}} e^{2\pi i a\frac{Q(g)}{cN^2}}$$
$$\sum_{\substack{g_1 \pmod{cN} \\ (cA)g_1\equiv 0 \pmod{cN}}} e^{2\pi i\frac{{}^{\mathrm{t}}g_1(cA)g}{(cN)^2}}\theta(cA,P,g_1,cz+d)$$

$$= \frac{i^k(-1)^{k+r}}{\sqrt{\det(cA)}} \sum_{\substack{g \pmod{cN} \\ g\equiv h \pmod N}} e^{2\pi i a\frac{Q(g)}{cN^2}}$$
$$\sum_{\substack{g_1 \pmod{cN} \\ (cA)g_1\equiv 0 \pmod{cN}}} e^{2\pi i\frac{{}^{\mathrm{t}}g_1(cA)g}{(cN)^2}}e^{2\pi i d\frac{Q(g_1)}{cN^2}}\theta(cA,P,g_1,cz)$$

$$= \frac{i^k(-1)^{k+r}}{\sqrt{\det(cA)}} \sum_{\substack{g_1 \pmod{cN} \\ (cA)g_1\equiv 0 \pmod{cN}}}$$
$$\Bigg(\sum_{\substack{g \pmod{cN} \\ g\equiv h \pmod N}} e^{2\pi i\left(\frac{aQ(g)+{}^{\mathrm{t}}g_1Ag+dQ(g_1)}{cN^2}\right)}\Bigg)\theta(cA,P,g_1,cz)$$

$$= \frac{i^k(-1)^{k+r}}{\sqrt{\det(cA)}} \sum_{\substack{g_1 \pmod{cN} \\ (cA)g_1\equiv 0 \pmod{cN}}} s_\alpha(g_1,h)\theta(cA,P,g_1,cz)$$

$$= \frac{i^k(-1)^{k+r}}{\sqrt{\det(cA)}} \sum_{\substack{g_1 \pmod{cN} \\ Ag_1\equiv 0 \pmod{N}}} s_\alpha(g_1,h)\theta(cA,P,g_1,cz)$$

$$= \frac{i^k(-1)^{k+r}}{\sqrt{\det(cA)}} \sum_{\substack{g_1 \in \mathbb{Z}^f/N\mathbb{Z}^f \\ Ag_1 \equiv 0 \pmod N}} \sum_{m \in N\mathbb{Z}^f/cN\mathbb{Z}^f} s_\alpha(g_1 + m, h)\theta(cA, P, g_1 + m, cz)$$

$$= \frac{i^k(-1)^{k+r}}{\sqrt{\det(cA)}} \sum_{\substack{g_1 \in \mathbb{Z}^f/N\mathbb{Z}^f \\ Ag_1 \equiv 0 \pmod N}} s_\alpha(g_1, h) \sum_{m \in N\mathbb{Z}^f/cN\mathbb{Z}^f} \theta(cA, P, g_1 + m, cz)$$

$$= \frac{i^k(-1)^{k+r}}{\sqrt{\det(cA)}} \sum_{\substack{g_1 \in \mathbb{Z}^f/N\mathbb{Z}^f \\ Ag_1 \equiv 0 \pmod N}} s_\alpha(g_1, h) \sum_{\substack{g' \pmod{cN} \\ g' \equiv g_1 \pmod N}} \theta(cA, P, g', cz)$$

$$= \frac{i^k(-1)^{k+r}}{\sqrt{\det(cA)}} \sum_{\substack{g_1 \in \mathbb{Z}^f/N\mathbb{Z}^f \\ Ag_1 \equiv 0 \pmod N}} s_\alpha(g_1, h) \sum_{\substack{g' \pmod{cN} \\ g' \equiv g_1 \pmod N}} \theta(cA, P, g', cz)$$

$$= \frac{1}{i^{k+2r} c^k \sqrt{\det(A)}} \sum_{\substack{g_1 \pmod N \\ Ag_1 \equiv 0 \pmod N}} s_\alpha(g_1, h) \cdot \theta(A, P, g_1, z).$$

Here, we used Lemma 2.7.2. □

The action of $\Gamma_0(N)$

Lemma 2.7.4. *Let f be an even positive integer, let $A \in \mathrm{M}(f, \mathbb{Z})$ be a positive-definite even integral symmetric matrix and let N be the level of A. Let*

$$Y(A) = \{g \in \mathbb{Z}^f : Ag \equiv 0 \pmod N\}.$$

Define a function

$$s : Y(A) \longrightarrow \mathbb{C}$$

by

$$s(g) = \sum_{\substack{q \pmod N \\ Aq \equiv 0 \pmod N}} e^{2\pi i \frac{{}^t g A q}{N^2}} = \sum_{q \in Y(A)/N\mathbb{Z}^f} e^{2\pi i \frac{{}^t g A q}{N^2}}$$

for $g \in Y(A)$. The function s is well-defined and

$$s(g) = \begin{cases} 0 & \text{if } g \not\equiv 0 \pmod N, \\ \#Y(A)/N\mathbb{Z}^f & \text{if } g \equiv 0 \pmod N \end{cases}$$

for $g \in Y(A)$.

Proof. To see that s is well defined, let $g, q_1, q_2 \in Y$ and assume that $q_2 = q_1 + Nq_3$ for some $q_3 \in \mathbb{Z}^f$. Then

$$\begin{aligned} {}^t g A q_2 &= {}^t g A q_1 + N\, {}^t g A q_3 \\ &= {}^t g A q_1 + N\, {}^t (Ag) A q_3 \end{aligned}$$

$$\equiv {}^{\mathrm{t}}gAq_1 \pmod{N^2}$$

because $Ag \equiv 0 \pmod N$. This implies that

$$e^{2\pi i \frac{{}^{\mathrm{t}}gAq_1}{N^2}} = e^{2\pi i \frac{{}^{\mathrm{t}}gAq_2}{N^2}},$$

so that s is well-defined. To prove the second assertion, asssume first that $g \equiv 0 \pmod N$. Write $g = Nm$ for some $m \in \mathbb{Z}^f$. Let $q \in Y(A)$. Then

$$\begin{aligned} {}^{\mathrm{t}}gAq &= N\,{}^{\mathrm{t}}m(Aq) \\ &\equiv 0 \pmod{N^2} \end{aligned}$$

since $Aq \equiv 0 \pmod N$ because $q \in Y(A)$. It follows that

$$s(g) = \sum_{q \in Y(A)/N\mathbb{Z}^f} e^{2\pi i \frac{{}^{\mathrm{t}}gAq}{N^2}} = \sum_{q \in Y(A)/N\mathbb{Z}^f} 1 = \#Y(A)/N\mathbb{Z}^f.$$

Finally, assume that $g \not\equiv 0 \pmod N$. Then there exists $m \in \mathbb{Z}^f$ such that ${}^{\mathrm{t}}gm \not\equiv 0 \pmod N$. This implies that ${}^{\mathrm{t}}gNm \not\equiv 0 \pmod{N^2}$. Let $q_1 = NA^{-1}m$. Then $q \in Y(A)$ because $Aq = Nm \equiv 0 \pmod N$. Also,

$${}^{\mathrm{t}}gAq_1 = {}^{\mathrm{t}}gNm \not\equiv 0 \pmod{N^2}.$$

This implies that $e^{2\pi i \frac{{}^{\mathrm{t}}gAq_1}{N^2}} \neq 1$. Since the function $Y(A)/N\mathbb{Z}^f \to \mathbb{C}^\times$ defined by $q \mapsto e^{2\pi i \frac{{}^{\mathrm{t}}gAq}{N^2}}$ is a character, and since this character is non-trivial at q_1, it follows that summing this character over the elements of $Y(A)/N\mathbb{Z}^f$ gives 0; this means that $s(g) = 0$. □

Proposition 2.7.5. *Let f be a positive even integer, and define $k = f/2$. Let $A \in \mathrm{M}(f, \mathbb{Z})$ be an even symmetric positive-definite matrix, and let N be the level of A. Define the quadratic form $Q(x)$ in f variables by*

$$Q(x) = \frac{1}{2}\,{}^{\mathrm{t}}xAx.$$

Let r be a non-negative integer, and let $P \in \mathcal{H}_r(A)$. Let $h \in \mathbb{Z}^f$ be such that

$$Ah \equiv 0 \pmod N.$$

Let

$$\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$$

and assume that d is a positive integer. Then

$$\theta(A, P, h, z)\big|_{k+r} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \Big(\frac{1}{d^k} \sum_{\substack{q \pmod{dN} \\ q \equiv h \pmod N}} e^{2\pi i \cdot \frac{bQ(q)}{dN^2}}\Big) \cdot \theta(A, P, ah, z). \quad (2.14)$$

Proof. We will abbreviate

$$\alpha = \begin{bmatrix} b & -a \\ d & -c \end{bmatrix}.$$

Applying first Lemma 2.7.3 (note that $d > 0$), and then (2.4), we obtain:

$$\begin{aligned}
&\theta(A,P,h,z)\big|_{k+r}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
&= \big(\theta(A,P,h,z)\big|_{k+r}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}\big)\big|_{k+r}\begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \\
&= \big(\theta(A,P,h,z)\big|_{k+r}\begin{bmatrix} b & a \\ d & -c \end{bmatrix}\big)\big|_{k+r}\begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \\
&= \frac{1}{i^{k+2r} d^k \sqrt{\det(A)}} \sum_{\substack{q \pmod N \\ Aq \equiv 0 \pmod N}} s_\alpha(q,h)\theta(A,P,q,z)\big|_{k+r}\begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \\
&= \frac{1}{i^{2r} d^k \det(A)} \sum_{\substack{q \pmod N \\ Aq \equiv 0 \pmod N}} \sum_{\substack{g \pmod N \\ Ag \equiv 0 \pmod N}} s_\alpha(q,h) e^{2\pi i \frac{{}^t g A q}{N^2}} \theta(A,P,g,z) \\
&= \frac{1}{i^{2r} d^k \det(A)} \sum_{\substack{g \pmod N \\ Ag \equiv 0 \pmod N}} \Big(\sum_{\substack{q \pmod N \\ Aq \equiv 0 \pmod N}} s_\alpha(q,h) e^{2\pi i \frac{{}^t g A q}{N^2}} \Big) \theta(A,P,g,z).
\end{aligned}$$

We can calculate the inner sum as follows:

$$\begin{aligned}
&\sum_{\substack{q \pmod N \\ Aq \equiv 0 \pmod N}} s_\alpha(q,h) e^{2\pi i \frac{{}^t g A q}{N^2}} \\
&= \sum_{\substack{q \pmod N \\ Aq \equiv 0 \pmod N}} s_\alpha(0,h-cq) e^{-2\pi i \left(\frac{-a\,{}^t h A q + acQ(q)}{N^2}\right)} e^{2\pi i \frac{{}^t g A q}{N^2}} \quad \text{(cf. (2.12))} \\
&= s_\alpha(0,h) \sum_{\substack{q \pmod N \\ Aq \equiv 0 \pmod N}} e^{2\pi i \left(\frac{{}^t(ah+g)Aq}{N^2}\right)} e^{2\pi i \left(\frac{-acQ(q)}{N^2}\right)} \\
&= s_\alpha(0,h) \sum_{\substack{q \pmod N \\ Aq \equiv 0 \pmod N}} e^{2\pi i \left(\frac{{}^t(ah+g)Aq}{N^2}\right)} \quad \text{(cf. Lemma 1.5.8)} \\
&= s_\alpha(0,h) s(g+ah) \quad \text{(cf. Lemma 2.7.4)} \\
&= s_\alpha(0,h) \times \begin{cases} 0 & \text{if } g \not\equiv -ah \pmod N, \\ \#Y(A)/N\mathbb{Z}^f & \text{if } g \equiv -ah \pmod N \end{cases} \quad \text{(cf. Lemma 2.7.4).}
\end{aligned}$$

It follows that

$$\theta(A,P,h,z)\big|_{k+r}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2.15}$$

$$= \frac{\#Y(A)/N\mathbb{Z}^f}{i^{2r}d^k\det(A)} \cdot s_\alpha(0,h)\cdot\theta(A,P,-ah,z)$$
$$= \frac{(-1)^r\#Y(A)/N\mathbb{Z}^f}{i^{2r}d^k\det(A)} \cdot s_\alpha(0,h)\cdot\theta(A,P,ah,z) \quad \text{(cf. (2.3))}$$
$$= \frac{\#Y(A)/N\mathbb{Z}^f}{d^k\det(A)} \cdot s_\alpha(0,h)\cdot\theta(A,P,ah,z). \tag{2.16}$$

The definition of s_α asserts that:

$$s_\alpha(0,h) = \sum_{\substack{q \pmod{dN} \\ q\equiv h \pmod{N}}} e^{2\pi i\left(\frac{bQ(q)}{dN^2}\right)}.$$

Finally, to determine $\#Y(A)/N\mathbb{Z}^f$, assume that $h=0$, $r=0$, and that P is the element of $\mathcal{H}_0(A)$ such that $P(X_1,\ldots,X_f)=1$. Then the function

$$\theta(A,1,0,z) = \sum_{n\in\mathbb{Z}^f} e^{2\pi izQ(n)}$$

is not identically zero. Also, let

$$\begin{bmatrix} a & b \\ c & d\end{bmatrix} = \begin{bmatrix} 1 & \\ & 1\end{bmatrix}, \quad \text{so that} \quad \alpha = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}.$$

Then $s_\alpha(0,0)=1$, and (2.16) asserts that:

$$\theta(A,1,0,z) = \frac{\#Y(A)/N\mathbb{Z}^f}{\det(A)}\cdot\theta(A,1,0,z).$$

We conclude that

$$\#Y(A)/N\mathbb{Z}^f = \det(A).$$

This completes the proof. □

Lemma 2.7.6. *Let f be a positive even integer, let $A\in \mathrm{M}(f,\mathbb{Z})$ be an even symmetric positive-definite matrix, and let N be the level of A. Let*

$$Y(A) = \{h\in\mathbb{Z}^f : Ah\equiv 0 \pmod{N}\}.$$

Then

$$\#Y(A)/N\mathbb{Z}^f = \det(A).$$

Proof. This was proven in the proof of Proposition 2.7.5. □

Lemma 2.7.7. *Let f be a positive even integer, and define $k=f/2$. Let $A\in\mathrm{M}(f,\mathbb{Z})$ be an even symmetric positive-definite matrix, and let N be the level of A. Assume that $N>1$. Define the quadratic form $Q(x)$ in f variables by*

$$Q(x) = \frac{1}{2}\,{}^{\mathrm{t}}xAx.$$

Define

$$\chi_A : \mathbb{Z} \longrightarrow \mathbb{C}$$

by

$$\chi_A(d) = \frac{1}{d^k} \cdot \sum_{m \in \mathbb{Z}^f / d\mathbb{Z}^f} e^{2\pi i \cdot \frac{Q(m)}{d}}$$

for $d \in \mathbb{Z}$ *with* $(d, N) = 1$ *and* $d > 0$*, by*

$$\chi_A(d) = (-1)^k \chi_A(-d)$$

for $d \in \mathbb{Z}$ *with* $(d, N) = 1$ *and* $d < 0$*, and by* $\chi(d) = 0$ *for* $d \in \mathbb{Z}$ *with* $(d, N) > 1$*. Then* χ_A *is a well-defined real-valued Dirichlet character modulo* N*. Moreover, if* r *is a non-negative integer,* $h \in \mathbb{Z}^f$ *is such that* $Ah \equiv 0 \pmod{N}$*, and* $P \in \mathcal{H}_r(A)$*, then*

$$\theta(A, P, h, z)\big|_{k+r} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = e^{2\pi i \cdot \frac{abQ(h)}{N^2}} \cdot \chi_A(d) \cdot \theta(A, P, ah, z) \tag{2.17}$$

for

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N).$$

Proof. Define a function

$$\alpha : \Gamma_0(N) \longrightarrow \mathbb{C}$$

in the following way. Let

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N). \tag{2.18}$$

If $d > 0$, then define

$$\alpha(g) = \frac{1}{d^k} \sum_{q \in \mathbb{Z}^f / d\mathbb{Z}^f} e^{2\pi i \cdot \frac{bQ(q)}{d}} \tag{2.19}$$

and if $d < 0$, define

$$\alpha(g) = (-1)^k \alpha(\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}) = (-1)^k \alpha(\begin{bmatrix} -1 & \\ & -1 \end{bmatrix} g). \tag{2.20}$$

Note that $d \neq 0$ since $ad - bc = 1$ and $N > 1$ (by assumption). Our first goal will be to prove that α takes values in $\mathbb{Q}^\times$ and is in fact a homomorphism from $\Gamma_0(N)$ to $\mathbb{Q}^\times$. Let $P = 1 \in \mathcal{H}_0(A)$ be the polynomial in f variables such that $P(X_1, \dots, X_f) = 1$. Let g be as in (2.18), and assume $d > 0$. Then by (2.14) we have

$$\theta(A, 1, 0, z)\big|_k g = \Big(\frac{1}{d^k} \sum_{\substack{q \in \mathbb{Z}^f / dN\mathbb{Z}^f \\ q \equiv 0 \pmod{N}}} e^{2\pi i \cdot \frac{bQ(q)}{dN^2}}\Big) \cdot \theta(A, 1, 0, z)$$

$$= \Big(\frac{1}{d^k} \sum_{q \in \mathbb{Z}^f/d\mathbb{Z}^f} e^{2\pi i \cdot \frac{bQ(Nq)}{dN^2}}\Big) \cdot \theta(A,1,0,z)$$
$$= \Big(\frac{1}{d^k} \sum_{q \in \mathbb{Z}^f/d\mathbb{Z}^f} e^{2\pi i \cdot \frac{bQ(q)}{d}}\Big) \cdot \theta(A,1,0,z)$$
$$\theta(A,1,0,z)\big|_k g = \alpha(g) \cdot \theta(A,1,0,z).$$

Assume that $d < 0$. Then by what we just proved,

$$\begin{aligned} \theta(A,1,0,z)\big|_k g &= \theta(A,1,0,z)\big|_k \begin{bmatrix} -1 & \\ & -1 \end{bmatrix} \begin{bmatrix} -1 & \\ & -1 \end{bmatrix} g \\ &= (-1)^k \theta(A,1,0,z)\big|_k \begin{bmatrix} -1 & \\ & -1 \end{bmatrix} g \\ &= (-1)^k \alpha(-g)\theta(A,1,0,z) \\ &= \alpha(g) \cdot \theta(A,1,0,z). \end{aligned}$$

Thus,

$$\theta(A,1,0,z)\big|_k g = \alpha(g) \cdot \theta(A,1,0,z)$$

for all $g \in \Gamma_0(N)$. Since $\theta(A,1,0,z)$ is non-zero, this formula also implies that $\alpha(g) \neq 0$ for all $g \in \Gamma_0(N)$. Thus, α actually takes values in $\mathbb{C}^\times$. Let $g, g' \in \Gamma_0(N)$. Then

$$\begin{aligned} \theta(A,1,0,z)\big|_k (gg') &= \big(\theta(A,1,0,z)\big|_k g\big)\big|_k g' \\ \alpha(gg')\theta(A,1,0,z) &= \alpha(g) \cdot \theta(A,1,0,z)\big|_k g' \\ \alpha(gg')\theta(A,1,0,z) &= \alpha(g)\alpha(g')\theta(A,1,0,z). \end{aligned}$$

Since $\theta(A,1,0,z) \neq 0$, we have

$$\alpha(gg') = \alpha(g)\alpha(g') \tag{2.21}$$

for $g, g' \in \Gamma_0(N)$. We have already noted that $\alpha(g)$ is non-zero for all $g \in \Gamma_0(N)$; we will now show that α takes values in $\mathbb{Q}^\times$. To prove this it will suffice to prove that $\alpha(g) \in \mathbb{Q}$ for g as in (2.18) with $d > 0$. Fix such a g. If $d = 1$ then it is clear that $\alpha(g) \in \mathbb{Q}$. Assume that $d > 1$. Then $c \neq 0$ (recall that $ad - bc = 1$). Let n be an integer such that $nc + d > 0$. Then

$$\begin{aligned} \alpha(\begin{bmatrix} 1 & n \\ & 1 \end{bmatrix})\alpha(g) &= \alpha(\begin{bmatrix} 1 & n \\ & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}) \\ 1 \cdot \alpha(g) &= \alpha(\begin{bmatrix} a & an+b \\ c & cn+d \end{bmatrix}) \\ \alpha(g) &= \alpha(\begin{bmatrix} a & an+b \\ c & cn+d \end{bmatrix}). \end{aligned}$$

By the definition of α, this implies that

$$\alpha(g) = \frac{1}{(cn+d)^k} \sum_{q \in \mathbb{Z}^f/d\mathbb{Z}^f} e^{2\pi i \cdot \frac{(an+b)Q(q)}{cn+d}}.$$

It is clear from this formula that

$$\alpha(g) \in \mathbb{Q}(\zeta_{nc+d})$$

where $\zeta_{nc+d} = e^{2\pi i/(nc+d)}$ is a primitive $nc+d$-th root of unity. Assume that $c > 0$. Then $c + d > 0$, and

$$\alpha(g) \in \mathbb{Q}(\zeta_d) \cap \mathbb{Q}(\zeta_{c+d}).$$

Since c and d are non-zero and relatively prime (because $ad - bc = 1$), d and $c+d$ are relatively prime. This implies that $\mathbb{Q}(\zeta_d) \cap \mathbb{Q}(\zeta_{c+d}) = \mathbb{Q}$, so that $\alpha(g) \in \mathbb{Q}$. Assume that $c < 0$. Then $(-1)c + d > 0$, and

$$\alpha(g) \in \mathbb{Q}(\zeta_d) \cap \mathbb{Q}(\zeta_{-c+d}).$$

Since $-c$ and d are non-zero and relatively prime, d and $-c + d$ are relatively prime, and $\mathbb{Q}(\zeta_d) \cap \mathbb{Q}(\zeta_{-c+d}) = \mathbb{Q}$, so that $\alpha(g) \in \mathbb{Q}$. This completes the argument that $\alpha(g) \in \mathbb{Q}$ for $g \in \Gamma_0(N)$.

Now we prove the claims about χ_A. We need to prove that the four conditions of Lemma 1.1.1 hold for χ_A. It is immediate from the formula for χ_A that $\chi_A(1) = 1$; this proves the first condition. The third condition, that $\chi_A(d) = 0$ for $d \in \mathbb{Z}$ such that $(d, N) > 1$, follows from the definition of χ_A.

To prove the remaining conditions we first make a connection to α. We will prove that if $d \in \mathbb{Z}$ with $(d, N) = 1$, and

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$$

then

$$\chi_A(d) = \alpha(\begin{bmatrix} a & b \\ c & d \end{bmatrix}). \tag{2.22}$$

Assume first that $d > 0$. By definition,

$$\alpha(g) = \frac{1}{d^k} \sum_{q \in \mathbb{Z}^f/d\mathbb{Z}^f} e^{2\pi i \cdot \frac{bQ(q)}{d}}$$

The summands in this formula are contained in $\mathbb{Q}(\zeta_d)$, where $\zeta_d = e^{2\pi i/d}$. Since $(b, d) = 1$, there exists an element σ of $\mathrm{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q})$ such that $\sigma(\zeta_d) = \zeta_d^b$. We have $\sigma^{-1}(\zeta_d^b) = \zeta_d$. Applying σ^{-1} to both sides of the above formula, and using that $\alpha(g) \in \mathbb{Q}$, we obtain:

$$\alpha(g) = \frac{1}{d^k} \sum_{q \in \mathbb{Z}^f/d\mathbb{Z}^f} e^{2\pi i \cdot \frac{Q(q)}{d}}$$

$$\alpha(g) = \chi_A(d).$$

This proves (2.22) for the case $d > 0$. Assume that $d < 0$. Using the previous case, and the definition of α, we have:

$$\chi_A(d) = (-1)^k \chi_A(-d)$$

$$
\begin{aligned}
&= (-1)^k \alpha(\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}) \\
&= (-1)^k \alpha(\begin{bmatrix} -1 & \\ & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}) \\
\chi_A(d) &= \alpha(\begin{bmatrix} a & b \\ c & d \end{bmatrix}).
\end{aligned}
$$

This proves (2.22) in all cases.

Now we will prove the fourth condition of Lemma 1.1.1, which asserts that $\chi_A(d) = \chi_A(d+N)$ for all $d \in \mathbb{Z}$. Let $d \in \mathbb{Z}$. If $(d,N) > 1$, then $(d+N,N) > 1$, and $\chi_A(d) = 0 = \chi_A(d+N)$. Assume that $(d,N) = 1$. Then there exists $a, b \in \mathbb{Z}$ such that $ad - bN = 1$. By (2.22),

$$
\begin{aligned}
\alpha(\begin{bmatrix} a & b \\ N & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}) &= \alpha(\begin{bmatrix} a & b \\ N & d \end{bmatrix}) \alpha(\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}) \\
\alpha(\begin{bmatrix} a & a+b \\ N & d+N \end{bmatrix} &= \chi_A(d) \cdot 1 \\
\chi_A(d+N) &= \chi_A(d). \quad \text{(cf. (2.22))}
\end{aligned}
$$

To prove the remaining second condition of Lemma 1.1.1 let $d_1, d_2 \in \mathbb{Z}$. If $(d_1, N) > 0$ or $(d_2, N) > 0$, then evidently $\chi_A(d_1 d_2) = 0 = \chi_A(d_1)\chi_A(d_2)$. Assume, therefore, that $(d_1, N) = (d_2, N) = 1$. There exist $a_1, b_1, a_2, b_2 \in \mathbb{Z}$ and $\varepsilon_2 \in \{\pm 1\}$ such that be such that $a_1 d_1 - b_1 N = 1$, $a_2 d_2 - b_2 \varepsilon_2 N = 1$, and $b_2 \geq 0$. Then

$$
\begin{aligned}
\alpha(\begin{bmatrix} a_1 & b_1 \\ N & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ \varepsilon_2 N & d_2 \end{bmatrix}) &= \alpha(\begin{bmatrix} a_1 a_2 + b_1 \varepsilon_2 N & a_1 b_2 + b_1 d_2 \\ a_2 N + d_1 \varepsilon_2 N & d_1 d_2 + b_2 N \end{bmatrix}) \\
\alpha(\begin{bmatrix} a_1 & b_1 \\ N & d_1 \end{bmatrix}) \alpha(\begin{bmatrix} a_2 & b_2 \\ \varepsilon_2 N & d_2 \end{bmatrix}) &= \alpha(\begin{bmatrix} a_1 a_2 + b_1 \varepsilon_2 N & a_1 b_2 + b_1 d_2 \\ a_2 N + d_1 \varepsilon_2 N & d_1 d_2 + b_2 N \end{bmatrix}) \\
\chi_A(d_1)\chi_A(d_2) &= \chi_A(d_1 d_2 + b_2 N) \\
\chi_A(d_1)\chi_A(d_2) &= \chi_A(d_1 d_2 + \underbrace{N + \cdots + N}_{b_2}) \\
\chi_A(d_1)\chi_A(d_2) &= \chi_A(d_1 d_2) \quad \text{(fourth condition)}.
\end{aligned}
$$

We have proven that all the conditions of Lemma 1.1.1; by this lemma χ_A is a Dirichlet character modulo N. Since (2.22) holds, and since $\alpha(g) \in \mathbb{Q}^\times$ for all $g \in \Gamma_0(N)$, it follows that χ_A is real-valued.

It remains to prove (2.17). Let

$$
g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)
$$

and let $h \in Y(A)$, i.e., $h \in \mathbb{Z}^f$ with $Ah \equiv 0 \pmod{N}$. First assume that $d > 0$. We have:

$$
\frac{1}{d^k} \sum_{\substack{q \pmod{dN} \\ q \equiv h \pmod{N}}} e^{2\pi i \cdot \frac{bQ(q)}{dN^2}}
$$

$$
\begin{aligned}
&= \frac{1}{d^k} \sum_{\substack{q \in \mathbb{Z}^f/dN\mathbb{Z}^f \\ q \equiv h \pmod{N}}} e^{2\pi i \cdot \frac{bQ(q)}{dN^2}} \\
&= \frac{1}{d^k} \sum_{\substack{q \in \mathbb{Z}^f/dN\mathbb{Z}^f \\ q \equiv ad\cdot h \pmod{N}}} e^{2\pi i \cdot \frac{bQ(q)}{dN^2}} \quad (ad \equiv 1 \pmod{N}) \\
&= \frac{1}{d^k} \sum_{\substack{q \in \mathbb{Z}^f/N\mathbb{Z}^f \\ q \equiv ad\cdot h \pmod{N}}} \sum_{q_1 \in N\mathbb{Z}^f/dN\mathbb{Z}^f} e^{2\pi i \cdot \frac{bQ(q+q_1)}{dN^2}} \\
&= \frac{1}{d^k} \sum_{q_1 \in N\mathbb{Z}^f/dN\mathbb{Z}^f} e^{2\pi i \cdot \frac{bQ(ad\cdot h) + b\,{}^{\mathrm{t}}(ad\cdot h)Aq_1 + bQ(q_1)}{dN^2}} \\
&= \frac{1}{d^k} \sum_{m \in \mathbb{Z}^f/d\mathbb{Z}^f} e^{2\pi i \cdot \frac{ba^2d^2Q(h) + abdN\,{}^{\mathrm{t}}hAm + bN^2Q(m)}{dN^2}} \\
&= \frac{1}{d^k} \cdot e^{2\pi i \cdot \frac{ab\cdot ad\cdot Q(h)}{N^2}} \cdot \sum_{m \in \mathbb{Z}^f/d\mathbb{Z}^f} e^{2\pi i \cdot \frac{ab\,{}^{\mathrm{t}}(Ah)m}{N}} \cdot e^{2\pi i \cdot \frac{bQ(m)}{d}} \\
&= e^{2\pi i \cdot \frac{ab\cdot ad\cdot Q(h)}{N^2}} \cdot \frac{1}{d^k} \cdot \sum_{m \in \mathbb{Z}^f/d\mathbb{Z}^f} e^{2\pi i \cdot \frac{bQ(m)}{d}} \quad (\text{since } Ah \equiv 0 \pmod{N}) \\
&= e^{2\pi i \cdot \frac{abQ(h)}{N^2}} \cdot \frac{1}{d^k} \cdot \sum_{m \in \mathbb{Z}^f/d\mathbb{Z}^f} e^{2\pi i \cdot \frac{bQ(m)}{d}} \quad (ad = 1 + bc,\ N|c,\ \text{Lemma 1.5.8}) \\
&= e^{2\pi i \cdot \frac{abQ(h)}{N^2}} \cdot \alpha(g) \\
&= e^{2\pi i \cdot \frac{abQ(h)}{N^2}} \cdot \chi_A(d) \quad (\text{cf. } (2.22)).
\end{aligned}
$$

In summary, if $d > 0$, then

$$
\frac{1}{d^k} \sum_{\substack{q \pmod{dN} \\ q \equiv h \pmod{N}}} e^{2\pi i \cdot \frac{bQ(q)}{dN^2}} = e^{2\pi i \cdot \frac{abQ(h)}{N^2}} \cdot \chi_A(d).
$$

This equality and (2.14) now imply (2.17) if $d > 0$. Assume that $d < 0$. We then have:

$$
\begin{aligned}
&\theta(A, P, h, z)\big|_{k+r} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
&= \theta(A, P, h, z)\big|_{k+r} \begin{bmatrix} -1 & \\ & -1 \end{bmatrix} \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} \\
&= (-1)^{k+r} \theta(A, P, h, z)\big|_{k+r} \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} \\
&= (-1)^{k+r} e^{2\pi i \cdot \frac{(-a)(-b)Q(h)}{N^2}} \cdot \chi_A(-d) \cdot \theta(A, P, (-a)h, z) \\
&= (-1)^{k+r} e^{2\pi i \cdot \frac{abQ(h)}{N^2}} (-1)^k \cdot \chi_A(d) \cdot (-1)^r \theta(A, P, ah, z) \quad (\text{cf. } (2.3))
\end{aligned}
$$

$$= e^{2\pi i\cdot\frac{abQ(h)}{N^2}}\cdot\chi_A(d)\cdot\theta(A,P,ah,z).$$

This completes the proof. □

Calculation of χ_A

Lemma 2.7.8. *Let p be a prime, and let $\chi : (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character modulo p. We define the* ***Gauss sum*** $\mathrm{W}(\chi)$ *to be the complex number*

$$\mathrm{W}(\chi) = \sum_{a=0}^{p-1}\chi(a)e^{2\pi i\frac{a}{p}} = \sum_{a\in\mathbb{Z}/p\mathbb{Z}}\chi(a)e^{2\pi i\frac{a}{p}}.$$

If χ is trivial, then $\mathrm{W}(\chi)=0$. If χ is non-trivial, then

$$\mathrm{W}(\chi)\mathrm{W}(\bar{\chi}) = \chi(-1)p.$$

Proof. Let G be a finite group. In this proof we will the following fact:

$$\text{If } \eta\in\mathrm{Hom}(G,\mathbb{C}^\times) \text{ and } \eta\neq 1, \text{ then } \sum_{g\in G}\eta(g)=0. \tag{2.23}$$

Assume that $\chi=1$. Consider the function $\mathbb{Z}/p\mathbb{Z}\to\mathbb{C}^\times$ defined by $a\mapsto e^{2\pi i\frac{a}{p}}$. This function is a non-trivial element of $\mathrm{Hom}(\mathbb{Z}/p\mathbb{Z},\mathbb{C}^\times)$. The assertion $\mathrm{W}(\chi)=0$ follows from (2.23).

Next, assume that χ is non-trivial. In the following computation, if $b\in(\mathbb{Z}/p\mathbb{Z})^\times$, then we will denote the inverse of b in $(\mathbb{Z}/p\mathbb{Z})^\times$ by b', so that $bb'=1$. We have

$$\begin{aligned}
\mathrm{W}(\chi)\mathrm{W}(\bar{\chi}) &= (\sum_{a\in\mathbb{Z}/p\mathbb{Z}}\chi(a)e^{2\pi i\frac{a}{p}})\cdot(\sum_{b\in\mathbb{Z}/p\mathbb{Z}}\overline{\chi(b)}e^{2\pi i\frac{b}{p}})\\
&= (\sum_{a\in\mathbb{Z}/p\mathbb{Z}}\chi(a)e^{2\pi i\frac{a}{p}})\cdot(\sum_{b\in(\mathbb{Z}/p\mathbb{Z})^\times}\chi(b)^{-1}e^{2\pi i\frac{b}{p}})\\
&= (\sum_{a\in\mathbb{Z}/p\mathbb{Z}}\chi(a)e^{2\pi i\frac{a}{p}})\cdot\sum_{b\in(\mathbb{Z}/p\mathbb{Z})^\times}\chi(b')e^{2\pi i\frac{b}{p}})\\
&= \sum_{b\in(\mathbb{Z}/p\mathbb{Z})^\times}\sum_{a\in\mathbb{Z}/p\mathbb{Z}}\chi(ab')e^{2\pi i\frac{a+b}{p}}\\
&= \sum_{b\in(\mathbb{Z}/p\mathbb{Z})^\times}\sum_{a\in\mathbb{Z}/p\mathbb{Z}}\chi(abb')e^{2\pi i\frac{ab+b}{p}}\\
&= \sum_{b\in(\mathbb{Z}/p\mathbb{Z})^\times}\sum_{a\in\mathbb{Z}/p\mathbb{Z}}\chi(a)e^{2\pi i\frac{(a+1)b}{p}}\\
&= \sum_{a\in\mathbb{Z}/p\mathbb{Z}}\chi(a)\sum_{b\in(\mathbb{Z}/p\mathbb{Z})^\times}e^{2\pi i\frac{(a+1)b}{p}}\\
&= \sum_{a\in\mathbb{Z}/p\mathbb{Z}}\chi(a)\big(-1+\sum_{b\in\mathbb{Z}/p\mathbb{Z}}e^{2\pi i\frac{(a+1)b}{p}}\big)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{a \in \mathbb{Z}/p\mathbb{Z} \\ a+1 \equiv 0 \pmod p}} \chi(a)\big(-1 + \sum_{b \in \mathbb{Z}/p\mathbb{Z}} e^{2\pi i \frac{(a+1)b}{p}}\big) \\
&\quad + \sum_{\substack{a \in \mathbb{Z}/p\mathbb{Z} \\ a+1 \not\equiv 0 \pmod p}} \chi(a)\big(-1 + \sum_{b \in \mathbb{Z}/p\mathbb{Z}} e^{2\pi i \frac{(a+1)b}{p}}\big) \\
&= \chi(-1)\big(-1+p\big) \\
&\quad + \sum_{\substack{a \in \mathbb{Z}/p\mathbb{Z} \\ a+1 \not\equiv 0 \pmod p}} \chi(a)\big(-1+0\big) \quad \text{(cf. (2.23))} \\
&= \chi(-1)(p-1) - \sum_{\substack{a \in \mathbb{Z}/p\mathbb{Z} \\ a+1 \not\equiv 0 \pmod p}} \chi(a) \\
&= \chi(-1)(p-1) - \big(-\chi(-1) + \sum_{a \in \mathbb{Z}/p\mathbb{Z}} \chi(a)\big) \\
&= \chi(-1)(p-1) - \big(-\chi(-1) + 0\big) \quad \text{(cf. (2.23))} \\
&= p\chi(-1).
\end{aligned}$$

This completes the proof. □

Lemma 2.7.9. *Let f be a positive even integer, and define $k = f/2$. Let $A \in \mathrm{M}(f, \mathbb{Z})$ be an even symmetric positive-definite matrix, and let N be the level of A. Assume that $N > 1$. We recall from Lemma 1.5.4 that N divides $\det(A)$, and that $\det(A)$ and N have the same set of prime divisors. Define $\chi_A : \mathbb{Z} \to \mathbb{C}$ as in Lemma 2.7.7; by this lemma, χ_A is a Dirichlet character modulo N. Let $\Delta = \Delta(A) = (-1)^k \det(A)$ be the discriminant of A. Let $\big(\frac{\Delta}{\cdot}\big)$ be the Kronecker symbol from section 1.4, which is a Dirichlet character modulo $\det(A)$ by Proposition 1.4.2 and Lemma 1.5.2. Then the diagram*

$$\begin{array}{ccc}
(\mathbb{Z}/\det(A)\mathbb{Z})^\times & \longrightarrow & (\mathbb{Z}/N\mathbb{Z})^\times \\
& \searrow^{(\frac{\Delta}{\cdot})} & \downarrow \chi_A \\
& & \{\pm 1\}
\end{array}$$

commutes. We have

$$\chi_A(d) = \Big(\frac{\Delta}{d}\Big) = \Big(\frac{(-1)^k \det(A)}{d}\Big) \tag{2.24}$$

for $d \in \mathbb{Z}$.

Proof. By Lemma 1.5.4, N divides $\det(A)$, and $\det(A)$ and N have the same set of prime divisors. To prove the assertions of this lemma it will suffice to prove that $\chi_A(d) = \big(\frac{\Delta}{d}\big)$ for $d \in \mathbb{Z}$ with $(d, N) = 1$. Let $d \in \mathbb{Z}$ with $(d, N) = 1$; then $(d, \det(A)) = 1$. By Dirichlet's theorem about infinitely many primes in arithmetic progressions (see, for example, Theorem 155 on p. 125 of [14]), there exists an odd prime p such that $p \equiv d \pmod{\det(A)}$. Then $(p, N) = 1$ and $p \equiv d \pmod{N}$. Regard A as an element of $\mathrm{M}(f, \mathbb{Z}/p\mathbb{Z})$. We have $\det(A) \in (\mathbb{Z}/p\mathbb{Z})^\times$. It follows that there exists a matrix $U \in \mathrm{M}(f, \mathbb{Z})$ and $a_1, \ldots, a_f \in \mathbb{Z}$ such that $(a_1, p) = \cdots = (a_f, p) = 1$, $(\det(U), p) = 1$, and

$$^{\mathrm{t}}UAU \equiv \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_f \end{bmatrix} \pmod{p}.$$

We have

$$\begin{aligned}
\chi_A(d) &= \chi_A(p) \\
&= \frac{1}{p^k} \cdot \sum_{m \in \mathbb{Z}^f/p\mathbb{Z}^f} e^{2\pi i \cdot \frac{Q(m)}{p}} \\
&= \frac{1}{p^k} \cdot \sum_{m \in \mathbb{Z}^f/p\mathbb{Z}^f} e^{2\pi i \cdot \frac{Q(2m)}{p}} \\
&= \frac{1}{p^k} \cdot \sum_{m \in (\mathbb{Z}/p\mathbb{Z})^f} e^{2\pi i \cdot \frac{4\, ^{\mathrm{t}}mAm}{2p}} \\
&= \frac{1}{p^k} \cdot \sum_{m \in (\mathbb{Z}/p\mathbb{Z})^f} e^{2\pi i \cdot \frac{2 \cdot ^{\mathrm{t}}mAm}{p}} \\
&= \frac{1}{p^k} \cdot \sum_{m \in (\mathbb{Z}/p\mathbb{Z})^f} e^{2\pi i \cdot \frac{2\, ^{\mathrm{t}}(Um)A(Um)}{p}} \\
&= \frac{1}{p^k} \cdot \sum_{m \in (\mathbb{Z}/p\mathbb{Z})^f} e^{2\pi i \cdot \frac{2\, ^{\mathrm{t}}m\, ^{\mathrm{t}}UAUm}{p}} \\
&= \frac{1}{p^k} \cdot \sum_{m \in (\mathbb{Z}/p\mathbb{Z})^f} e^{2\pi i \cdot \frac{2(a_1 m_1^2 + \cdots + a_f m_f^2)}{p}} \\
&= \frac{1}{p^k} \cdot \prod_{1 \le i \le f} \sum_{m_i \in \mathbb{Z}/p\mathbb{Z}} e^{2\pi i \cdot \frac{2a_i m_i^2}{p}} \\
&= \frac{1}{p^k} \cdot \prod_{1 \le i \le f} \sum_{m_i \in \mathbb{Z}/p\mathbb{Z}} \left(1 + \left(\frac{m_i}{p}\right)\right) \cdot e^{2\pi i \cdot \frac{2a_i m_i}{p}} \\
&= \frac{1}{p^k} \cdot \prod_{1 \le i \le f} \left(\sum_{m_i \in \mathbb{Z}/p\mathbb{Z}} e^{2\pi i \cdot \frac{2a_i m_i}{p}} + \sum_{m_i \in \mathbb{Z}/p\mathbb{Z}} \left(\frac{m_i}{p}\right) e^{2\pi i \cdot \frac{2a_i m_i}{p}} \right) \\
&= \frac{1}{p^k} \cdot \prod_{1 \le i \le f} \sum_{m_i \in \mathbb{Z}/p\mathbb{Z}} \left(\frac{m_i}{p}\right) e^{2\pi i \cdot \frac{2a_i m_i}{p}} \quad (\text{cf. } (2.23)) \\
&= \frac{1}{p^k} \cdot \prod_{1 \le i \le f} \sum_{m_i \in \mathbb{Z}/p\mathbb{Z}} \left(\frac{2a_i m_i}{p}\right) e^{2\pi i \cdot \frac{m_i}{p}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{p^k} \cdot \prod_{1 \le i \le f} \Big(\frac{2a_i}{p}\Big) \sum_{m_i \in \mathbb{Z}/p\mathbb{Z}} \Big(\frac{m_i}{p}\Big) e^{2\pi i \cdot \frac{m_i}{p}} \\
&= \frac{1}{p^k} \cdot \prod_{1 \le i \le f} \Big(\frac{2a_i}{p}\Big) \mathrm{W}\big(\big(\frac{\cdot}{p}\big)\big) \\
&= \frac{\mathrm{W}((\frac{\cdot}{p}))^f}{p^k} \cdot \prod_{1 \le i \le f} \Big(\frac{2a_i}{p}\Big) \\
&= \frac{\big(\mathrm{W}((\frac{\cdot}{p}))^2\big)^k}{p^k} \cdot \Big(\frac{2^f a_1 \cdots a_f}{p}\Big) \\
&= \frac{\big(p(\frac{-1}{p})\big)^k}{p^k} \cdot \Big(\frac{2^f \det(U)^2 \det(A)}{p}\Big) \quad \text{(cf. Lemma 2.7.8)} \\
&= \Big(\frac{(-1)^k}{p}\Big) \cdot \Big(\frac{\det(A)}{p}\Big) \\
&= \Big(\frac{(-1)^k \det(A)}{p}\Big) \\
&= \Big(\frac{\Delta}{p}\Big) \\
&= \Big(\frac{\Delta}{d}\Big).
\end{aligned}
$$

This completes the proof. □

Theorem 2.7.10. *Let f be a positive even integer, and define $k = f/2$. Let $A \in \mathrm{M}(f, \mathbb{Z})$ be an even symmetric positive-definite matrix, and let N be the level of A. Define the quadratic form $Q(x)$ in f variables by*

$$Q(x) = \frac{1}{2}\,{}^{\mathrm{t}}xAx.$$

Let r be a non-negative integer, and let $P \in \mathcal{H}_r(A)$. Let $h \in \mathbb{Z}^f$ be such that

$$Ah \equiv 0 \pmod{N}.$$

The analytic function $\theta(A, P, h, z)$ on $\mathbb{H}_1$ defined by

$$\theta(A, P, h, z) = \sum_{\substack{m \in \mathbb{Z}^f \\ n \equiv 0 \pmod{N}}} P(n) e^{2\pi i z \frac{Q(n)}{N^2}}$$

for $z \in \mathbb{H}_1$ from Lemma 2.4.1 is a modular form of weight $k + r$ with respect to $\Gamma(N)$. If $r > 0$, then $\theta(A, P, h, z)$ is a cusp form.

Proof. The case $N = 1$ is Proposition 2.5.1. We may thus assume that $N > 1$. Let

$$\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(N).$$

Then $\alpha \in \Gamma_0(N)$. By (2.17), we have

$$\theta(A, P, h, z)\big|_{k+r}\alpha = e^{2\pi i \cdot \frac{abQ(h)}{N^2}} \cdot \chi_A(d) \cdot \theta(A, P, ah, z).$$

Since $\alpha \in \Gamma(N)$ we have $a \equiv d \equiv 1 \pmod N$ and $b \equiv c \equiv 0 \pmod N$. By Lemma 2.7.7, χ_A is a Dirichlet character modulo N; hence, $\chi_A(d) = 1$. By Lemma 1.5.8, $Q(h) \equiv 0 \pmod N$. Hence, $abQ(h) \equiv 0 \pmod{N^2}$; this implies that $e^{2\pi i \cdot \frac{abQ(h)}{N^2}} = 1$. Since $a \equiv 1 \pmod N$, we see that $ah \equiv h \pmod N$; by (2.2), this implies that $\theta(A, P, ah, z) = \theta(A, P, h, z)$. We now have

$$\theta(A, P, h, z)\big|_{k+r}\alpha = \theta(A, P, h, z).$$

To prove that $\theta(A, P, h, z)$ is a modular form of weight $k + r$ with respect to $\Gamma(N)$ we still need to prove that $\theta(A, P, h, z)$ is holomorphic at the cusps of $\Gamma(N)$, as defined in section 1.8. Clearly, N is the smallest positive integer M such that $\Gamma(M) \subset \Gamma(N)$. To prove that $\theta(A, P, h, z)$ is holomorphic at the cusps of $\Gamma(N)$, and is a cusp form if $r > 0$, it will suffice to prove that for each $\sigma \in \mathrm{SL}(2, \mathbb{Z})$ there exists a power series

$$\sum_{m=0}^{\infty} a(m)q^m$$

that converges in $D(1) = \{q \in \mathbb{C} : |q| < 1\}$ such that

$$\theta(A, P, h, z)\big|_{k+r}\sigma = \sum_{m=0}^{\infty} a(m)e^{2\pi i m/N}$$

for $z \in \mathbb{H}_1$, and $a(0) = 0$ if $r > 0$. Let

$$\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

We recall the set $Y(A) = \{g \in \mathbb{Z}^f : Ag \equiv 0 \pmod N\}$, and the finite-dimensional vector space $V(A, P)$ spanned by the theta series $\theta(A, P, g, z)$ for $g \in Y(A)/N\mathbb{Z}^f$ from Lemma 2.4.1. By Lemma 2.4.1 the vector space $V(A, P)$ is preserved by $\mathrm{SL}(2, \mathbb{Z})$ under the $\big|_{k+r}$ action. It follows that there exist constants $c(g) \in \mathbb{C}$ for $g \in Y(A)/N\mathbb{Z}^f$ such that

$$\theta(A, P, h, z)\big|_{k+r}\sigma = \sum_{g \in Y(A)/N\mathbb{Z}^f} c(g) \cdot \theta(A, P, g, z). \tag{2.25}$$

Let $g \in Y(A)$. By Lemma 1.5.8, for every $n \in \mathbb{Z}^f$ with $n \equiv g \pmod N$, the number $Q(n)/N$ is a non-negative integer. Consequently, we may consider the power series

$$\sum_{\substack{n \in \mathbb{Z}^f \\ n \equiv g \pmod N}} P(n)q^{\frac{Q(n)}{N}} \tag{2.26}$$

in the complex variable q. Let $q \in D(1)$. There exists $z \in \mathbb{H}_1$ such that $q = e^{2\pi i z/N}$. Since

$$\sum_{\substack{n \in \mathbb{Z}^f \\ n \equiv g \pmod N}} P(n) q^{\frac{Q(n)}{N}} = \sum_{\substack{n \in \mathbb{Z}^f \\ n \equiv g \pmod N}} P(n) e^{2\pi i z \frac{Q(n)}{N^2}} = \theta(A, P, g, z)$$

converges absolutely by Lemma 2.4.1, it follows that the power series (2.26) converges absolutely at q. Hence, the radius of convergence of (2.26) is at least 1. Consequently, the radius of convergence of the finite linear combination of power series

$$\sum_{g \in Y(A)/N\mathbb{Z}^f} c(g) \sum_{\substack{n \in \mathbb{Z}^f \\ n \equiv g \pmod N}} P(n) q^{\frac{Q(n)}{N}} \tag{2.27}$$

is also at least 1. Denote this power series by

$$\sum_{m=0}^{\infty} a(m) q^m.$$

By construction,

$$\theta(A, P, h, z)\big|_{k+r} \sigma = \sum_{m=0}^{\infty} a(m) e^{2\pi i m/N}$$

for $z \in \mathbb{H}_1$. This proves that $\theta(A, h, P, z)$ is a modular form of weight $k+r$ with respect to $\Gamma(N)$. Finally, assume that $r > 0$; we need to prove that $a(0) = 0$. From above,

$$\begin{aligned}
a(0) &= \sum_{g \in Y(A)/N\mathbb{Z}^f} c(g) \sum_{\substack{n \in \mathbb{Z}^f \\ n \equiv g \pmod N \\ \frac{Q(n)}{N} = 0}} P(n) \\
&= \sum_{g \in Y(A)/N\mathbb{Z}^f} c(g) \sum_{\substack{n \in \mathbb{Z}^f \\ n \equiv g \pmod N \\ n = 0}} P(n) \\
&= c(0)P(0) \\
&= c(0) \cdot 0 \\
&= 0.
\end{aligned}$$

Here, $P(0) = 0$ because P is a homogeneous polynomial in $r > 0$ variables. □

2.8 Example: the quadratic form $x_1^2 + x_2^2 + x_3^2 + x_4^2$

In this example we let

$$A = \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ & & & 2 \end{bmatrix}$$

so that

$$Q(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

Evidently,

$$N = 4 \quad \text{and} \quad k = 2.$$

Also, χ_A is the trivial character of $(\mathbb{Z}/4\mathbb{Z})^\times$. We will simplify the notation for $\theta(A, 1, h, z)$ for $h \in Y(A)$, and write:

$$\theta(h) = \theta(A, 1, h, z).$$

Let V be the $\mathbb{C}$ vector space spanned the $\theta(h)$ for $h \in Y(A)$:

$$V = \langle \theta(h) : h \in Y(A) \rangle.$$

By Theorem 2.7.10, we have $V \subset M_2(\Gamma(4))$. If $h \in \mathbb{Z}^4$, then $h \in Y(A)$ if and only if $Ah \equiv 0 \pmod 4$, i.e., $h \equiv 0 \pmod 2$. Define the following elements of $Y(A)$:

$$h_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, h_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad h_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad h_3 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \quad h_4 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}.$$

The vector space V is spanned by the five modular forms

$$\theta(h_0), \quad \theta(h_1), \quad \theta(h_2), \quad \theta(h_3), \quad \theta(h_4).$$

For $z \in \mathbb{H}_1$, define

$$q_4 = e^{2\pi i z/4}.$$

We have:

$$\begin{aligned}
\theta(h_0) &= \sum_{m \in \mathbb{Z}^4} q_4^{4m_1^2 + 4m_2^2 + 4m_3^2 + 4m_4^2}, \\
\theta(h_1) &= \sum_{m \in \mathbb{Z}^4} q_4^{(2m_1+1)^2 + 4m_2^2 + 4m_3^2 + 4m_4^2}, \\
\theta(h_2) &= \sum_{m \in \mathbb{Z}^4} q_4^{(2m_1+1)^2 + (2m_2+1)^2 + 4m_3^2 + 4m_4^2}, \\
\theta(h_3) &= \sum_{m \in \mathbb{Z}^4} q_4^{(2m_1+1)^2 + (2m_2+1)^2 + (2m_3+1)^2 + 4m_4^2}, \\
\theta(h_4) &= \sum_{m \in \mathbb{Z}^4} q_4^{(2m_1+1)^2 + (2m_2+1)^2 + (2m_3+1)^2 + (2m_4+1)^2}.
\end{aligned}$$

Calculations show that:

$$\begin{aligned}
\theta(h_0) &= 1 + 8q_4^4 + 24q_4^8 + 32q_4^{12} + 24q_4^{16} + 48q_4^{20} + \cdots, \\
\theta(h_1) &= 2q_4 + 12q_4^5 + 26q_4^9 + 28q_4^{13} + 36q_4^{17} + 64q_4^{21} + \cdots,
\end{aligned}$$

$$\begin{aligned}
\theta(h_2) &= 4q_4^2 + 16q_4^6 + 24q_4^{10} + 32q_4^{14} + 52q_4^{18} + 48q_4^{22} + \cdots,\\
\theta(h_3) &= 8q_4^3 + 16q_4^7 + 24q_4^{11} + 48q_4^{15} + 40q_4^{19} + 48q_4^{23} + \cdots,\\
\theta(h_4) &= 16q_4^4 + 64q_4^{12} + 96q_4^{20} + 128q_4^{28} + 208q_4^{36} + 192q_4^{44} + \cdots.
\end{aligned}$$

These expansions show that $\theta(h_0), \dots, \theta(h_4)$ are linearly independent, so that

$$\dim_{\mathbb{C}} V = 5.$$

Lemma 2.8.1. *We have*

$$\dim M_2(\Gamma_0(2)) = 1 \quad \textit{and} \quad \dim M_2(\Gamma_0(4)) = 2.$$

Proof. See, for example, Proposition 1.40 on page 23, Proposition 1.43 on page 24, and Theorem 2.23 on page 46 of [27]. □

Proposition 2.8.2. *Let*

$$V_1 = \langle \theta(h_0) + \theta(h_4),\ \theta(h_2)\rangle, \quad V_2 = \langle \theta(h_0) - \theta(h_4),\ \theta(h_1),\ \theta(h_3)\rangle,$$

so that

$$V = V_1 \oplus V_2.$$

Then V_1 and V_2 are irreducible $\mathrm{SL}(2,\mathbb{Z})$ *subspaces of V. Moreover,*

$$\begin{aligned}
M_2(\Gamma_0(4)) &= \langle \theta(h_0),\ \theta(h_4)\rangle,\\
M_2(\Gamma_0(2)) &= \langle \theta(h_0) + \theta(h_4)\rangle.
\end{aligned}$$

Proof. By (2.4) we have

$$\begin{aligned}
\theta(h_0)\big|_2 \begin{bmatrix} & 1\\ -1 & \end{bmatrix} &= -\frac{1}{4}\big(\theta(h_0) + 4\cdot\theta(h_1) + 6\cdot\theta(h_2) + 4\cdot\theta(h_3) + \theta(h_4)\big),\\
\theta(h_1)\big|_2 \begin{bmatrix} & 1\\ -1 & \end{bmatrix} &= -\frac{1}{4}\big(\theta(h_0) + 2\cdot\theta(h_1) - 2\cdot\theta(h_3) - \theta(h_4)\big),\\
\theta(h_2)\big|_2 \begin{bmatrix} & 1\\ -1 & \end{bmatrix} &= -\frac{1}{4}\big(\theta(h_0) - 2\cdot\theta(h_2) + \theta(h_4)\big)\\
\theta(h_3)\big|_2 \begin{bmatrix} & 1\\ -1 & \end{bmatrix} &= -\frac{1}{4}\big(\theta(h_0) - 2\cdot\theta(h_1) + 2\cdot\theta(h_3) - \theta(h_4)\big),\\
\theta(h_4)\big|_2 \begin{bmatrix} & 1\\ -1 & \end{bmatrix} &= -\frac{1}{4}\big(\theta(h_0) - 4\cdot\theta(h_1) + 6\cdot\theta(h_2) - 4\cdot\theta(h_3) + \theta(h_4)\big).
\end{aligned}$$

By (2.5) we have:

$$\begin{aligned}
\theta(h_0)\big|_2 \begin{bmatrix} 1 & 1\\ & 1 \end{bmatrix} &= \theta(h_0),\\
\theta(h_1)\big|_2 \begin{bmatrix} 1 & 1\\ & 1 \end{bmatrix} &= i\theta(h_1),
\end{aligned}$$

$$\theta(h_2)\big|_2 \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} = -\theta(h_2),$$
$$\theta(h_3)\big|_2 \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} = -i\theta(h_3),$$
$$\theta(h_4)\big|_2 \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} = \theta(h_4).$$

Since $\mathrm{SL}(2,\mathbb{Z})$ is generated by

$$\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$$

the above equations imply that V_1 and V_2 are $\mathrm{SL}(2,\mathbb{Z})$ subspaces of V.

To see that V_1 is irreducible as an $\mathrm{SL}(2,\mathbb{Z})$ space, let $W \subset V_1$ be a $\mathrm{SL}(2,\mathbb{Z})$ subspace. We need to prove that $W = 0$ or $W = V_1$, and to prove this it suffices to prove that $\dim W \neq 1$. Assume that $\dim W = 1$; we will obtain a contradiction. Let $a, b \in \mathbb{C}$ be such that $F_1 = a(\theta(h_0) + \theta(h_4)) + b\theta(h_2)$ is a basis for W. Since W is one-dimensional, $\mathrm{SL}(2,\mathbb{Z})$ acts on W by a character $\beta : \mathrm{SL}(2,\mathbb{Z}) \to \mathbb{C}^\times$. F_1 is fixed by $\mathrm{SL}(2,\mathbb{Z})$. Now

$$F_1\big|_2 \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} = \beta(\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix})F_1$$
$$a(\theta(h_0) + \theta(h_4)) - b\theta(h_2) = a\beta(\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix})(\theta(h_0) + \theta(h_4)) + b\beta(\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix})\theta(h_2).$$

This equality implies that $a = 0$ or $b = 0$. If $a = 0$ and $b \neq 0$, then

$$F_1\big|_2 \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} = \beta(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix})F_1$$
$$-\frac{b}{4}\big(\theta(h_0) - 2\cdot\theta(h_2) + \theta(h_4)\big) = \beta(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix})b\theta(h_2).$$

This is a contradiction. Similarly, the case $a \neq 0$ and $b = 0$ leads to a contradiction. Thus, V_1 is irreducible.

To prove that V_2 is irreducible, let W be a non-zero $\mathrm{SL}(2,\mathbb{Z})$ subspace of V_2; we need to prove that $W = V_2$. An argument similar to that in the last paragraph proves that W cannot be one-dimensional. Assume that W is two-dimensional; we will obtain a contradiction. The formulas for the action of

$$\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$$

show that W can contain at most one of $\theta(h_0) - \theta(h_4)$, $\theta(h_1)$ and $\theta(h_3)$; otherwise, $W = V_2$, a contradiction. Consider the quotient V_2/W. This $\mathrm{SL}(2,\mathbb{Z})$ space is one-dimensional. Hence, $\mathrm{SL}(2,\mathbb{Z})$ acts on V_2/W by a character $\delta : \mathrm{SL}(2,\mathbb{Z}) \to \mathbb{C}^\times$. Let $p : V_2 \to V_2/W$ be the projection map. We have The formulas for the action of

$$\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$$

imply that

$$p(\theta(h_0) - \theta(h_4)) = \delta(\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix})p(\theta(h_0) - \theta(h_4)),$$
$$ip(\theta(h_1)) = \delta(\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix})p(\theta(h_1)),$$
$$-ip((\theta(h_3)) = \delta(\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix})p((\theta(h_3)).$$

Since at least two of $p(\theta(h_0)-\theta(h_4)), p(\theta(h_1))$, and $p(\theta(h_3))$ are non-zero, these equations imply that

$$\delta(\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix})$$

is equal to at least two distinct elements of $\{1, i, -i\}$, a contradiction. Thus, V_2 is irreducible.

By Lemma 2.8.1 we have $\dim M_2(\Gamma_0(4)) = 2$ and $\dim M_2(\Gamma_0(2)) = 1$. By Lemma 2.7.7 and Theorem 2.7.10, the functions $\theta(h_0)$ and $\theta(h_4)$ are contained in $M_2(\Gamma_0(4))$. Since $\theta(h_0)$ and $\theta(h_4)$ are linearly independent, $\theta(h_0)$ and $\theta(h_4)$ form a basis for $M_2(\Gamma_0(4))$. Finally, we need to prove that

$$F = \theta(h_0) + \theta(h_4)$$

is contained in $M_2(\Gamma_0(2))$. It will suffice to prove that

$$F\big|_2\gamma = F \quad \text{for } \gamma \in \Gamma_0(2)$$

for $\gamma \in \Gamma_0(2)$. We begin with some preliminary calculations. Let $h \in Y(A)$; we write $h = 2h'$ for some $h' \in \mathbb{Z}^4$. Let

$$\alpha = \begin{bmatrix} 1 & \\ 2 & 1 \end{bmatrix}.$$

By (2.13),

$$\begin{aligned} \theta(h)\big|_2 \begin{bmatrix} 1 & \\ 2 & 1 \end{bmatrix} &= \frac{1}{i^k 2^2\sqrt{\det(A)}} \sum_{g\in Y(A)/4\mathbb{Z}^4} s_\alpha(g,h)\theta(g) \\ &= \frac{1}{-2^4} \sum_{g\in Y(A)/4\mathbb{Z}^4} s_\alpha(g,h)\theta(g). \end{aligned} \tag{2.28}$$

Let $g \in Y(A)$, and write $g = 2g'$ for some $g' \in \mathbb{Z}^4$. We obtain

$$\begin{aligned} s_\alpha(g,h) &= \sum_{\substack{x\in\mathbb{Z}^4/8\mathbb{Z}^4 \\ x\equiv h \pmod 4}} e^{2\pi i\left(\frac{Q(x)+{}^{\mathrm{t}}gAx+Q(g)}{32}\right)} \\ &= e^{2\pi i\left(\frac{Q(g)}{32}\right)} \sum_{\substack{x\in\mathbb{Z}^4/8\mathbb{Z}^4 \\ x\equiv h \pmod 4}} e^{2\pi i\left(\frac{Q(x)+{}^{\mathrm{t}}gAx}{32}\right)} \end{aligned}$$

$$
\begin{aligned}
&= e^{2\pi i\left(\frac{Q(g)}{32}\right)} \sum_{y\in\mathbb{Z}^4/2\mathbb{Z}^4} e^{2\pi i\left(\frac{Q(h+4y)+{}^{\mathrm{t}}gA(h+4y)}{32}\right)}\\
&= e^{2\pi i\left(\frac{Q(g)}{32}\right)} \sum_{y\in\mathbb{Z}^4/2\mathbb{Z}^4} e^{2\pi i\left(\frac{Q(h)+2\,{}^{\mathrm{t}}gh+8\,{}^{\mathrm{t}}(g+h)y+16Q(y)}{32}\right)}\\
&= e^{2\pi i\left(\frac{Q(g)+Q(h)+2\,{}^{\mathrm{t}}gh}{32}\right)} \sum_{y\in\mathbb{Z}^4/2\mathbb{Z}^4} e^{2\pi i\left(\frac{8\,{}^{\mathrm{t}}(g+h)y+16Q(y)}{32}\right)}\\
&= e^{2\pi i\left(\frac{Q(g+h)}{32}\right)} \sum_{y\in\mathbb{Z}^4/2\mathbb{Z}^4} e^{2\pi i\left(\frac{16\,{}^{\mathrm{t}}(g'+h')y+16Q(y)}{32}\right)}\\
&= e^{2\pi i\left(\frac{Q(g+h)}{32}\right)} \sum_{y\in\mathbb{Z}^4/2\mathbb{Z}^4} e^{2\pi i\left(\frac{{}^{\mathrm{t}}(g'+h')y+Q(y)}{2}\right)}\\
&= e^{2\pi i\left(\frac{Q(g+h)}{32}\right)} \sum_{y\in\mathbb{Z}^4/2\mathbb{Z}^4} e^{2\pi i\left(\frac{{}^{\mathrm{t}}(g'+h')y+Q(y)}{2}\right)}.
\end{aligned}
$$

The function $\mathbb{Z}^4/2\mathbb{Z}^4 \to \mathbb{C}^\times$ defined by

$$y \mapsto e^{2\pi i\left(\frac{{}^{\mathrm{t}}(g'+h')y+Q(y)}{2}\right)}$$

is a homomorphism. This homomorphism is trivial if and only if every entry of $g'+h'$ is odd, or equivalently, $g+h\equiv h_4 \pmod 4$. Therefore,

$$s_\alpha(g,h) = e^{2\pi i\left(\frac{Q(g+h)}{32}\right)} \sum_{y\in\mathbb{Z}^4/2\mathbb{Z}^4} e^{2\pi i\left(\frac{{}^{\mathrm{t}}(g'+h')y+Q(y)}{2}\right)}$$

$$s_\alpha(g,h) = \begin{cases} -2^4 & \text{if } g+h\equiv h_4 \pmod 4,\\ 0 & \text{if } g+h\not\equiv h_4 \pmod 4.\end{cases}$$

Consequently,

$$
\begin{aligned}
\theta(h)\big|_2 \begin{bmatrix}1 & \\ 2 & 1\end{bmatrix} &= \frac{1}{-2^4}\sum_{g\in Y(A)/4\mathbb{Z}^4} s_\alpha(g,h)\theta(g)\\
&= \theta(h_4-h).
\end{aligned}
$$

This implies that:

$$
\begin{aligned}
\theta(h_0)\big|_2 \begin{bmatrix}1 & \\ 2 & 1\end{bmatrix} &= \theta(h_4),\\
\theta(h_1)\big|_2 \begin{bmatrix}1 & \\ 2 & 1\end{bmatrix} &= \theta(h_3),\\
\theta(h_2)\big|_2 \begin{bmatrix}1 & \\ 2 & 1\end{bmatrix} &= \theta(h_2),\\
\theta(h_3)\big|_2 \begin{bmatrix}1 & \\ 2 & 1\end{bmatrix} &= \theta(h_1),
\end{aligned}
$$

$$\theta(h_4)\big|_2 \begin{bmatrix} 1 & \\ 2 & 1 \end{bmatrix} = \theta(h_0).$$

Since $F \in M_2(\Gamma_0(4)$, to prove that $F|_2\gamma = F$ for $\gamma \in \Gamma_0(2)$, it will suffices to prove that $F|_2\gamma = F$ for $\gamma \in \Gamma_0(2)$ of the form

$$\gamma = \begin{bmatrix} a & b \\ 2c & d \end{bmatrix}$$

where c is an odd integer; we note that since $ad - 2bc = 1$, d is also odd. Let $\gamma \in \Gamma_0(2)$ have this form. Then

$$\begin{aligned} F\big|_2\gamma &= \theta(h_0)\big|_2\gamma + \theta(h_4)\big|_2\gamma \\ &= \theta(h_0)\big|_2\gamma \begin{bmatrix} 1 & \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ 2 & 1 \end{bmatrix} + \theta(h_4)\big|_2\gamma \begin{bmatrix} 1 & \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ 2 & 1 \end{bmatrix} \\ &= \theta(h_0)\big|_2 \begin{bmatrix} a-2b & b \\ 2(c-d) & 2c+d \end{bmatrix} \begin{bmatrix} 1 & \\ 2 & 1 \end{bmatrix} + \theta(h_4)\big|_2 \begin{bmatrix} a-2b & b \\ 2(c-d) & 2c+d \end{bmatrix} \begin{bmatrix} 1 & \\ 2 & 1 \end{bmatrix} \\ &= \theta(h_0)\big|_2 \begin{bmatrix} 1 & \\ 2 & 1 \end{bmatrix} + \theta(h_4)\big|_2 \begin{bmatrix} 1 & \\ 2 & 1 \end{bmatrix} \qquad (c-d \text{ is even}) \\ &= \theta(h_4) + \theta(h_0) \\ &= F. \end{aligned}$$

This proves our claim about F. □

Proposition 2.8.3 (Jacobi's four square theorem)**.** *If n is a positive integer, then the number of $(x, y, z, w) \in \mathbb{Z}^4$ such*

$$x^2 + y^2 + z^2 + w^2 = n$$

is

$$8 \cdot \sum_{\substack{m>0,\ m|n, \\ m \not\equiv 0 \pmod 4}} m.$$

In particular, every positive integer is a sum of four squares.

Proof. We have

$$\theta(h_0, z) = \sum_{n=0}^{\infty} a(n) q^n$$

where

$$a(n) = \#\{m \in \mathbb{Z}^4 : Q(m) = n\}$$

for each non-negative integer n. The modular form $\theta(h_0, z)$ is contained in $M_2(\Gamma_0(4))$. By Lemma 2.8.1, the dimension of $M_2(\Gamma_0(4))$ is two, and the dimension of $M_2(\Gamma_0(2))$ is one. The vector space $M_2(\Gamma_0(2))$ is spanned by

$$E(z) = \frac{1}{24} + \sum_{n=1}^{\infty} b(n) q^n$$

where $q = e^{2\pi i z}$ for $z \in \mathbb{H}_1$; here, for positive integers n,

$$b(n) = \begin{cases} \sigma_1(n) - 2\sigma_1(n/2) & \text{if } n \text{ is even,} \\ \sigma_1(n) & \text{if } n \text{ is odd.} \end{cases}$$

For this, see Theorem 5.8 on page 88 of [28]. Trivially, the function $E(z)$ is contained in $M_2(\Gamma_0(4))$. The function

$$E(z)\big|_2 \begin{bmatrix} 2 & \\ & 1 \end{bmatrix} = E(2z)$$

is also contained in $M_2(\Gamma_0(4))$. We have

$$E(2z) = \frac{1}{24} + \sum_{n=1}^{\infty} c(n) q^n$$

where

$$c(n) = \begin{cases} \sigma_1(n/2) - 2\sigma_1(n/4) & \text{if } n \text{ is divisible by } 4, \\ \sigma_1(n/2) & \text{if } n \text{ is even and } n/2 \text{ is odd,} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

for positive integers n. The two modular forms $E(z)$ and $E(2z)$ form a basis for $M_2(\Gamma_0(4))$. Hence, there exist $c_1, c_2 \in \mathbb{C}$ such that

$$\theta(h_0, z) = c_1 \cdot E(z) + c_2 \cdot E(2z).$$

Calculations show that

$$\begin{aligned} \theta(h_0, z) &= 1 + 8q + 24q^2 + 32q^3 + 24q^4 + 48q^5 + 96q^6 + 64q^7 + \cdots, \\ E(z) &= \frac{1}{24} + q + q^2 + 4q^3 + q^4 + 6q^5 + 4q^6 + 8q^7 + \cdots, \\ E(2z) &= \frac{1}{24} + q^2 + q^4 + 4q^6 + q^8 + 6q^{10} + 4q^{12} + \cdots. \end{aligned}$$

Using these expansions to solve for c_1 and c_2, we find that:

$$\theta(h_0, z) = 8 \cdot E(z) + 16 \cdot E(2z).$$

It follows that

$$\begin{aligned} a(n) &= 8b(n) + 16c(n) \\ &= \begin{cases} 8\sigma_1(n) - 32\sigma_1(n/4) & \text{if } 4|n, \\ 8\sigma_1(n) & \text{if } n \text{ is even and } n/2 \text{ is odd,} \\ 8\sigma_1(n) & \text{if } n \text{ is odd,} \end{cases} \\ &= 8 \cdot \sum_{\substack{m>0,\, m|n, \\ m \not\equiv 0 \pmod 4}} m. \end{aligned}$$

This completes the proof. □

Chapter 3

Classical theta series on $\mathbb{H}_n$

3.1 Convergence

Let m and n be positive integers. If $A \in \mathrm{M}(m,\mathbb{C})$ and $X \in \mathrm{M}(m \times n,\mathbb{C})$, then we define

$$A[X] = {}^{\mathrm{t}}XAX.$$

Lemma 3.1.1. *Let m and n be positive integers, and let $A \in \mathrm{M}(m,\mathbb{Z})$ be an even positive-definite symmetric integral matrix. For every $N \in \mathrm{M}(m\times n,\mathbb{Z})$ the $n \times n$ integral matrix $A[N]$ is an even positive semi-definite symmetric matrix.*

Proof. Let $N \in \mathrm{M}(m \times n,\mathbb{Z})$. Set $B = A[N]$. It is clear that B is integral and symmetric. Let $x \in \mathbb{R}^n$. Then ${}^{\mathrm{t}}xBx = {}^{\mathrm{t}}(Nx)A(Nx) \geq 0$. It follows that B is positive semi-definite. □

Assume that $A \in \mathrm{M}(m,\mathbb{Z})$ and $B \in \mathrm{M}(n,\mathbb{Z})$ are even symmetric integral matrices. Assume further that A is positive-definite, and that B is positive semi-definite. We say that A **represents** B if there exists $N \in \mathrm{M}(m \times n,\mathbb{Z})$ such that

$$A[N] = B.$$

We let

$$r(A,B) = \#\{N \in \mathrm{M}(m \times n,\mathbb{Z}) : A[N] = B\}.$$

Lemma 3.1.2. *Let m and n be positive integers, and let $A \in \mathrm{M}(m,\mathbb{Z})$ and $B \in \mathrm{M}(n,\mathbb{Z})$ be even symmetric integral matrices with A positive-definite and B positive semi-definite. The set $\{N \in \mathrm{M}(m \times n,\mathbb{Z}) : A[N] = B\}$ is finite, so that $r(A,B)$ is a non-negative integer.*

Proof. By §1.5, there exists $T \in \mathrm{GL}(m,\mathbb{R})$ and positive numbers $\lambda_1,\dots,\lambda_m$

such that ${}^{\mathrm{t}}T = T$ and

$$D = {}^{\mathrm{t}}TAT = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_m \end{bmatrix}.$$

Define Let $N \in \mathrm{M}(m \times n, \mathbb{Z})$. We have $A[N] = B$ if and only if $D[TN] = B$. Write $TN = [(TN)_1 \cdots (TN)_n]$ where $(TN)_1, \dots, (TN)_n \in \mathbb{R}^m$ are column vectors. We have

$$B_{jj} = {}^{\mathrm{t}}(TN)_j D(TN)_j = \sum_{i=1}^{m} \lambda_i (TN)_{ij}^2$$

for $1 \le j \le n$. Let S be the set of $X \in \mathrm{M}(m \times n, \mathbb{R})$ such that

$$B_{jj} = \sum_{i=1}^{m} \lambda_i X_{ij}^2$$

for $1 \le j \le n$. It follows that $\{N \in \mathrm{M}(m \times n, \mathbb{Z}) : A[N] = B\}$ is contained in $T^{-1}S \cap \mathrm{M}(m \times n, \mathbb{Z})$. The set S is compact, so that $T^{-1}S$ is also compact. Since $T^{-1}S$ is compact and $\mathrm{M}(m \times n, \mathbb{Z})$ is a discrete subset of $\mathrm{M}(m \times n, \mathbb{R})$, the set $T^{-1}S \cap \mathrm{M}(m \times n, \mathbb{Z})$ is finite. ☐

Lemma 3.1.3. *Let n be a positive integer. Let $S, T \in \mathrm{M}(n, \mathbb{R})$ be positive semi-definite symmetric matrices. Then* $\mathrm{tr}(ST) \ge 0$.

Proof. Arguing as before (1.7), there exist positive semi-definite symmetric matrices $U, V \in \mathrm{M}(n, \mathbb{R})$ such that $S = U^2$ and $T = V^2$. Now

$$\begin{aligned} \mathrm{tr}(ST) &= \mathrm{tr}(UUVV) \\ &= \mathrm{tr}(VUUV) \\ &= \mathrm{tr}({}^{\mathrm{t}}(V)\,{}^{\mathrm{t}}UUV) \\ &= \mathrm{tr}({}^{\mathrm{t}}(UV)UV). \end{aligned}$$

Let $W = UV$. Then

$$\begin{aligned} \mathrm{tr}(ST) &= \mathrm{tr}({}^{\mathrm{t}}WW) \\ &= \sum_{k=1}^{n} (\sum_{j=1} ({}^{\mathrm{t}}W)_{kj} W_{jk}) \\ &= \sum_{k=1}^{n} (\sum_{j=1} W_{jk} W_{jk}) \\ &= \sum_{k=1}^{n} (\sum_{j=1} W_{jk}^2) \end{aligned}$$

$$\geq 0.$$

This completes the proof. □

Lemma 3.1.4. *Let K be a compact subset of $\mathrm{Sym}(n,\mathbb{R})$. Assume that $S > 0$ for $S \in K$. Then there exists $\delta > 0$ such that $S - \delta > 0$ for all $S \in K$.*

Proof. Let $S \in K$. Since S is positive-definite, there exists $T \in \mathrm{GL}(n,\mathbb{R})$ such that ${}^{\mathrm{t}}TT = T\,{}^{\mathrm{t}}T = 1$ and

$$A = {}^{\mathrm{t}}T \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_n \end{bmatrix} T$$

for some positive numbers $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. Let $\epsilon_S > 0$ be a positive number such and $\lambda_1 > \epsilon_S, \dots, \lambda_n > \epsilon_S$. Let $x \in \mathbb{R}^n$ with $x \neq 0$. Then

$$\begin{aligned} {}^{\mathrm{t}}x(S - \epsilon_S)x &= {}^{\mathrm{t}}x\,{}^{\mathrm{t}}T \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_n \end{bmatrix} Tx - \epsilon_S\,{}^{\mathrm{t}}xx \\ &= {}^{\mathrm{t}}(Tx) \begin{bmatrix} \lambda_1 - \epsilon_S & & & & \\ & \lambda_2 - \epsilon_S & & & \\ & & \lambda_3 - \epsilon_S & & \\ & & & \ddots & \\ & & & & \lambda_n - \epsilon_S \end{bmatrix} Tx \\ &> 0. \end{aligned}$$

It follows that $S - \epsilon_S > 0$. Hence, $S \in \epsilon_S + \mathrm{Sym}(n,\mathbb{R})^+$. By Lemma 1.10.1, set $\mathrm{Sym}(n,\mathbb{R})^+$ is open in $\mathrm{Sym}(n,\mathbb{R})$. The sets $\epsilon_S + \mathrm{Sym}(n,\mathbb{R})^+$ form an open cover for K. Since K is compact, this cover has a finite subcover $\mathrm{Sym}(n,\mathbb{R})^+ + \epsilon_{S_1}, \dots, \mathrm{Sym}(n,\mathbb{R})^+ + \epsilon_{S_k}$ for some $S_1, \dots, S_k \in K$. Let $\delta = \min(\epsilon_{S_1}, \dots, \epsilon_{S_k})$. Now let $S \in K$. Then $S \in \mathrm{Sym}(n,\mathbb{R})^+ + \epsilon_{S_i}$ for some $i \in \{1, \dots, k\}$. Hence, $S - \epsilon_{S_i} \in \mathrm{Sym}(n,\mathbb{R})^+$. This implies that $S - \epsilon_{S_i} > 0$, so that $S > \epsilon_{S_i} \geq \delta$, as desired. □

Lemma 3.1.5. *Let m and n be positive integers. Let $M, N \in \mathrm{M}(m \times n,\mathbb{R})$. Then*

$$|\mathrm{tr}({}^{\mathrm{t}}MN)| \leq \sum_{i=1}^{n} \|M_i\| \|N_i\|.$$

Here, for $P \in \mathrm{M}(m \times n,\mathbb{R})$, we write $P = [P_1 \cdots P_n]$, where $P_i \in \mathbb{R}^m$ for $1 \leq i \leq n$ are column vectors.

Proof. We have

$$\begin{aligned}|\mathrm{tr}({}^{\mathrm{t}}MN)| &= |\mathrm{tr}({}^{\mathrm{t}}[M_1 \cdots M_n][N_1 \cdots N_n])| \\ &= |\sum_{i=1}^{n} {}^{\mathrm{t}}M_i N_i| \\ &\leq \sum_{i=1}^{n} |\,{}^{\mathrm{t}}M_i N_i| \\ &\leq \sum_{i=1}^{n} \|M_i\| \|N_i\|,\end{aligned}$$

where in the last step we used the Cauchy-Schwarz inequality. □

Lemma 3.1.6. *Let k be a positive integer, and let $\delta > 0$ and $M > 0$ be positive real numbers. Then there exists positive numbers $R > 0$ and $\epsilon > 0$ such that if $x_1 \geq 0,\ \ldots,\ x_k \geq 0$ and*

$$x_1^2 + \cdots + x_k^2 \geq R,$$

then

$$-\delta(x_1^2 + \cdots + x_k^2) + 2M(x_1 + \cdots + x_k) + M \leq -\epsilon(x_1^2 + \cdots + x_k^2).$$

Proof. Let ϵ be any positive number such that $0 < \epsilon < \delta$. Let $m \in \mathbb{R}$ be such that

$$m \leq (\delta - \epsilon)x^2 - 2Mx - M$$

for all $x \in \mathbb{R}$. There exists a positive number T such that if $x \geq T$, then

$$-(k-1)m \leq (\delta - \epsilon)x^2 + 2Mx - M.$$

Now define $R = T^2 k$. Assume that $x_1 \geq 0, \ldots, x_k \geq 0$ and $x_1^2 + \cdots + x_k^2 \geq R$. Then for some $i \in \{1, \ldots, k\}$ we have $x_i^2 \geq R/k$, i.e., $x_i \geq \sqrt{R/k} = T$. It follows that

$$\begin{aligned}&(\delta - \epsilon)(x_1^2 + \cdots + x_k^2) - 2M(x_1 + \cdots + x_k) - M \\ &\geq (\delta - \epsilon)x_i^2 - 2Mx_i - M + (k-1)m \\ &\geq -(k-1)m + (k-1)m \\ &\geq 0.\end{aligned}$$

This completes the proof. □

Lemma 3.1.7. *Let m and n be positive integers, and let $A \in \mathrm{M}(m, \mathbb{R})$ be a positive-definite symmetric matrix. Let K be a compact subset of $\mathbb{H}_n$, and let K_1 and K_2 be compact subsets of $\mathrm{M}(m \times n, \mathbb{C})$. There exists a positive real number $R > 0$ and a positive constant ϵ such that such that*

$$\mathrm{Re}\big(\pi i \mathrm{tr}(ZA[N - Y]) + 2\pi i \mathrm{tr}({}^{\mathrm{t}}NX) - \pi i \mathrm{tr}({}^{\mathrm{t}}XY)\big) \leq -\epsilon \cdot \sum_{i=1}^{n} \|N_i\|^2$$

for $Z \in K$, $X \in K_1$, $Y \in K_2$ and $N \in \mathrm{M}(m \times n, \mathbb{R})$ with

$$\sum_{i=1}^{n} \|N_i\|^2 \geq R.$$

Here, for $N \in \mathrm{M}(m \times n, \mathbb{R})$, we write $N = [N_1 \cdots N_n]$, where $N_i \in \mathbb{R}^m$ for $1 \leq i \leq n$ are column vectors.

Proof. We first prove that we may assume that $A = 1$. To see this, assume that the assertion holds for $1 = 1_m$. Since A is positive-definite, there exists a positive-definite symmetric matrix $B \in \mathrm{M}(n, \mathbb{R})$ such that $A = B^2$ (see (1.7)). Define $K_1' = B^{-1}(K_1)$ and $K_2' = B(K_2)$. Since we are assuming that the assertion holds for $1 = 1_m$, there exists a positive real number $R > 0$ and a positive constant ϵ such that

$$\mathrm{Re}\big(\pi i \mathrm{tr}(Z\,{}^{\mathrm{t}}(N' - Y')(N' - Y')) + 2\pi i \mathrm{tr}({}^{\mathrm{t}}N'X') - \pi i \mathrm{tr}({}^{\mathrm{t}}X'Y')\big) \leq -\epsilon \cdot \sum_{i=1}^{n} \|N_i'\|^2$$

for $Z \in K$, $X' \in K_1' = B(K_1)$, $Y' \in B^{-1}(K_2)$ and $N' \in \mathrm{M}(m \times n, \mathbb{R})$ with

$$\sum_{i=1}^{n} \|N_i'\|^2 \geq R.$$

Regard the matrix B^{-1} as operator from $\mathbb{R}^m$ to $\mathbb{R}^m$. Then B is continuous and hence bounded. Therefore, there exists a positive constant $\|B^{-1}\|$ such that

$$\|B^{-1}(g)\| \leq \|B^{-1}\| \|g\|$$

for $g \in \mathbb{R}^m$. Define $T = \|B^{-1}\|^2 R$. Let $N \in \mathrm{M}(m \times n, \mathbb{R})$ with

$$\sum_{i=1}^{n} \|N_i\|^2 \geq T.$$

Define $N' = BN$. Then

$$\begin{aligned}
\sum_{i=1}^{n} \|N_i'\|^2 &= \sum_{i=1}^{n} \|(BN)_i\|^2 \\
&= \sum_{i=1}^{n} \|BN_i\|^2 \\
&= \sum_{i=1}^{n} \|B^{-1}\|^{-2} \|B^{-1}\|^2 \|BN_i\|^2 \\
&\geq \sum_{i=1}^{n} \|B^{-1}\|^{-2} \|B^{-1}BN_i\|^2 \\
&= \sum_{i=1}^{n} \|B^{-1}\|^{-2} \|N_i\|^2
\end{aligned}$$

$$\begin{aligned}
&= \|B^{-1}\|^{-2}\sum_{i=1}^{n}\|N_i\|^2 \\
&\geq \|B^{-1}\|^{-2}T \\
&= R.
\end{aligned}$$

Let $Z \in K$, $X \in K_1$ and $Y \in K_2$. Then $X' = B^{-1}(X) \in K_1'$ and $Y' = B(Y) \in K_2'$. Since

$$\begin{aligned}
&\mathrm{Re}\big(\pi i \mathrm{tr}(Z\,{}^{\mathrm{t}}(N'-Y')(N'-Y')) + 2\pi i \mathrm{tr}({}^{\mathrm{t}}N'X') - \pi i \mathrm{tr}({}^{\mathrm{t}}X'Y')\big) \\
&= \mathrm{Re}\big(\pi i \mathrm{tr}(Z\,{}^{\mathrm{t}}(BN-BY)(BN-BY)) + 2\pi i \mathrm{tr}({}^{\mathrm{t}}(BN)B^{-1}X) \\
&\qquad - \pi i \mathrm{tr}({}^{\mathrm{t}}(B^{-1}X)BY)\big) \\
&= \mathrm{Re}\big(\pi i \mathrm{tr}(Z\,{}^{\mathrm{t}}(N-Y)BB(N-Y)) + 2\pi i \mathrm{tr}({}^{\mathrm{t}}NX) - \pi i \mathrm{tr}({}^{\mathrm{t}}XY)\big) \\
&= \mathrm{Re}\big(\pi i \mathrm{tr}(Z\,{}^{\mathrm{t}}(N-Y)A(N-Y)) + 2\pi i \mathrm{tr}({}^{\mathrm{t}}NX) - \pi i \mathrm{tr}({}^{\mathrm{t}}XY)\big) \\
&= \mathrm{Re}\big(\pi i \mathrm{tr}(ZA[N-Y]) + 2\pi i \mathrm{tr}({}^{\mathrm{t}}NX) - \pi i \mathrm{tr}({}^{\mathrm{t}}XY)\big),
\end{aligned}$$

and,

$$\begin{aligned}
-\epsilon\cdot\sum_{i=1}^{n}\|N_i'\|^2 &= -\epsilon\cdot\sum_{i=1}^{n}\|BN_i\|^2 \\
&= -\epsilon\cdot\sum_{i=1}^{n}\|B^{-1}\|^{-2}\|B^{-1}\|^{2}\|BN_i\|^2 \\
&\leq -\epsilon\cdot\sum_{i=1}^{n}\|B^{-1}\|^{-2}\|N_i\|^2 \\
&= -\epsilon\|B^{-1}\|^{-2}\cdot\sum_{i=1}^{n}\|N_i\|^2.
\end{aligned}$$

we conclude that

$$\mathrm{Re}\big(\pi i \mathrm{tr}(ZA[N-Y]) + 2\pi i \mathrm{tr}({}^{\mathrm{t}}NX) - \pi i \mathrm{tr}({}^{\mathrm{t}}XY)\big) \leq -\epsilon\|B^{-1}\|^{-2}\cdot\sum_{i=1}^{n}\|N_i\|^2.$$

It follows that we may assume that $A = 1 = 1_m$.

We now prove the lemma for $A = 1 = 1_m$. Since K, K_1 and K are compact, there exists a positive number $M > 0$ such that

$$\begin{gathered}
\|(V\,{}^{\mathrm{t}}Y_1 + U\,{}^{\mathrm{t}}Y_2 - {}^{\mathrm{t}}X_2)_i\| \leq M, \quad \text{for } 1 \leq i \leq n, \\
|\mathrm{tr}({}^{\mathrm{t}}X_1Y_2 + {}^{\mathrm{t}}X_2Y_1 - U({}^{\mathrm{t}}Y_1Y_2 + {}^{\mathrm{t}}Y_2Y_1)) - V({}^{\mathrm{t}}Y_1Y_1 + {}^{\mathrm{t}}Y_2Y_2))| \leq M
\end{gathered}$$

for $Z = U + iV \in K$, $X = X_1 + iX_2 \in K_1$ and $Y = Y_1 + iY_2 \in K_2$ where U, V, X_1, X_2, Y_1 and Y_2 are real matrices. By Lemma 3.1.4 there exists $\delta > 0$ such that $\mathrm{Im}(Z) - \delta > 0$ for all $Z \in K$. Let $N \in \mathrm{M}(m \times n, \mathbb{R})$. Then ${}^{\mathrm{t}}NN \geq 0$.

Hence, by Lemma 3.1.3, we have $\mathrm{tr}((\mathrm{Im}(Z)-\delta)\,{}^{\mathrm{t}}NN) \geq 0$ for $N \in \mathrm{M}(m\times n,\mathbb{R})$, or equivalently,

$$-\mathrm{tr}((\mathrm{Im}(Z)\,{}^{\mathrm{t}}NN) \leq -\delta\mathrm{tr}({}^{\mathrm{t}}NN) \quad \text{for } N \in \mathrm{M}(m\times n,\mathbb{R}). \tag{3.1}$$

Let $Z \in K$, $X \in K_1$ and $Y \in K_2$. Write $Z = U + iV$ for $U, V \in \mathrm{M}(n\times n,\mathbb{R})$ with ${}^{\mathrm{t}}U = U$, ${}^{\mathrm{t}}V = V$, and $V > 0$. Also, write $X = X_1 + iX_2$ and $Y = Y_1 + iY_2$ for $X_1, X_2, Y_1, Y_2 \in \mathrm{M}(m\times n,\mathbb{R})$. We have

$$\begin{aligned}
&\pi^{-1}\mathrm{Re}\big(\pi i\mathrm{tr}(Z\,{}^{\mathrm{t}}(N-Y)(N-Y)) + 2\pi i\mathrm{tr}({}^{\mathrm{t}}NX) - \pi i\mathrm{tr}({}^{\mathrm{t}}XY)\big)\\
&= -\pi^{-1}\mathrm{Im}\big(\pi\mathrm{tr}(Z\,{}^{\mathrm{t}}(N-Y)(N-Y)) + 2\pi\mathrm{tr}({}^{\mathrm{t}}NX) - \pi\mathrm{tr}({}^{\mathrm{t}}XY)\big)\\
&= -\mathrm{tr}(V\,{}^{\mathrm{t}}NN) + 2\mathrm{tr}(V\,{}^{\mathrm{t}}Y_1N) + 2\mathrm{tr}(U\,{}^{\mathrm{t}}Y_2N) - 2\mathrm{tr}({}^{\mathrm{t}}NX_2)\\
&\qquad + \mathrm{tr}({}^{\mathrm{t}}X_1Y_2 + {}^{\mathrm{t}}X_2Y_1 - U({}^{\mathrm{t}}Y_1Y_2 + {}^{\mathrm{t}}Y_2Y_1)) - V({}^{\mathrm{t}}Y_1Y_1 + {}^{\mathrm{t}}Y_2Y_2))\\
&= -\mathrm{tr}(V\,{}^{\mathrm{t}}NN) + 2\mathrm{tr}((V\,{}^{\mathrm{t}}Y_1 + U\,{}^{\mathrm{t}}Y_2 - {}^{\mathrm{t}}X_2)N)\\
&\qquad + \mathrm{tr}({}^{\mathrm{t}}X_1Y_2 + {}^{\mathrm{t}}X_2Y_1 - U({}^{\mathrm{t}}Y_1Y_2 + {}^{\mathrm{t}}Y_2Y_1)) - V({}^{\mathrm{t}}Y_1Y_1 + {}^{\mathrm{t}}Y_2Y_2))\\
&\leq -\delta\mathrm{tr}({}^{\mathrm{t}}NN) + 2|\mathrm{tr}((V\,{}^{\mathrm{t}}Y_1 + U\,{}^{\mathrm{t}}Y_2 - {}^{\mathrm{t}}X_2)N)|\\
&\qquad + |\mathrm{tr}({}^{\mathrm{t}}X_1Y_2 + {}^{\mathrm{t}}X_2Y_1 - U({}^{\mathrm{t}}Y_1Y_2 + {}^{\mathrm{t}}Y_2Y_1)) - V({}^{\mathrm{t}}Y_1Y_1 + {}^{\mathrm{t}}Y_2Y_2))|\\
&= -\delta\sum_{i=1}^{n}\|N_i\|^2 + 2|\mathrm{tr}((V\,{}^{\mathrm{t}}Y_1 + U\,{}^{\mathrm{t}}Y_2 - {}^{\mathrm{t}}X_2)N)|\\
&\qquad + |\mathrm{tr}({}^{\mathrm{t}}X_1Y_2 + {}^{\mathrm{t}}X_2Y_1 - U({}^{\mathrm{t}}Y_1Y_2 + {}^{\mathrm{t}}Y_2Y_1)) - V({}^{\mathrm{t}}Y_1Y_1 + {}^{\mathrm{t}}Y_2Y_2))|\\
&\leq -\delta\sum_{i=1}^{n}\|N_i\|^2 + 2\sum_{i=1}^{n}\|(V\,{}^{\mathrm{t}}Y_1 + U\,{}^{\mathrm{t}}Y_2 - {}^{\mathrm{t}}X_2)_i\|\|N_i\|\\
&\qquad + |\mathrm{tr}({}^{\mathrm{t}}X_1Y_2 + {}^{\mathrm{t}}X_2Y_1 - U({}^{\mathrm{t}}Y_1Y_2 + {}^{\mathrm{t}}Y_2Y_1)) - V({}^{\mathrm{t}}Y_1Y_1 + {}^{\mathrm{t}}Y_2Y_2))|\\
&\leq -\delta\sum_{i=1}^{n}\|N_i\|^2 + 2M\sum_{i=1}^{n}\|N_i\| + M.
\end{aligned}$$

By Lemma 3.1.6, there exists positive numbers $R > 0$ and $\epsilon > 0$ such that

$$-\delta\sum_{i=1}^{n}\|N_i\|^2 + 2M\sum_{i=1}^{n}\|N_i\| + M \leq -\epsilon\sum_{i=1}^{n}\|N_i\|^2$$

for

$$\sum_{i=1}^{n}\|N_i\|^2 \geq R.$$

This completes the proof. □

Proposition 3.1.8. *Let m and n be positive integers, and let $A \in \mathrm{M}(m,\mathbb{R})$ be a positive-definite symmetric matrix. For $Z \in \mathbb{H}_n$, $X, Y \in \mathrm{M}(m\times n,\mathbb{C})$, define*

$$\theta(A,Z,X,Y) = \sum_{N\in\mathrm{M}(m\times n,\mathbb{Z})} \exp\big(\pi i\mathrm{tr}(ZA[N-Y]) + 2\pi i\mathrm{tr}({}^{\mathrm{t}}NX) - \pi i\mathrm{tr}({}^{\mathrm{t}}XY)\big).$$

If D, D_1 and D_2 are products of closed disks in $\mathbb{C}$ such that $D \subset \mathbb{H}_n$ and $D_1, D_2 \subset \mathrm{M}(m \times n, \mathbb{C})$, then the series $\theta(A, Z, X, Y)$ converges absolutely and uniformly on $D \times D_1 \times D_2$. The resulting function $\theta(A, Z, X, Y)$ defined on $\mathbb{H}_n \times \mathrm{M}(m \times n, \mathbb{C}) \times \mathrm{M}(m \times n, \mathbb{C})$ is analytic in each complex variable.

Proof. Let D, D_1 and D_2 be products of closed disks in $\mathbb{C}$ such that $D \subset \mathbb{H}_n$ and $D_1, D_2 \subset \mathrm{M}(m \times n, \mathbb{C})$. By there exists a positive real number $R > 0$ and a positive constant ϵ such that such that

$$\mathrm{Re}\big(\pi i \mathrm{tr}(ZA[N-Y]) + 2\pi i \mathrm{tr}({}^{\mathrm{t}}NX) - \pi i \mathrm{tr}({}^{\mathrm{t}}XY)\big) \leq -\epsilon \cdot \sum_{i=1}^{n} \|N_i\|^2$$

for $Z \in D$, $X \in D_1$, $Y \in D_2$ and $N \in \mathrm{M}(m \times n, \mathbb{R})$ with

$$\sum_{i=1}^{n} \|N_i\|^2 \geq R.$$

Hence,

$$\begin{aligned} &|\exp\big(\pi i \mathrm{tr}(ZA[N-Y]) + 2\pi i \mathrm{tr}({}^{\mathrm{t}}NX) - \pi i \mathrm{tr}({}^{\mathrm{t}}XY)\big)| \\ &= \exp\big(\mathrm{Re}\big(\pi i \mathrm{tr}(ZA[N-Y]) + 2\pi i \mathrm{tr}({}^{\mathrm{t}}NX) - \pi i \mathrm{tr}({}^{\mathrm{t}}XY)\big)\big) \\ &\leq \exp\big(-\epsilon \cdot \sum_{i=1}^{n} \|N_i\|^2\big) \end{aligned}$$

for $Z \in D$, $X \in D_1$, $Y \in D_2$ and all but finitely many $N \in \mathrm{M}(m \times n, \mathbb{Z})$. The series

$$\sum_{N \in \mathrm{M}(m \times n, \mathbb{Z})} \exp\big(-\epsilon \cdot \sum_{i=1}^{n} \|N_i\|^2\big)$$

converges. The Weierstrass M-test (see [17], p. 160) now implies that the series $\theta(A, Z, X, Y)$ converges absolutely and uniformly on $D \times D_1 \times D_2$. Since for each $N \in \mathrm{M}(m \times n, \mathbb{Z})$ the function on $\mathbb{H}_n \times \mathrm{M}(m \times n, \mathbb{C}) \times \mathrm{M}(m \times n, \mathbb{C})$ defined by

$$(Z, X, Y) \mapsto \exp\big(\pi i \mathrm{tr}(ZA[N-Y]) + 2\pi i \mathrm{tr}({}^{\mathrm{t}}NX) - \pi i \mathrm{tr}({}^{\mathrm{t}}XY)\big)$$

is an analytic function in each complex variable and since our series converges absolutely and uniformly on all products of closed disks, the function $\theta(A, Z, X, Y)$ is analytic in each variable (see [17], p. 162). $\square$

Corollary 3.1.9. *Let m and n be positive integers, and let $A \in \mathrm{M}(m, \mathbb{Z})$ be an even positive-definite symmetric integral matrix. For $Z \in \mathbb{H}_n$, define*

$$\theta(A, Z) = \sum_{N \in \mathrm{M}(m \times n, \mathbb{Z})} \exp\big(\pi i \mathrm{tr}(A[N]Z)\big).$$

If D is a product of closed disks in $\mathbb{C}$ such that $D \subset \mathbb{H}_n$ then the series $\theta(A, Z)$ converges absolutely and uniformly on D. The resulting function $\theta(A, Z)$ defined *on $\mathbb{H}_n$ is analytic in each complex variable. Moreover,*

$$\theta(A, Z) = \sum_{\substack{B \in \mathrm{Sym}(n,\mathbb{Z})_{\mathrm{even}}, \\ B \geq 0}} r(A, B) \exp\big(\pi i \mathrm{tr}(BZ)\big).$$

3.2 The Eichler lemma

Let k be a positive integer. For $Z \in \mathbb{H}_k$, and $X, Y \in \mathrm{M}(k, 1, \mathbb{C})$ we will consider the series

$$\begin{aligned} &\theta(Z, X, Y) \\ &\qquad = \sum_{R \in \mathrm{M}(k,1,\mathbb{Z})} \exp\big(\pi i\, {}^{\mathrm{t}}(R - Y)Z(R - Y) + 2\pi i\, {}^{\mathrm{t}}RX - \pi i\, {}^{\mathrm{t}}XY\big). \end{aligned} \tag{3.2}$$

This series is actually an example of the series considered in Proposition 3.1.8 with $m = 1$ and $k = n$. To see this, we note that if $W_1, W_2 \in \mathrm{M}(k, 1, \mathbb{C})$, then

$${}^{\mathrm{t}}W_1 W_2 = \mathrm{tr}({}^{\mathrm{t}}({}^{\mathrm{t}}W_1)\, {}^{\mathrm{t}}W_2).$$

Therefore, for $Z \in \mathbb{H}_k$, and $X, Y \in \mathrm{M}(k, 1, \mathbb{C})$,

$$\begin{aligned} \theta(Z, X, Y) &= \sum_{R \in \mathrm{M}(k,1,\mathbb{Z})} \exp\big(\pi i\, {}^{\mathrm{t}}(R - Y)Z(R - Y) + 2\pi i\, {}^{\mathrm{t}}RX - \pi i\, {}^{\mathrm{t}}XY\big) \\ &= \sum_{R \in \mathrm{M}(k,1,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}({}^{\mathrm{t}}({}^{\mathrm{t}}(R - Y))\, {}^{\mathrm{t}}(Z(R - Y))) + 2\pi i \mathrm{tr}({}^{\mathrm{t}}({}^{\mathrm{t}}R)\, {}^{\mathrm{t}}X) \\ &\qquad - \pi i \mathrm{tr}({}^{\mathrm{t}}({}^{\mathrm{t}}X)\, {}^{\mathrm{t}}Y)\big) \\ &= \sum_{R \in \mathrm{M}(k,1,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}({}^{\mathrm{t}}({}^{\mathrm{t}}R - {}^{\mathrm{t}}Y)({}^{\mathrm{t}}R - {}^{\mathrm{t}}Y)\, {}^{\mathrm{t}}Z) + 2\pi i \mathrm{tr}({}^{\mathrm{t}}({}^{\mathrm{t}}R)\, {}^{\mathrm{t}}X) \\ &\qquad - \pi i \mathrm{tr}({}^{\mathrm{t}}({}^{\mathrm{t}}X)\, {}^{\mathrm{t}}Y)\big) \\ &= \sum_{R \in \mathrm{M}(k,1,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(Z\, {}^{\mathrm{t}}({}^{\mathrm{t}}R - {}^{\mathrm{t}}Y)({}^{\mathrm{t}}R - {}^{\mathrm{t}}Y)) + 2\pi i \mathrm{tr}({}^{\mathrm{t}}({}^{\mathrm{t}}R)\, {}^{\mathrm{t}}X) \\ &\qquad - \pi i \mathrm{tr}({}^{\mathrm{t}}({}^{\mathrm{t}}X)\, {}^{\mathrm{t}}Y)\big) \\ &= \sum_{N \in \mathrm{M}(1,k,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(Z \cdot 1[N - {}^{\mathrm{t}}Y]) + 2\pi i \mathrm{tr}({}^{\mathrm{t}}N\, {}^{\mathrm{t}}X) - \pi i \mathrm{tr}({}^{\mathrm{t}}({}^{\mathrm{t}}X)\, {}^{\mathrm{t}}Y)\big) \\ &= \theta(1, Z, {}^{\mathrm{t}}X, {}^{\mathrm{t}}Y), \end{aligned}$$

where 1 is the 1×1 matrix with entry 1. It follows that $\theta(Z, X, Y)$ for $Z \in \mathbb{H}_k$, and $X, Y \in \mathrm{M}(k, 1, \mathbb{C})$ has the convergence properties mentioned in Proposition 3.1.8. For $Z \in \mathbb{H}_k$, $R \in \mathrm{M}(k, 1, \mathbb{R})$, and $X, Y \in \mathrm{M}(k, 1, \mathbb{C})$ define

$$g(Z, R, X, Y) = \exp\big(\pi i\, {}^{\mathrm{t}}(R - Y)Z(R - Y) + 2\pi i\, {}^{\mathrm{t}}RX - \pi i\, {}^{\mathrm{t}}XY\big) \tag{3.3}$$

Lemma 3.2.1. *Let k be a positive integer, $U \in \mathrm{Sym}(k,\mathbb{R})^+$ and $X, Y \in \mathrm{M}(k,1,\mathbb{C})$. The function $g(iU,\cdot,X,Y)$ is contained in the Schwartz space*

$$\mathcal{S}(\mathrm{M}(k,1,\mathbb{R})) = \mathcal{S}(\mathbb{R}^k)$$

(see section 2.2 for the definition of the Schwartz space).

Proof. Write $X = X_1 + iX_2$ and $Y = Y_1 + iY_2$ for $X_1, X_2, Y_1, Y_2 \in \mathrm{M}(k,1,\mathbb{R})$. Also, write $U = V^2$ for some $V \in \mathrm{Sym}(k,\mathbb{R})^+$ (see (1.7)). Since $\exp(-\pi i\,{}^{\mathrm{t}}XY)$ is constant, it suffices to prove that the function defined by

$$R \mapsto \exp\big(-\pi\,{}^{\mathrm{t}}(R-Y)U(R-Y) + 2\pi i\,{}^{\mathrm{t}}RX\big)$$

is contained $\mathcal{S}(\mathrm{M}(k,1,\mathbb{R}))$. Since $\mathcal{S}(\mathrm{M}(k,1,\mathbb{R}))$ is mapped to itself by the map induced by $R \mapsto R + Y_2$, we may assume that our function has the form

$$R \mapsto \exp\big(-\pi\,{}^{\mathrm{t}}(R-iY_2)U(R-iY_2) + 2\pi i\,{}^{\mathrm{t}}RX\big)$$

Let $R \in \mathrm{M}(k,1,\mathbb{R})$. Then

$$\begin{aligned}
&\exp\big(-\pi\,{}^{\mathrm{t}}(R-Y)U(R-Y) + 2\pi i\,{}^{\mathrm{t}}RX\big)\\
&\quad= \exp\big(-\pi\,{}^{\mathrm{t}}(R-iY_2)\,{}^{\mathrm{t}}VV(R-iY_2) + 2\pi i\,{}^{\mathrm{t}}RX\big)\\
&\quad= \exp\big(-\pi\,{}^{\mathrm{t}}(VR-iVY_2)(VR-iVY_2) + 2\pi i\,{}^{\mathrm{t}}RX\big).
\end{aligned}$$

Since $\mathcal{S}(\mathrm{M}(k,1,\mathbb{R}))$ is mapped to itself by the map induced by $R \mapsto V^{-1}R$, we may assume that our function has the form

$$R \mapsto \exp\big(-\pi\,{}^{\mathrm{t}}(R-iY_2)(R-iY_2) + 2\pi i\,{}^{\mathrm{t}}RX\big)$$

For $R \in \mathrm{M}(k,1,\mathbb{R})$ we have:

$$\begin{aligned}
&\exp\big(-\pi\,{}^{\mathrm{t}}(R-iY_2)(R-iY_2) + 2\pi i\,{}^{\mathrm{t}}RX\big)\\
&\quad= \exp\big(-\pi\,{}^{\mathrm{t}}RR - 2\pi\,{}^{\mathrm{t}}RX_2 + \pi\,{}^{\mathrm{t}}Y_2Y_2 + i(2\pi\,{}^{\mathrm{t}}RX_1 + \pi\,{}^{\mathrm{t}}RY_2 + \pi\,{}^{\mathrm{t}}Y_2R)\big).
\end{aligned}$$

Since $\exp(\pi\,{}^{\mathrm{t}}Y_2Y_2)$ is constant, we see that it suffices to prove that the function $h : \mathrm{M}(k,1,\mathbb{R}) \to \mathbb{C}$ defined by

$$h(R) = \exp\big(-\pi\,{}^{\mathrm{t}}RR - 2\pi\,{}^{\mathrm{t}}RX_2 + i(2\pi\,{}^{\mathrm{t}}RX_1 + \pi\,{}^{\mathrm{t}}RY_2 + \pi\,{}^{\mathrm{t}}Y_2R)\big)$$

is contained $\mathcal{S}(\mathrm{M}(k,1,\mathbb{R}))$. Let $\alpha = (\alpha_1,\dots,\alpha_k) \in \mathbb{Z}^k_{\geq 0}$ and $P(X_1,\dots,X_k) \in \mathbb{C}[X_1,\dots,X_k]$; we need to prove that $|P(R)(D^\alpha h)(R)|$ is bounded as a function of $R \in \mathrm{M}(k,1,\mathbb{R})$. To see this, we note that there exists a polynomial $Q_\alpha(X_1,\dots,X_k) \in \mathbb{C}[X_1,\dots,X_k]$ such that

$$(D^\alpha h)(R) = Q_\alpha(R)h(R).$$

for $R \in \mathrm{M}(k,1,\mathbb{R})$. For $R \in \mathrm{M}(k,1,\mathbb{R})$ we have

$$|P(R)(D^\alpha h)(R)| = |P(R)Q_\alpha(R)\exp\big(-\pi\,{}^{\mathrm{t}}RR - 2\pi\,{}^{\mathrm{t}}RX_2\big)|$$

$$= |P(R)Q_\alpha(R)\exp\big(-\pi\,{}^{\mathrm{t}}(R+X_2)(R+X_2)-\pi\,{}^{\mathrm{t}}X_2X_2\big)|$$
$$= |\exp(-\pi\,{}^{\mathrm{t}}X_2X_2)P(R)Q_\alpha(R)\exp\big(-\pi\,{}^{\mathrm{t}}(R+X_2)(R+X_2)\big)|. \tag{3.4}$$

It is well-known that the function

$$R \mapsto \exp\big(-\pi\,{}^{\mathrm{t}}RR\big)$$

is contained $\mathcal{S}(\mathrm{M}(k,1,\mathbb{R}))$. As above, this implies that

$$\exp\big(-\pi\,{}^{\mathrm{t}}(R+X_2)(R+X_2)\big)$$

is also contained $\mathcal{S}(\mathrm{M}(k,1,\mathbb{R}))$. This implies that (3.4) is bounded. □

Lemma 3.2.2. *Let k be a positive integer. Let $U \in \mathrm{Sym}(k,\mathbb{R})^+$ and $X, Y \in \mathrm{M}(k,1,\mathbb{C})$. The Fourier transform (see section 2.2) of the Schwartz function $g(iU,\cdot,X,Y)$ is given by*

$$\mathcal{F}(g(iU,\cdot,X,Y))(R) = \det(U)^{-1/2}g(-(iU)^{-1},-R,Y,-X).$$

Proof. Let $R \in \mathrm{M}(k,1,\mathbb{R})$. We recall that for $Z \in \mathbb{H}_k$, the function g is given by:

$$g(Z,R,X,Y) = \exp\big(\pi i\,{}^{\mathrm{t}}(R-Y)Z(R-Y)+2\pi i\,{}^{\mathrm{t}}RX-\pi i\,{}^{\mathrm{t}}XY\big).$$

Therefore,

$$\begin{aligned}
&\mathcal{F}(g(iU,\cdot,X,Y))(R)\\
&= \int\limits_{\mathbb{R}^k} \exp\big(-\pi\,{}^{\mathrm{t}}(r-Y)U(r-Y)+2\pi i\,{}^{\mathrm{t}}rX-\pi i\,{}^{\mathrm{t}}XY\big)\exp(-2\pi i\,{}^{\mathrm{t}}Rr)\,dr\\
&= \exp(-\pi i\,{}^{\mathrm{t}}XY)\int\limits_{\mathbb{R}^k} \exp\Big(-\pi\Big[{}^{\mathrm{t}}(r-Y)U(r-Y)-2i\,{}^{\mathrm{t}}rX+2i\,{}^{\mathrm{t}}Rr\Big]\Big)\,dr.
\end{aligned}$$

Write $U = V^2$ for some $V \in \mathrm{Sym}(k,\mathbb{R})^+$ (see (1.7)). Then:

$$\begin{aligned}
&\int\limits_{\mathbb{R}^k} \exp\Big(-\pi\Big[{}^{\mathrm{t}}(r-Y)U(r-Y)-2i\,{}^{\mathrm{t}}rX+2i\,{}^{\mathrm{t}}Rr\Big]\Big)\,dr\\
&= \int\limits_{\mathbb{R}^k} \exp\Big(-\pi\Big[{}^{\mathrm{t}}(r-Y)U(r-Y)+2i\,{}^{\mathrm{t}}r(-X+R)\Big]\Big)\,dr\\
&= \int\limits_{\mathbb{R}^k} \exp\Big(-\pi\Big[{}^{\mathrm{t}}(r-Y)\,{}^{\mathrm{t}}VV(r-Y)+2i\,{}^{\mathrm{t}}r\,{}^{\mathrm{t}}V\,{}^{\mathrm{t}}V^{-1}(-X+R)\Big]\Big)\,dr\\
&= \int\limits_{\mathbb{R}^k} \exp\Big(-\pi\Big[{}^{\mathrm{t}}(Vr-VY)(Vr-VY)+2i\,{}^{\mathrm{t}}(Vr)\,{}^{\mathrm{t}}V^{-1}(-X+R)\Big]\Big)\,dr\\
&= \det(V)^{-1}\int\limits_{\mathbb{R}^k} \exp\Big(-\pi\Big[{}^{\mathrm{t}}(r-VY)(r-VY)+2i\,{}^{\mathrm{t}}r\,{}^{\mathrm{t}}V^{-1}(-X+R)\Big]\Big)\,dr
\end{aligned}$$

$$= \det(U)^{-1/2} \exp(-\pi\, {}^{\mathrm{t}}(VY)(VY)) \int\limits_{\mathbb{R}^k} \exp\big(-\pi\big[\,{}^{\mathrm{t}}rr + 2\,{}^{\mathrm{t}}rQ\big]\big)\, dr,$$

where

$$Q = -VY + i\,{}^{\mathrm{t}}V^{-1}(-X+R) = -VY - i\,{}^{\mathrm{t}}V^{-1}X + i\,{}^{\mathrm{t}}V^{-1}R.$$

For the penultimate equality, we used the formula for a linear change of variables (see Theorem 2.20, (e) on page 50 and section 2.23 of [24]). Completing the square, we obtain:

$$\begin{aligned}
&\det(U)^{-1/2} \exp(-\pi\, {}^{\mathrm{t}}(VY)(VY)) \int\limits_{\mathbb{R}^k} \exp\big(-\pi\big[\,{}^{\mathrm{t}}rr + 2\,{}^{\mathrm{t}}rQ\big]\big)\, dr \\
&= \det(U)^{-1/2} \exp(-\pi\, {}^{\mathrm{t}}YUY) \int\limits_{\mathbb{R}^k} \exp\big(-\pi\big[\,{}^{\mathrm{t}}rr + 2\,{}^{\mathrm{t}}rQ + {}^{\mathrm{t}}QQ - {}^{\mathrm{t}}QQ\big]\big)\, dr \\
&= \det(U)^{-1/2} \exp(-\pi\, {}^{\mathrm{t}}YUY) \int\limits_{\mathbb{R}^k} \exp\big(-\pi\big[\,{}^{\mathrm{t}}(r+Q)(r+Q) - {}^{\mathrm{t}}QQ\big]\big)\, dr \\
&= \det(U)^{-1/2} \exp(-\pi\, {}^{\mathrm{t}}YUY + \pi\, {}^{\mathrm{t}}QQ) \int\limits_{\mathbb{R}^k} \exp\big(-\pi\, {}^{\mathrm{t}}(r+Q)(r+Q))\big)\, dr \\
&= \det(U)^{-1/2} \exp(-\pi\, {}^{\mathrm{t}}YUY + \pi\, {}^{\mathrm{t}}QQ) \int\limits_{\mathbb{R}^k} \exp(-\pi\, {}^{\mathrm{t}}rr)\, dr \\
&= \det(U)^{-1/2} \exp(-\pi\, {}^{\mathrm{t}}YUY + \pi\, {}^{\mathrm{t}}QQ).
\end{aligned}$$

For the penultimate equality, we used Lemma 2.2.2. Therefore,

$$\begin{aligned}
&\mathcal{F}(g(iU,\cdot,X,Y))(R) \\
&= \det(U)^{-1/2} \exp(-\pi i\, {}^{\mathrm{t}}XY) \exp(-\pi\, {}^{\mathrm{t}}YUY + \pi\, {}^{\mathrm{t}}QQ) \\
&= \det(U)^{-1/2} \exp\big(-i\pi\, {}^{\mathrm{t}}XY - \pi\, {}^{\mathrm{t}}XV^{-1}\, {}^{\mathrm{t}}V^{-1}X + \pi\, {}^{\mathrm{t}}RV^{-1}\, {}^{\mathrm{t}}V^{-1}X \\
&\quad + i\pi\, {}^{\mathrm{t}}Y\, {}^{\mathrm{t}}V\, {}^{\mathrm{t}}V^{-1}X - \pi\, {}^{\mathrm{t}}YUY + \pi\, {}^{\mathrm{t}}XV^{-1}\, {}^{\mathrm{t}}V^{-1}R \\
&\quad + i\pi\, {}^{\mathrm{t}}XV^{-1}VY - \pi\, {}^{\mathrm{t}}RV^{-1}\, {}^{\mathrm{t}}V^{-1}R - i\pi\, {}^{\mathrm{t}}RV^{-1}VY \\
&\quad - i\pi\, {}^{\mathrm{t}}Y\, {}^{\mathrm{t}}V\, {}^{\mathrm{t}}V^{-1}R + \pi\, {}^{\mathrm{t}}Y\, {}^{\mathrm{t}}VVY\big) \\
&= \det(U)^{-1/2} \exp\big(-i\pi\, {}^{\mathrm{t}}XY - \pi\, {}^{\mathrm{t}}XU^{-1}X + \pi\, {}^{\mathrm{t}}RU^{-1}X \\
&\quad + i\pi\, {}^{\mathrm{t}}YX - \pi\, {}^{\mathrm{t}}YUY + \pi\, {}^{\mathrm{t}}XU^{-1}R \\
&\quad + i\pi\, {}^{\mathrm{t}}XY - \pi\, {}^{\mathrm{t}}RU^{-1}R - i\pi\, {}^{\mathrm{t}}RY \\
&\quad - i\pi\, {}^{\mathrm{t}}YR + \pi\, {}^{\mathrm{t}}YUY\big) \\
&= \det(U)^{-1/2} \exp\big(-\pi\big[\,{}^{\mathrm{t}}XU^{-1}X - {}^{\mathrm{t}}RU^{-1}X - {}^{\mathrm{t}}XU^{-1}R + {}^{\mathrm{t}}RU^{-1}R\big] \\
&\quad - 2i\pi\, {}^{\mathrm{t}}RY + i\pi\, {}^{\mathrm{t}}YX\big) \\
&= \det(U)^{-1/2} \exp\big(-\pi\big[\,{}^{\mathrm{t}}(R-X)U^{-1}(R-X)\big]
\end{aligned}$$

$$
\begin{aligned}
&- 2i\pi\, {}^{\mathrm{t}}RY - i\pi\, {}^{\mathrm{t}}Y(-X)\big)\\
&= \det(U)^{-1/2} \exp\big(\pi i\Big[\,{}^{\mathrm{t}}(R-X)(-(iU)^{-1})(R-X)\Big]\\
&\quad - 2i\pi\, {}^{\mathrm{t}}RY - i\pi\, {}^{\mathrm{t}}Y(-X)\big)\\
&= \det(U)^{-1/2} \exp\big(\pi i\Big[\,{}^{\mathrm{t}}(-R-(-X))(-(iU)^{-1})(-R-(-X))\Big]\\
&\quad + 2i\pi\, {}^{\mathrm{t}}(-R)Y - i\pi\, {}^{\mathrm{t}}Y(-X)\big)\\
&= \det(U)^{-1/2} g(-(iU)^{-1}, -R, Y, -X).
\end{aligned}
$$

This completes the proof. □

Lemma 3.2.3. *Let k be a positive integer. There exists an eighth root of unity ξ such that for $Z \in \mathbb{H}_k$ and $X, Y \in \mathrm{M}(k,1,\mathbb{C})$ we have*

$$
\theta(Z,X,Y) = \xi s(\begin{bmatrix} & 1\\ -1 & \end{bmatrix}, Z)^{-1} \theta(-Z^{-1}, Y, -X).
$$

Here, $s(\left[\begin{smallmatrix} & 1\\ -1 & \end{smallmatrix}\right], Z)$ for $Z \in \mathbb{H}_k$ is defined as in Proposition 1.10.8, and has the property

$$
s(\begin{bmatrix} & 1\\ -1 & \end{bmatrix}, Z)^2 = j(\begin{bmatrix} & 1\\ -1 & \end{bmatrix}, Z) = \det(-Z^{-1}).
$$

for $Z \in \mathbb{H}_k$.

Proof. Let the function g be as in (3.3). Let $U \in \mathrm{Sym}(k,\mathbb{R})^+$ and $X, Y \in \mathrm{M}(k,1,\mathbb{C})$. By Lemma 3.2.1 the function $g(iU,\cdot,X,Y)$ is in $\mathcal{S}(\mathrm{M}(k,1,\mathbb{R}))$. By Theorem 2.2.4, Lemma 3.2.2, and Proposition 1.10.8, we have:

$$
\begin{aligned}
\sum_{R\in \mathrm{M}(k,1,\mathbb{Z})} g(iU,R,X,Y) &= \sum_{R\in \mathrm{M}(k,1,\mathbb{Z})} (\mathcal{F}g)(iU,R,X,Y)\\
\theta(iU,X,Y) &= \det(U)^{-1/2} \sum_{R\in \mathrm{M}(k,1,\mathbb{Z})} g(-(iU)^{-1}, -R, Y, -X)\\
\theta(iU,X,Y) &= \det(U)^{-1/2} \theta(-(iU)^{-1}, Y, -X)\\
\theta(iU,X,Y) &= \xi s(\begin{bmatrix} & 1\\ -1 & \end{bmatrix}, iU)^{-1} \theta(-(iU)^{-1}, Y, -X).
\end{aligned}
$$

The assertion of the lemma follows now from Lemma 1.10.5. □

Let k be a positive integer. Let V be the be $\mathbb{C}$ vector space of all functions from $\mathbb{H}_k \times \mathrm{M}(k,1,\mathbb{C}) \times \mathrm{M}(k,1,\mathbb{C})$ to $\mathbb{C}$. For $g = \left[\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right] \in \mathrm{Sp}(2n,\mathbb{Z})$ and $F \in V$ we define another element $F\big|g$ of V by the formula

$$
(F\big|g)(Z,X,Y) = s(g,Z)^{-1} F(g\cdot Z, AX+BY, CX+DY)
$$

for $X \in \mathbb{H}_k$ and $X, Y \in \mathrm{M}(k,1,\mathbb{C})$. We define an equivalence relation $\sim$ on the set V by defining $F_1, F_2 \in V$ to be equivalent if there exists an eighth root of unity ζ such that $F_2 = \zeta F_1$. If $F \in V$, then the equivalence class determined by F will be denoted by $[F]$. For $F \in V$ and $g \in \mathrm{Sp}(2k, \mathbb{Z})$, we define another equivalence class in $V/\sim$ by

$$[F]\big|g = [F|g].$$

It is easy to see that $[F]\big|g$ is well-defined, and a calculation using Corollary 1.10.9 and Lemma 1.10.7 shows that

$$[F]\big|(gh) = ([F]\big|g)\big|h$$

for $F \in V$ and $g, h \in \mathrm{Sp}(2k, \mathbb{Z})$. We define a function

$$T : \mathbb{Z}^{2k} \longrightarrow V/\sim \tag{3.5}$$

by

$$T(m) = [\exp\big(-\pi i\,{}^{\mathrm{t}}m_1 X/2 + \pi i\,{}^{\mathrm{t}}m_2 Y/2\big)\theta(Z, X + m_2/2, Y + m_1/2)]$$

where $m \in \mathbb{Z}^{2k}$ is (as usual) regarded as a column vector, and $m = \left[\begin{smallmatrix} m_1 \\ m_2 \end{smallmatrix}\right]$ with $m_1, m_2 \in \mathbb{Z}^k$.

Lemma 3.2.4. *Let k be a positive integer. Then*

$$T(m + 2n) = T(m)$$

for $m, n \in \mathbb{Z}^{2k}$.

Proof. We begin with an observation about θ. Let $X_0, Y_0 \in \mathrm{M}(k, 1, \mathbb{Z})$. Then for $Z \in \mathbb{H}_k$ and $X, Y \in \mathrm{M}(k, 1, \mathbb{C})$ we have:

$$\begin{aligned}
&\theta(Z, X + X_0, Y + Y_0)\\
&= \sum_{R \in \mathrm{M}(k,1,\mathbb{Z})} \exp\big(\pi Z[R - Y - Y_0] + 2\pi i\,{}^{\mathrm{t}}R(X + X_0) - \pi i\,{}^{\mathrm{t}}(X + X_0)(Y + Y_0)\big)\\
&= \sum_{R \in \mathrm{M}(k,1,\mathbb{Z})} \exp\big(\pi Z[R - Y] + 2\pi i\,{}^{\mathrm{t}}(R + Y_0)(X + X_0)\\
&\qquad - \pi i\,{}^{\mathrm{t}}(X + X_0)(Y + Y_0)\big)\\
&= \sum_{R \in \mathrm{M}(k,1,\mathbb{Z})} \exp\big(\pi Z[R - Y] + 2\pi i\,{}^{\mathrm{t}}RX + 2\pi i\,{}^{\mathrm{t}}RX_0 + 2\pi i\,{}^{\mathrm{t}}Y_0 X + 2\pi i\,{}^{\mathrm{t}}Y_0 X_0\\
&\qquad - \pi i\,{}^{\mathrm{t}}XY - \pi i\,{}^{\mathrm{t}}XY_0 - \pi i\,{}^{\mathrm{t}}X_0 Y - \pi i\,{}^{\mathrm{t}}X_0 Y_0\big)\\
&= \sum_{R \in \mathrm{M}(k,1,\mathbb{Z})} \exp\big(\pi Z[R - Y] + 2\pi i\,{}^{\mathrm{t}}RX + \pi i\,{}^{\mathrm{t}}Y_0 X +\\
&\qquad - \pi i\,{}^{\mathrm{t}}XY - \pi i\,{}^{\mathrm{t}}X_0 Y - \pi i\,{}^{\mathrm{t}}X_0 Y_0\big) \qquad (\text{since } {}^{\mathrm{t}}RX_0,\ {}^{\mathrm{t}}Y_0 X_0 \in \mathbb{Z})\\
&= \exp\big(\pi i\,{}^{\mathrm{t}}Y_0 X - \pi i\,{}^{\mathrm{t}}X_0 Y - \pi i\,{}^{\mathrm{t}}X_0 Y_0\big)\\
&\qquad \times \sum_{R \in \mathrm{M}(k,1,\mathbb{Z})} \exp\big(\pi Z[R - Y] + 2\pi i\,{}^{\mathrm{t}}RX - \pi i\,{}^{\mathrm{t}}XY\big)
\end{aligned}$$

$$= \exp\big(\pi i\,{}^{t}Y_0 X - \pi i\,{}^{t}X_0 Y - \pi i\,{}^{t}X_0 Y_0\big)\theta(Z,X,Y).$$

It follows that

$$[\theta(Z, X+X_0, Y+Y_0)] = [\exp\big(\pi i\,{}^{t}Y_0 X - \pi i\,{}^{t}X_0 Y\big)\theta(Z,X,Y)]$$

because $\exp(-\pi i\,{}^{t}X_0 Y_0)$ is an eighth root of unity. Now let $m, n \in \mathbb{Z}^{2k}$. Then

$$\begin{aligned}
&T(m+2n)\\
&= [\exp\big(-\pi i\,{}^{t}(m_1+2n_1)X/2 + \pi i\,{}^{t}(m_2+2n_2)Y/2\big)\\
&\qquad \times \theta(Z, X+m_2/2+n_2, Y+m_1/2+n_1)]\\
&= [\exp\big(-\pi i\,{}^{t}m_1 X/2 - \pi i\,{}^{t}n_1 X + \pi i\,{}^{t}m_2 Y/2 + \pi i\,{}^{t}n_2 Y\big)\\
&\qquad \times \exp\big(\pi i\,{}^{t}n_1(X+m_2/2) - \pi i\,{}^{t}n_2(Y+m_1/2)\big)\\
&\qquad \times \theta(Z, X+m_2/2, Y+m_1/2)]\\
&= [\exp\big(-\pi i\,{}^{t}m_1 X/2 - \pi i\,{}^{t}n_1 X + \pi i\,{}^{t}m_2 Y/2 + \pi i\,{}^{t}n_2 Y\big)\\
&\qquad \times \exp\big(\pi i\,{}^{t}n_1 X + \pi i\,{}^{t}n_1 m_2/2 - \pi i\,{}^{t}n_2 Y - \pi i\,{}^{t}n_2 m_1/2\big)]\\
&\qquad \times \theta(Z, X+m_1/2, Y+m_2/2)\\
&= [\exp\big(-\pi i\,{}^{t}m_1 X/2 + \pi i\,{}^{t}m_2 Y/2\big)\\
&\qquad \times \exp\big(\pi i\,{}^{t}n_1 m_2/2 - \pi i\,{}^{t}n_2 m_1/2\big)\\
&\qquad \times \theta(Z, X+m_2/2, Y+m_1/2)]\\
&= [\exp\big(-\pi i\,{}^{t}m_1 X/2 + \pi i\,{}^{t}m_2 Y/2\big)\theta(Z, X+m_2/2, Y+m_1/2)]\\
&= T(m),
\end{aligned}$$

because $\exp(\pi i\,{}^{t}n_1 m_2/2 - \pi i\,{}^{t}n_2 m_1/2)$ is an eighth root of unity. $\square$

By Lemma 3.2.4, the function T induces a function

$$T : (\mathbb{Z}/2\mathbb{Z})^{2k} \longrightarrow V/\sim,$$

which we denote by the same name.

Next, if $H : (\mathbb{Z}/2\mathbb{Z})^{2k} \to V/\sim$ is a function and $g \in \mathrm{Sp}(2n,\mathbb{Z})$, then we define a new function $H\big|g : (\mathbb{Z}/2\mathbb{Z})^{2k} \to V/\sim$ by

$$(H\big|g)(m) = H(g\{m\})\big|g$$

for $m \in (\mathbb{Z}/2\mathbb{Z})^{2k}$; here, $g\{m\}$ is defined in Proposition 1.11.2, where it is proven that this defines an action of $\mathrm{Sp}(2k,\mathbb{Z})$ on $(\mathbb{Z}/2\mathbb{Z})^{2k}$. It is easy to verify that

$$H\big|(gh) = (H\big|g)\big|h \tag{3.6}$$

for $g, h \in \mathrm{Sp}(2k,\mathbb{Z})$ and a function $H : (\mathbb{Z}/2\mathbb{Z})^{2k} \to V/\sim$.

Theorem 3.2.5. *Let k be a positive integer. Then*

$$T\big|g = T$$

for $g \in \mathrm{Sp}(2k,\mathbb{Z})$.

Proof. Since (3.6) holds, it suffices to prove that $T\big|g = T$ for the generators of $\mathrm{Sp}(2k,\mathbb{Z})$ from Theorem 1.9.6. Let $B \in \mathrm{Sym}(k,\mathbb{Z})$ and $m \in (\mathbb{Z}/2\mathbb{Z})^{2k}$. Then, using that

$$
\begin{aligned}
&(T|\begin{bmatrix}1 & B\\ & 1\end{bmatrix})(m)\\
&= T(\begin{bmatrix}1 & B\\ & 1\end{bmatrix}\{m\})\big|\begin{bmatrix}1 & B\\ & 1\end{bmatrix}\\
&= T(\begin{bmatrix}m_1\\ -Bm_1+m_2+\mathrm{diag}(B)\end{bmatrix})\big|\begin{bmatrix}1 & B\\ & 1\end{bmatrix}\\
&= [\exp\big(-\pi i\,{}^{\mathrm{t}}m_1(X+BY)/2 + \pi i\,{}^{\mathrm{t}}(-Bm_1+m_2+\mathrm{diag}(B))Y/2\big)\\
&\qquad \times \theta(Z, X - Bm_1/2 + m_2/2 + \mathrm{diag}(B)/2, Y + m_1/2)]\big|\begin{bmatrix}1 & B\\ & 1\end{bmatrix}\\
&= [\exp\big(-\pi i\,{}^{\mathrm{t}}m_1(X+BY)/2 + \pi i\,{}^{\mathrm{t}}(-Bm_1+m_2+\mathrm{diag}(B))Y/2\big)\\
&\qquad \times \theta(Z+B, X + BY - Bm_1/2 + m_2/2 + \mathrm{diag}(B)/2, Y + m_1/2)]\\
&\qquad (\text{use } s([\begin{smallmatrix}1 & B\\ & 1\end{smallmatrix}], Z)^2 = 1, \text{ so that } s([\begin{smallmatrix}1 & B\\ & 1\end{smallmatrix}], Z) \text{ is identically } 1 \text{ or } -1)\\
&= [\exp\big(-\pi i\,{}^{\mathrm{t}}m_1(X+BY)/2 + \pi i\,{}^{\mathrm{t}}(-Bm_1+m_2+\mathrm{diag}(B))Y/2)\\
&\qquad \times \sum_{R\in \mathrm{M}(k,1,\mathbb{Z})} \exp\big(\pi i(Z+B)[R - Y - m_1/2]\\
&\qquad + 2\pi i\,{}^{\mathrm{t}}R(X + BY - Bm_1/2 + m_2/2 + \mathrm{diag}(B)/2)\\
&\qquad - \pi i\,{}^{\mathrm{t}}(X + BY - Bm_1/2 + m_2/2 + \mathrm{diag}(B)/2)(Y + m_1/2)\big)]\\
&= [\exp\big(-\pi i\,{}^{\mathrm{t}}m_1(X+BY)/2 + \pi i\,{}^{\mathrm{t}}(-Bm_1+m_2+\mathrm{diag}(B))Y/2)\\
&\qquad \times \sum_{R\in \mathrm{M}(k,1,\mathbb{Z})} \exp\big(\pi i Z[R - Y - m_1/2] + 2\pi i\,{}^{\mathrm{t}}R(X + m_2/2)\\
&\qquad - \pi i\,{}^{\mathrm{t}}(X + m_2/2)(Y + m_1/2)\big)\\
&\qquad \times \exp\big(\pi i B[R - Y - m_1/2] + 2\pi i\,{}^{\mathrm{t}}R(BY - Bm_1/2 + \mathrm{diag}(B)/2)\\
&\qquad - \pi i\,{}^{\mathrm{t}}(BY - Bm_1/2 + \mathrm{diag}(B)/2)(Y + m_1/2)\big)]\\
&= [\exp\big(-\pi i\,{}^{\mathrm{t}}m_1(X+BY)/2 + \pi i\,{}^{\mathrm{t}}(-Bm_1+m_2+\mathrm{diag}(B))Y/2)\\
&\qquad \times \sum_{R\in \mathrm{M}(k,1,\mathbb{Z})} \exp\big(\pi i(Z+B)[R - Y - m_1/2]\\
&\qquad + 2\pi i\,{}^{\mathrm{t}}R(X + BY - Bm_1/2 + m_2/2 + \mathrm{diag}(B)/2)\\
&\qquad \times \exp\big(\pi i\,{}^{\mathrm{t}}(R - Y - m_1/2)B(R - Y - m_1/2)\\
&\qquad + 2\pi i\,{}^{\mathrm{t}}R(BY - Bm_1/2 + \mathrm{diag}(B)/2)\\
&\qquad - \pi i\,{}^{\mathrm{t}}(BY - Bm_1/2 + \mathrm{diag}(B)/2)(Y + m_1/2)\big)]\\
&= [\exp\big(-\pi i\,{}^{\mathrm{t}}m_1X/2 - \pi i\,{}^{\mathrm{t}}m_1BY/2\\
&\qquad - \pi i\,{}^{\mathrm{t}}m_1BY/2 + \pi i\,{}^{\mathrm{t}}m_2Y/2 + \pi i\,{}^{\mathrm{t}}\mathrm{diag}(B)Y/2)
\end{aligned}
$$

$$
\begin{aligned}
&\times \sum_{R\in \mathrm{M}(k,1,\mathbb{Z})} \exp\big(\pi i(Z+B)[R-Y-m_1/2]\\
&+2\pi i\,{}^{\mathrm{t}}R(X+BY-Bm_1/2+m_2/2+\mathrm{diag}(B)/2)\\
&\times \exp\big(\pi i\,{}^{\mathrm{t}}RBR-\pi i\,{}^{\mathrm{t}}RBY-\pi i\,{}^{\mathrm{t}}RBm_1/2\\
&-\pi i\,{}^{\mathrm{t}}YBR+\pi i\,{}^{\mathrm{t}}YBY+\pi i\,{}^{\mathrm{t}}YBm_1/2\\
&-\pi i\,{}^{\mathrm{t}}m_1BR/2+\pi i\,{}^{\mathrm{t}}m_1BY/2+\pi i\,{}^{\mathrm{t}}m_1Bm_1/4\\
&+2\pi i\,{}^{\mathrm{t}}RBY-2\pi i\,{}^{\mathrm{t}}RBm_1/2+2\pi i\,{}^{\mathrm{t}}R\mathrm{diag}(B)/2\\
&-\pi i\,{}^{\mathrm{t}}YBY-\pi i\,{}^{\mathrm{t}}YBm_1/2\\
&+\pi i\,{}^{\mathrm{t}}m_1BY/2+\pi i\,{}^{\mathrm{t}}m_1Bm_1/4\\
&-\pi i\,{}^{\mathrm{t}}\mathrm{diag}(B)Y/2-\pi i\,{}^{\mathrm{t}}\mathrm{diag}(B)m_1/4\big)]\\
=&[\exp\big(-\pi i\,{}^{\mathrm{t}}m_1X/2+\pi i\,{}^{\mathrm{t}}m_2Y/2\big)\\
&\times\exp\big(+\pi i\,{}^{\mathrm{t}}m_1Bm_1/2-\pi i\,{}^{\mathrm{t}}\mathrm{diag}(B)m_1/4\big)\\
&\times \sum_{R\in \mathrm{M}(k,1,\mathbb{Z})} \exp\big(\pi iZ[R-Y-m_1/2]+2\pi i\,{}^{\mathrm{t}}R(X+m_2/2)\\
&-\pi i\,{}^{\mathrm{t}}(X+m_2/2)(Y+m_1/2)\big)\\
&\times\exp\big(\pi i({}^{\mathrm{t}}RBR+{}^{\mathrm{t}}R\mathrm{diag}(B))-2\pi i\,{}^{\mathrm{t}}RBm_1\big)]\\
=&[\exp\big(-\pi i\,{}^{\mathrm{t}}m_1X/2+\pi i\,{}^{\mathrm{t}}m_2Y/2\big)\\
&\times\exp\big(\pi i\,{}^{\mathrm{t}}m_1Bm_1/2-\pi i\,{}^{\mathrm{t}}\mathrm{diag}(B)m_1/4\big)\\
&\times \sum_{R\in \mathrm{M}(k,1,\mathbb{Z})} \exp\big(\pi iZ[R-Y-m_1/2]+2\pi i\,{}^{\mathrm{t}}R(X+m_2/2)\\
&-\pi i\,{}^{\mathrm{t}}(X+m_2/2)(Y+m_1/2)\big)] \qquad \text{(See Lemma 1.11.1)}\\
=&[\exp\big(-\pi i\,{}^{\mathrm{t}}m_1X/2+\pi i\,{}^{\mathrm{t}}m_2Y/2\big)\theta(Z,X+m_2/2,Y+m_1/2)]\\
=&T(m).
\end{aligned}
$$

And:

$$
\begin{aligned}
&(T|\begin{bmatrix} & 1\\ -1 & \end{bmatrix})(m)\\
&=T(\begin{bmatrix} & 1\\ -1 & \end{bmatrix}\{m\})|\begin{bmatrix} & 1\\ -1 & \end{bmatrix}\\
&=T(\begin{bmatrix} m_2\\ -m_1\end{bmatrix})|\begin{bmatrix} & 1\\ -1 & \end{bmatrix}\\
&=[\exp\big(-\pi i\,{}^{\mathrm{t}}m_2X/2-\pi i\,{}^{\mathrm{t}}m_1Y\big)\theta(Z,X-m_1/2,Y+m_2/2)]|\begin{bmatrix} & 1\\ -1 & \end{bmatrix}\\
&=[s(\begin{bmatrix} & 1\\ -1 & \end{bmatrix},Z)^{-1}\exp\big(-\pi i\,{}^{\mathrm{t}}m_2Y/2+\pi i\,{}^{\mathrm{t}}m_1X/2\big)\\
&\quad\times\theta(-Z^{-1},Y-m_1/2,-X+m_2/2)]\\
&=[\exp\big(-\pi i\,{}^{\mathrm{t}}m_2Y/2+\pi i\,{}^{\mathrm{t}}m_1X/2\big)
\end{aligned}
$$

$$\begin{aligned}
&\times \theta(Z, X - m_2/2, Y - m_1/2)] \qquad \text{(by Lemma 3.2.3)}\\
&= [\exp\big(-\pi i\, {}^{\mathrm{t}}(-m_1)X/2 + \pi i\, {}^{\mathrm{t}}(-m_2)Y/2\big)\theta(Z, X - m_2/2, Y - m_1/2)]\\
&= T(-m)\\
&= T(m).
\end{aligned}$$

This completes the proof. □

Corollary 3.2.6. *Let k be a positive integer, and let Γ_θ be the theta group, as defined in sect. 1.11. Let μ_8 be the group of all eighth roots of unity. There exists a function $\chi : \Gamma_\theta \to \mu_8$ such that*

$$\theta(Z, X, Y) = \chi(g)s(g, Z)^{-1}\theta(g \cdot Z, AX + BY, CX + DY)$$

for $Z \in \mathbb{H}_k$, $X, Y \in \mathrm{M}(k, 1, \mathbb{C})$, and $g = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \Gamma_\theta$.

Proof. Let $g \in \Gamma_\theta$. By Theorem 3.2.5 we have $T\big|g = T$. Evaluating at $m = 0 \in (\mathbb{Z}/2\mathbb{Z})^{2k}$, we obtain:

$$\begin{aligned}
T(0) &= (T\big|g)(0)\\
[\theta(Z, X, Y)] &= T(g\{0\})\big|g\\
&= T(0)\big|g\\
&= [\theta(Z, X, Y)]\big|g\\
[\theta(Z, X, Y)] &= [s(g, Z)^{-1}\theta(g \cdot Z, AX + B, CX + D)].
\end{aligned}$$

It follows that there exists $\xi \in \mu_8$ such that

$$\theta(Z, X, Y) = \xi s(g, Z)^{-1}\theta(g \cdot Z, AX + B, CX + D)$$

for all $Z \in \mathbb{H}_k$ and $X, Y \in \mathrm{M}(k, 1, \mathbb{C})$. □

3.3 Application to general theta series

Lemma 3.3.1. *Let m and n be positive integers. If $A \in \mathrm{M}(m, \mathbb{C})$ and $B \in \mathrm{M}(n, \mathbb{C})$, then we define an element $A \otimes B \in \mathrm{M}(mn, \mathbb{C})$ by*

$$A \otimes B = \begin{bmatrix} b_{11}A & \cdots & b_{1n}A \\ \vdots & & \vdots \\ b_{n1}A & \cdots & b_{nn}A \end{bmatrix}.$$

Let $A, A' \in \mathrm{M}(m, \mathbb{C})$ and $B, B' \in \mathrm{M}(m, \mathbb{C})$. Then

$$(A \otimes B)(A' \otimes B') = AA' \otimes BB', \tag{3.7}$$

$$\det(A \otimes B) = (\det A)^n (\det B)^m, \tag{3.8}$$

$$ {}^{\mathrm{t}}(A \otimes B) = {}^{\mathrm{t}}A \otimes {}^{\mathrm{t}}B. \tag{3.9}$$

If A and B are invertible, then $A \otimes B$ is invertible, and

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}. \tag{3.10}$$

If $A \in \mathrm{Sym}(m, \mathbb{R})^+$ and $B \in \mathrm{Sym}(n, \mathbb{R})^+$, then $A \otimes B \in \mathrm{Sym}(mn, \mathbb{R})^+$.

Proof. We write $B = (b_{ij})_{1 \le i,j \le n}$ and $B = (b'_{ij})_{1 \le i,j \le n}$. Then

$$\begin{aligned}
(A \otimes B)(A' \otimes B') &= \begin{bmatrix} b_{11}A & \cdots & b_{1n}A \\ \vdots & & \vdots \\ b_{n1}A & \cdots & b_{nn}A \end{bmatrix} \begin{bmatrix} b'_{11}A' & \cdots & b'_{1n}A' \\ \vdots & & \vdots \\ b'_{n1}A' & \cdots & b'_{nn}A' \end{bmatrix} \\
&= \begin{bmatrix} (\sum_{j=1}^n b_{1j}b'_{j1})AA' & \cdots & (\sum_{j=1}^n b_{1j}b'_{jn})AA' \\ \vdots & & \vdots \\ (\sum_{j=1}^n b_{nj}b'_{j1})AA' & \cdots & (\sum_{j=1}^n b_{nj}b'_{jn})AA' \end{bmatrix} \\
&= AA' \otimes BB'.
\end{aligned}$$

Next,

$$\begin{aligned}
&\det(A \otimes B) \\
&= \det((A \otimes 1_n)(1_m \otimes B)) \\
&= \det(A \otimes 1_n)\det(1_m \otimes B) \\
&= \det(\begin{bmatrix} A & & \\ & \ddots & \\ & & A \end{bmatrix}) \det(\begin{bmatrix} \begin{bmatrix} b_{11} & & \\ & \ddots & \\ & & b_{11} \end{bmatrix} & \cdots & \begin{bmatrix} b_{1n} & & \\ & \ddots & \\ & & b_{1n} \end{bmatrix} \\ \vdots & & \vdots \\ \begin{bmatrix} b_{n1} & & \\ & \ddots & \\ & & b_{n1} \end{bmatrix} & \cdots & \begin{bmatrix} b_{nn} & & \\ & \ddots & \\ & & b_{nn} \end{bmatrix} \end{bmatrix} \\
&= \det(A)^n \det(B)^m.
\end{aligned}$$

We have

$$\begin{aligned}
{}^{\mathrm{t}}(A \otimes B) &= {}^{\mathrm{t}}\begin{bmatrix} b_{11}A & \cdots & b_{1n}A \\ \vdots & & \vdots \\ b_{n1}A & \cdots & b_{nn}A \end{bmatrix} \\
&= \begin{bmatrix} b_{11}\,{}^{\mathrm{t}}A & \cdots & b_{n1}\,{}^{\mathrm{t}}A \\ \vdots & & \vdots \\ b_{1n}\,{}^{\mathrm{t}}A & \cdots & b_{nn}\,{}^{\mathrm{t}}A \end{bmatrix} \\
&= {}^{\mathrm{t}}A \otimes {}^{\mathrm{t}}B.
\end{aligned}$$

Assume that A and B are invertible. Then

$$(A \otimes B)(A^{-1} \otimes B^{-1}) = AA^{-1} \otimes BB^{-1}$$

$$
\begin{aligned}
&= 1_m \otimes 1_n \\
&= 1_{mn}.
\end{aligned}
$$

This implies that $A\otimes B$ is invertible and has inverse $A^{-1}\otimes B^{-1}$. Finally, assume that $A \in \mathrm{Sym}(m,\mathbb{R})^+$ and $B \in \mathrm{Sym}(n,\mathbb{R})^+$. Since ${}^{\mathrm{t}}(A\otimes B) = {}^{\mathrm{t}}A \otimes {}^{\mathrm{t}}B = A\otimes B$, it follows that $A\otimes B$ is symmetric. By (1.5), there exist $T \in \mathrm{GL}(m,\mathbb{R})$ and $S \in \mathrm{GL}(n,\mathbb{R})$ such that $T^{-1} = {}^{\mathrm{t}}T$ and $S^{-1} = {}^{\mathrm{t}}S$, and there exist $\lambda_1 > 0, \dots, \lambda_m > 0$ and $\mu_1 > 0, \dots, \mu_n > 0$ such that

$$
{}^{\mathrm{t}}TAT = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix}, \qquad {}^{\mathrm{t}}SBS = \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{bmatrix}.
$$

We have:

$$
\begin{aligned}
{}^{\mathrm{t}}(T\otimes S)(A\otimes B)(T\otimes S) &= ({}^{\mathrm{t}}T \otimes {}^{\mathrm{t}}S)(A\otimes B)(T\otimes S) \\
&= {}^{\mathrm{t}}TAT \otimes {}^{\mathrm{t}}SBS \\
&= \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix} \otimes \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{bmatrix} \\
&= \begin{bmatrix} \mu_1\lambda_1 & & & & & & \\ & \ddots & & & & & \\ & & \mu_1\lambda_m & & & & \\ & & & \ddots & & & \\ & & & & \mu_n\lambda_1 & & \\ & & & & & \ddots & \\ & & & & & & \mu_n\lambda_m \end{bmatrix}.
\end{aligned}
$$

This equality implies that $A\otimes B$ is positive-definite. □

Lemma 3.3.2. *Let m and n be positive integers. Let $F \in \mathrm{Sym}(m,\mathbb{Z})$ be even and invertible, and let N be the level of F. Let*

$$
\Gamma_0(N) = \{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(2n,\mathbb{Z}) : C \equiv 0 \pmod{N} \}.
$$

Define a function

$$
t : \Gamma_0(N) \longrightarrow \Gamma_{\theta,2mn}
$$

by $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \mapsto \tilde{M}$, where

$$
\tilde{M} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} = \begin{bmatrix} 1_m \otimes A & F\otimes B \\ F^{-1}\otimes C & 1_m \otimes D \end{bmatrix}.
$$

The function t is a well-defined homomorphism.

Proof. We first verify that t is well-defined. Let $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \Gamma_0(N)$. By Lemma 1.9.2, we have

$$ {}^{\mathrm{t}}AC = {}^{\mathrm{t}}CA, \qquad {}^{\mathrm{t}}BD = {}^{\mathrm{t}}BD, \qquad {}^{\mathrm{t}}AD - {}^{\mathrm{t}}CB = 1_n, $$

and to see that $\tilde{M} \in \mathrm{Sp}(2mn, \mathbb{Z})$ it suffices to check that $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ are integral, and

$$ {}^{\mathrm{t}}\tilde{A}\tilde{C} = {}^{\mathrm{t}}\tilde{C}\tilde{A}, \qquad {}^{\mathrm{t}}\tilde{B}\tilde{D} = {}^{\mathrm{t}}\tilde{D}\tilde{B}, \qquad {}^{\mathrm{t}}\tilde{A}\tilde{D} - {}^{\mathrm{t}}\tilde{C}\tilde{B} = 1_{mn}. $$

It is clear that $\tilde{A}$, $\tilde{B}$ and $\tilde{D}$ are integral. Concerning $\tilde{C}$, we have:

$$ \tilde{C} = F^{-1} \otimes C = NF^{-1} \otimes N^{-1}C. $$

Since NF^{-1} and $N^{-1}C$ are integral, by the definition of the level of N and as $C \equiv 0 \pmod N$, it follows that $\tilde{C}$ is integral. Now

$$ \begin{aligned} {}^{\mathrm{t}}\tilde{A}\tilde{C} &= {}^{\mathrm{t}}(1_m \otimes A)(F^{-1} \otimes C) \\ &= (1_m \otimes {}^{\mathrm{t}}A)(F^{-1} \otimes C) \\ &= F^{-1} \otimes {}^{\mathrm{t}}AC \\ &= F^{-1} \otimes {}^{\mathrm{t}}CA \\ &= (F^{-1} \otimes {}^{\mathrm{t}}C)(1_m \otimes A) \\ &= ({}^{\mathrm{t}}F^{-1} \otimes {}^{\mathrm{t}}C)(1_m \otimes A) \\ &= {}^{\mathrm{t}}(F^{-1} \otimes C)(1_m \otimes A) \\ &= {}^{\mathrm{t}}\tilde{C}\tilde{A}. \end{aligned} $$

A similar calculation shows that ${}^{\mathrm{t}}\tilde{B}\tilde{D} = {}^{\mathrm{t}}\tilde{D}\tilde{B}$. Next,

$$ \begin{aligned} {}^{\mathrm{t}}\tilde{A}\tilde{D} - {}^{\mathrm{t}}\tilde{C}\tilde{B} &= (1_m \otimes {}^{\mathrm{t}}A)(1_m \otimes D) - ({}^{\mathrm{t}}F^{-1} \otimes {}^{\mathrm{t}}C)(F \otimes B) \\ &= 1_m \otimes {}^{\mathrm{t}}AD - 1_m \otimes {}^{\mathrm{t}}CB \\ &= 1_m \otimes ({}^{\mathrm{t}}AD - {}^{\mathrm{t}}CB) \\ &= 1_m \otimes 1_n \\ &= 1_{mn}. \end{aligned} $$

It follows that $\tilde{M} \in \mathrm{Sp}(2mn, \mathbb{Z})$. To now prove that $\tilde{M} \in \Gamma_{\theta,mn}$ it suffices to prove that

$$ \mathrm{diag}(\tilde{A}\,{}^{\mathrm{t}}\tilde{B}) \equiv 0 \pmod 2 \quad \text{and} \quad \mathrm{diag}(\tilde{C}\,{}^{\mathrm{t}}\tilde{D}) \equiv 0 \pmod 2. $$

We have

$$ \begin{aligned} \mathrm{diag}(\tilde{A}\,{}^{\mathrm{t}}\tilde{B}) &\equiv \mathrm{diag}((1_m \otimes A)\,{}^{\mathrm{t}}(F \otimes B) \pmod 2 \\ &\equiv \mathrm{diag}(F \otimes A\,{}^{\mathrm{t}}B) \pmod 2 \\ &\equiv 0 \pmod 2, \end{aligned} $$

by the definition of $\otimes$, and because $\mathrm{diag}(F) \equiv 0 \pmod 2$. And

$$\begin{aligned}\mathrm{diag}(\tilde{C}\,{}^{\mathrm{t}}\tilde{D}) &\equiv \mathrm{diag}((F^{-1} \otimes C)\,{}^{\mathrm{t}}(1_m \otimes D)) \pmod 2\\ &\equiv \mathrm{diag}(F^{-1} \otimes C\,{}^{\mathrm{t}}D) \pmod 2\\ &\equiv \mathrm{diag}(NF^{-1} \otimes N^{-1}C\,{}^{\mathrm{t}}D) \pmod 2\\ &\equiv 0 \pmod 2\end{aligned}$$

by the definition of $\otimes$, $\mathrm{diag}(NF^{-1}) \equiv 0 \pmod 2$, and $N^{-1}C\,{}^{\mathrm{t}}D \in \mathrm{M}(n, \mathbb{Z})$. Finally, we verify that t is a homomorphism. Let $\left[\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} A_2 & B_2 \\ C_2 & D_2 \end{smallmatrix}\right] \in \Gamma_0(N)$. Then

$$\begin{aligned}t(\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}) &= t(\begin{bmatrix} A_1A_2 + B_1C_2 & A_1B_2 + B_1D_2 \\ C_1A_2 + D_1C_2 & C_1B_2 + D_1D_2 \end{bmatrix})\\ &= t(\begin{bmatrix} 1_m \otimes (A_1A_2 + B_1C_2) & F \otimes (A_1B_2 + B_1D_2) \\ F^{-1} \otimes (C_1A_2 + D_1C_2) & 1_m \otimes (C_1B_2 + D_1D_2) \end{bmatrix})\\ &= t(\left[\begin{matrix} (1_m \otimes A_1)(1_m \otimes A_2) + (F \otimes B_1)(F^{-1} \otimes C_2) \\ (F^{-1} \otimes C_1)(1_m \otimes A_2) + (1 \otimes D_1)(F^{-1} \otimes C_2) \end{matrix}\right.\\ &\qquad \left.\begin{matrix} (1_m \otimes A_1)(F \otimes B_2) + (F \otimes B_1)(1_m \otimes D_2) \\ (F^{-1} \otimes C_1)(F \otimes B_2) + (1 \otimes D_1)(1 \otimes D_2) \end{matrix}\right])\\ &= \begin{bmatrix} 1_m \otimes A_1 & F \otimes B_1 \\ F^{-1} \otimes C_1 & 1_m \otimes D_1 \end{bmatrix}\begin{bmatrix} 1_m \otimes A_2 & F \otimes B_2 \\ F^{-1} \otimes C_2 & 1_m \otimes D_2 \end{bmatrix}\\ &= t(\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix})t(\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix})\end{aligned}$$

This completes the proof. □

Lemma 3.3.3. *Let m and n be positive integers, and let $F \in \mathrm{Sym}(m, \mathbb{R})^+$. For $Z \in \mathbb{H}_n$ and $Y \in \mathrm{M}(m, n, \mathbb{C})$ define*

$$\tilde{Z} = F \otimes Z, \qquad \tilde{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}$$

where $Y = [Y_1 \cdots Y_n]$ with $Y_1, \dots, Y_n \in \mathrm{M}(m, 1, \mathbb{C})$. We have

$$\begin{aligned}\tilde{Z} &\in \mathbb{H}_{mn},\\ \tilde{X} &\in \mathrm{M}(mn, 1, \mathbb{C}),\\ \tilde{Z}[\tilde{Y}] &= \mathrm{tr}(ZF[Y]),\\ {}^{\mathrm{t}}\tilde{X}\tilde{Y} &= \mathrm{tr}({}^{\mathrm{t}}XY),\\ \tilde{M} \cdot \tilde{Z} &= \widetilde{M \cdot Z},\\ \tilde{A}\tilde{X} + \tilde{B}\tilde{Y} &= X\,{}^{\mathrm{t}}\widetilde{A + FY}\,{}^{\mathrm{t}}B,\\ \tilde{C}\tilde{X} + \tilde{D}\tilde{Y} &= F^{-1}\widetilde{X\,{}^{\mathrm{t}}C} + Y\,{}^{\mathrm{t}}D,\end{aligned}$$

for $Z \in \mathbb{H}_n$, $X, Y \in \mathrm{M}(m,n,\mathbb{C})$, *and* $M \in \mathrm{Sp}(2n,\mathbb{Z})$. *Moreover, for every* $M \in \mathrm{Sp}(2n,\mathbb{Z})$ *there exists* $\varepsilon \in \{\pm 1\}$ *such that*

$$s(\tilde{M}, \tilde{Z}) = \varepsilon s(M,Z)^m$$

for $Z \in \mathbb{H}_n$.

Proof. Let $Z \in \mathbb{H}_n$ and $X, Y \in \mathrm{M}(m,n,\mathbb{C})$. We have ${}^{\mathrm{t}}\tilde{Z} = \tilde{Z}$ by Lemma 3.3.1. Write $Z = U + iV$ with $U, V \in \mathrm{Sym}(n,\mathbb{R})$ and $V > 0$. Then $\tilde{Z} = F \otimes (U + iV) = (F \otimes U) + i(F \otimes V)$. By Lemma 3.3.1 we have $F \otimes V > 0$. It follows that $Z \in \mathbb{H}_{mn}$. Next,

$$\begin{aligned}
\tilde{Z}[\tilde{Y}] &= {}^{\mathrm{t}}\begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} \begin{bmatrix} z_{11}F & \cdots z_{1n}F \\ \vdots & \vdots \\ z_{n1}F & \cdots z_{nn}F \end{bmatrix} \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} \\
&= \begin{bmatrix} {}^{\mathrm{t}}Y_1 & \cdots & {}^{\mathrm{t}}Y_n \end{bmatrix} \begin{bmatrix} z_{11}FY_1 + \cdots + z_{1n}FY_n \\ \vdots \\ z_{n1}FY_1 + \cdots + z_{nn}FY_n \end{bmatrix} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} z_{ij}\, {}^{\mathrm{t}}Y_i F Y_j.
\end{aligned}$$

And:

$$\begin{aligned}
\mathrm{tr}(ZF[Y]) &= \mathrm{tr}(Z\, {}^{\mathrm{t}}YFY) \\
&= \mathrm{tr}(Z\, {}^{\mathrm{t}}\begin{bmatrix} Y_1 & \cdots & Y_n \end{bmatrix} F \begin{bmatrix} Y_1 & \cdots & Y_n \end{bmatrix}) \\
&= \mathrm{tr}(Z \begin{bmatrix} {}^{\mathrm{t}}Y_1 \\ \vdots \\ {}^{\mathrm{t}}Y_n \end{bmatrix} F \begin{bmatrix} Y_1 & \cdots & Y_n \end{bmatrix}) \\
&= \mathrm{tr}(Z \begin{bmatrix} {}^{\mathrm{t}}Y_1F \\ \vdots \\ {}^{\mathrm{t}}Y_nF \end{bmatrix} \begin{bmatrix} Y_1 & \cdots & Y_n \end{bmatrix}) \\
&= \mathrm{tr}(\begin{bmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & & \vdots \\ z_{n1} & \cdots & z_{nn} \end{bmatrix} \begin{bmatrix} {}^{\mathrm{t}}Y_1FY_1 & \cdots & {}^{\mathrm{t}}Y_1FY_n \\ \vdots & & \vdots \\ {}^{\mathrm{t}}Y_nFY_1 & \cdots & {}^{\mathrm{t}}Y_nFY_n \end{bmatrix}) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} z_{ij}\, {}^{\mathrm{t}}Y_i F Y_j.
\end{aligned}$$

It follows that $\tilde{Z}[\tilde{Y}] = \mathrm{tr}(ZF[Y])$. Next, we have:

$${}^{\mathrm{t}}\tilde{X}\tilde{Y} = {}^{\mathrm{t}}\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}$$

$$= \begin{bmatrix} {}^{\mathrm{t}}X_1 & \cdots & {}^{\mathrm{t}}X_n \end{bmatrix} \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}$$
$$= \sum_{i=1}^{n} {}^{\mathrm{t}}X_i Y_i.$$

And:

$$\begin{aligned}
\mathrm{tr}({}^{\mathrm{t}}XY) &= \mathrm{tr}({}^{\mathrm{t}}\begin{bmatrix} X_1 & \cdots & X_n \end{bmatrix} \begin{bmatrix} Y_1 & \cdots & Y_n \end{bmatrix}) \\
&= \mathrm{tr}(\begin{bmatrix} {}^{\mathrm{t}}X_1 \\ \vdots \\ {}^{\mathrm{t}}X_n \end{bmatrix} \begin{bmatrix} Y_1 & \cdots & Y_n \end{bmatrix}) \\
&= \mathrm{tr}(\begin{bmatrix} {}^{\mathrm{t}}X_1Y_1 & \cdots & {}^{\mathrm{t}}X_1Y_n \\ \vdots & & \vdots \\ {}^{\mathrm{t}}X_nY_1 & \cdots & {}^{\mathrm{t}}X_nY_n \end{bmatrix}) \\
&= \sum_{i=1}^{n} {}^{\mathrm{t}}X_i Y_i.
\end{aligned}$$

It follows that ${}^{\mathrm{t}}\tilde{X}\tilde{Y} = \mathrm{tr}({}^{\mathrm{t}}XY)$. Let $M = [\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}] \in \mathrm{Sp}(2n, \mathbb{Z})$. Then

$$\begin{aligned}
\tilde{M} \cdot \tilde{Z} &= \begin{bmatrix} 1_m \otimes A & F \otimes B \\ F^{-1} \otimes C & 1_m \otimes D \end{bmatrix} \cdot (F \otimes Z) \\
&= ((1_m \otimes A)(F \otimes Z) + F \otimes B)((F^{-1} \otimes C)(F \otimes Z) + 1_m \otimes D)^{-1} \\
&= (F \otimes AZ + F \otimes B)(1_m \otimes CZ + 1_m \otimes D)^{-1} \\
&= (F \otimes (AZ + B))(1_m \otimes (CZ + D))^{-1} \\
&= (F \otimes (AZ + B))(1_m \otimes (CZ + D)^{-1}) \\
&= F \otimes (AZ + B)(CZ + D)^{-1} \\
&= F \otimes M \cdot Z \\
&= \widetilde{M \cdot Z}.
\end{aligned}$$

Now

$$\begin{aligned}
\tilde{A}\tilde{X} + \tilde{B}\tilde{Y} &= (1_m \otimes A) \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} + (F \otimes B) \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} \\
&= \begin{bmatrix} a_{11}1_m & \cdots & a_{1n}1_m \\ \vdots & & \vdots \\ a_{n1}1_m & \cdots & a_{nn}1_m \end{bmatrix} \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} + \begin{bmatrix} b_{11}F & \cdots & b_{1n}F \\ \vdots & & \vdots \\ b_{n1}F & \cdots & b_{nn}F \end{bmatrix} \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} \sum_{i=1}^n a_{1i}X_i \\ \vdots \\ \sum_{i=1}^n a_{ni}X_i \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^n b_{1i}FY_i \\ \vdots \\ \sum_{i=1}^n b_{ni}FY_i \end{bmatrix}.$$

And:

$$\begin{aligned} &X\,{}^{\mathrm{t}}\widetilde{A+F}Y\,{}^{\mathrm{t}}B = \begin{bmatrix} X_1 & \cdots & X_n \end{bmatrix}{}^{\mathrm{t}}\widetilde{A+F}\begin{bmatrix} Y_1 & \cdots & Y_n \end{bmatrix}{}^{\mathrm{t}}B \\ &= \widetilde{\begin{bmatrix} \sum_{i=1}^n a_{1i}X_i & \cdots & \sum_{i=1}^n a_{ni}X_i \end{bmatrix} + F}\begin{bmatrix} \sum_{i=1}^n b_{1i}Y_i & \cdots & \sum_{i=1}^n b_{ni}Y_i \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=1}^n a_{1i}X_i \\ \vdots \\ \sum_{i=1}^n a_{ni}X_i \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^n b_{1i}FY_i \\ \vdots \\ \sum_{i=1}^n b_{ni}FY_i \end{bmatrix}. \end{aligned}$$

Hence, $\tilde{A}\tilde{X}+\tilde{B}\tilde{Y} = X\,{}^{\mathrm{t}}\widetilde{A+F}Y\,{}^{\mathrm{t}}B$. The proof of $\tilde{C}\tilde{X}+\tilde{D}\tilde{Y} = F^{-1}X\,\widetilde{{}^{\mathrm{t}}C+}\,Y\,{}^{\mathrm{t}}D$ is similar. Finally, let $M \in \mathrm{Sp}(2n,\mathbb{Z})$. For $Z \in \mathbb{H}_n$ we have

$$\begin{aligned} s(\tilde{M},\tilde{Z})^2 &= \det(\tilde{C}\tilde{Z}+\tilde{D}) \\ &= \det((F^{-1}\otimes C)(F\otimes Z) + (1_m \otimes D)) \\ &= \det(1_m \otimes CZ + 1_m \otimes D) \\ &= \det(1_m \otimes (CZ+D)) \\ &= \det(CZ+D)^m \\ &= s(M,Z)^{2m}. \end{aligned}$$

It follows that for each $Z \in \mathbb{H}_n$ there exists $\varepsilon(Z) \in \{\pm 1\}$ such that $s(\tilde{M},\tilde{Z}) = \varepsilon(Z)s(M,Z)^m$. The function on $\mathbb{H}_n$ that sends Z to $\varepsilon(Z)$ is continuous and takes values in $\{\pm 1\}$. Since $\mathbb{H}_n$ is connected (see Proposition 1.10.3), the intermediate value theorem (see Theorem 6 on page 90 of [18]) implies now that this function is constant, which completes the proof of the lemma. □

Lemma 3.3.4. *Let m and n be positive integers, and let $F \in \mathrm{Sym}(m,\mathbb{R})^+$. For $Z \in \mathbb{H}_n$, $X, Y \in \mathrm{M}(m\times n,\mathbb{C})$, define*

$$\theta(F,Z,X,Y) = \sum_{R \in \mathrm{M}(m\times n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(ZF[R-Y]) + 2\pi i \mathrm{tr}({}^{\mathrm{t}}RX) - \pi i \mathrm{tr}({}^{\mathrm{t}}XY)\big).$$

By Lemma 3.1.8, this series converges absolutely and uniformly on compact subsets of $\mathbb{H}_n \times \mathrm{M}(m,n,\mathbb{C}) \times \mathrm{M}(m,n,\mathbb{C})$ and defines an analytic function on this set. With the notation of Lemma 3.3.3, we have

$$\theta(F,Z,X,Y) = \theta(\tilde{Z},\tilde{X},\tilde{Y}). \tag{3.11}$$

Proof. By definition,

$$\theta(\tilde{Z},\tilde{X},\tilde{Y}) = \sum_{R' \in \mathrm{M}(k,1,\mathbb{Z})} \exp\big(\pi i \tilde{Z}[R'-\tilde{Y}] + 2\pi i\,{}^{\mathrm{t}}R'\tilde{X} - \pi i\,{}^{\mathrm{t}}\tilde{X}\tilde{Y}\big).$$

The map $\mathrm{M}(m,n,\mathbb{Z}) \to \mathrm{M}(k,1,\mathbb{Z})$ defined by $R \mapsto \tilde{R}$ is an isomorphism of groups. Using this, and Lemma 3.3.3,

$$\begin{aligned}\theta(\tilde{Z},\tilde{X},\tilde{Y}) &= \sum_{R'\in\mathrm{M}(m,n,\mathbb{Z})} \exp\left(\pi i\tilde{Z}[\tilde{R}-\tilde{Y}] + 2\pi i\,{}^{\mathrm{t}}\tilde{R}\tilde{X} - \pi i\,{}^{\mathrm{t}}\tilde{X}\tilde{Y}\right)\\ &= \sum_{R\in\mathrm{M}(m,n,\mathbb{Z})} \exp\left(\pi i\mathrm{tr}(ZF[R-Y]) + 2\pi i\mathrm{tr}({}^{\mathrm{t}}RX) - \pi i\mathrm{tr}({}^{\mathrm{t}}XY)\right)\\ \theta(\tilde{Z},\tilde{X},\tilde{Y}) &= \theta(F,Z,X,Y).\end{aligned}$$

This completes the proof. ☐

Theorem 3.3.5. *Let m and n be positive integers, and let $F \in \mathrm{Sym}(m,\mathbb{Z})^+$ be even. Let N be the level of F. For $Z \in \mathbb{H}_n$, $X,Y \in \mathrm{M}(m\times n,\mathbb{C})$, define*

$$\theta(F,Z,X,Y) = \sum_{R\in\mathrm{M}(m\times n,\mathbb{Z})} \exp\left(\pi i\mathrm{tr}(ZF[R-Y]) + 2\pi i\mathrm{tr}({}^{\mathrm{t}}RX) - \pi i\mathrm{tr}({}^{\mathrm{t}}XY)\right).$$

By Lemma 3.1.8, this series converges absolutely and uniformly on compact subsets of $\mathbb{H}_n \times \mathrm{M}(m,n,\mathbb{C}) \times \mathrm{M}(m,n,\mathbb{C})$ and defines an analytic function on this set. Let μ_8 be the group of eighth roots of unity. There exists a function $\chi : \Gamma_0(N) \to \mu_8$ such that

$$\begin{aligned}&\chi(M)\theta(F,Z,X,Y)\\ &\qquad = s(M,Z)^{-m}\theta(F, M\cdot Z, X\,{}^{\mathrm{t}}A + FY\,{}^{\mathrm{t}}B, F^{-1}X\,{}^{\mathrm{t}}C + Y\,{}^{\mathrm{t}}D)\end{aligned}$$

for $M = \left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right] \in \Gamma_0(N)$, $Z \in \mathbb{H}_n$, and $X,Y \in \mathrm{M}(m,n,\mathbb{C})$.

Proof. Let $k = mn$. By Corollary 3.2.6 there exists a function $\mu : \Gamma_\theta \to \mu_8$ such that

$$\begin{aligned}&\mu(M')\theta(Z',X',Y')\\ &\qquad = s(M',Z')^{-1}\theta(M'\cdot Z', A'X'+B'Y', C'X'+D'Y') \quad (3.12)\end{aligned}$$

for $Z' \in \mathbb{H}_k$, $X',Y' \in \mathrm{M}(k,1,\mathbb{C})$, and $M' = \left[\begin{smallmatrix} A' & B' \\ C' & D'\end{smallmatrix}\right] \in \Gamma_{\theta,k}$. Here,

$$\theta(Z',X',Y') = \sum_{R'\in\mathrm{M}(k,1,\mathbb{Z})} \exp\left(\pi i Z'[R'-Y'] + 2\pi i\,{}^{\mathrm{t}}R'X - \pi i\,{}^{\mathrm{t}}X'Y'\right)$$

for $Z' \in \mathbb{H}_k$, $X',Y' \in \mathrm{M}(k,1,\mathbb{C})$. Let $M = \left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right] \in \Gamma_0(N)$, $Z \in \mathbb{H}_n$, and $X,Y \in \mathrm{M}(m,n,\mathbb{C})$. To prove the theorem we will substitute $\tilde{M}$ for M', $\tilde{Z}$ for Z', $\tilde{X}$ for X' and $\tilde{Y}$ for Y' in both sides of (3.12); note that $\tilde{M} \in \Gamma_{\theta,2k}$ by Lemma 3.3.2. Substituting in the left hand side, we have, by (3.11),

$$\theta(\tilde{Z},\tilde{X},\tilde{Y}) = \theta(F,Z,X,Y).$$

Substituting $\tilde{M}$ for M', $\tilde{Z}$ for Z', $\tilde{X}$ for X' and $\tilde{Y}$ for Y' in the right hand side of (3.12), using Lemma 3.3.3 again, and also (3.11), we get:

$$s(M',Z')^{-1}\theta(M'\cdot Z', A'X'+B'Y', C'X'+D'Y')$$

$$= s(\tilde{M},\tilde{Z})^{-1}\theta(\tilde{M}\cdot\tilde{Z},\tilde{A}\tilde{X}+\tilde{B}\tilde{Y},\tilde{C}\tilde{X}+\tilde{D}\tilde{Y})$$
$$= \varepsilon s(M,Z)^{-m}\theta(\widetilde{M\cdot Z},\widetilde{X\,{}^{\mathrm{t}}A+FY\,{}^{\mathrm{t}}B},\widetilde{F^{-1}X\,{}^{\mathrm{t}}C+Y\,{}^{\mathrm{t}}D})$$
$$= \varepsilon s(M,Z)^{-m}\theta(F,M\cdot Z,X\,{}^{\mathrm{t}}A+FY\,{}^{\mathrm{t}}B,F^{-1}X\,{}^{\mathrm{t}}C+Y\,{}^{\mathrm{t}}D).$$

Here, ε depends only on M. The theorem is proven. □

3.4 The multiplier

In this section we compute the multiplier $\chi(M)$ from Theorem 3.3.5 in the case that m is even.

Lemma 3.4.1. *Let m and n be positive integers, and assume that m is even. Let $F\in\mathrm{Sym}(m,\mathbb{Z})^+$ be even, and let N be the level of F. Let $\chi:\Gamma_0(N)\to\mu_8$ be as in Theorem 3.3.5. Then χ is a character.*

Proof. Let $M_1,M_2\in\Gamma_0(N)$. By Theorem 3.3.5, if $Z\in\mathbb{H}_n$, then:

$$\begin{aligned}
\chi(M_1M_2)\theta(F,Z) &= s(M_1M_2,Z)^{-m}\theta(F,(M_1M_2)\cdot Z)\\
&= j(M_1M_2,Z)^{-m/2}\theta(F,M_1\cdot(M_2\cdot Z))\\
&= j(M_1,M_2\cdot Z)^{-m/2}j(M_2,Z)^{-m/2}\\
&\qquad\times\chi(M_1)s(M_1,M_2\cdot Z)^{m}\theta(F,M_2\cdot Z)\\
&= j(M_1,M_2\cdot Z)^{-m/2}j(M_2,Z)^{-m/2}\\
&\qquad\times\chi(M_1)j(M_1,M_2\cdot Z)^{m/2}\theta(F,M_2\cdot Z)\\
&= j(M_2,Z)^{-m/2}\chi(M_1)\theta(F,M_2\cdot Z)\\
&= j(M_2,Z)^{-m/2}\chi(M_1)\chi(M_2)s(M_2,Z)^{m}\theta(F,Z)\\
&= j(M_2,Z)^{-m/2}\chi(M_1)\chi(M_2)j(M_2,Z)^{m/2}\theta(F,Z)\\
&= \chi(M_1)\chi(M_2)\theta(F,Z).
\end{aligned}$$

Since $\theta(F,\cdot)$ is not zero, we obtain $\chi(M_1M_2)=\chi(M_1)\chi(M_2)$. □

Lemma 3.4.2. *Let m and n be positive integers. Assume that m is even. Let $F\in\mathrm{Sym}(m,\mathbb{R})^+$. Then*

$$\theta(F,Z,X,Y)=\det(F)^{-n/2}\det(-iZ)^{-m/2}\theta(F^{-1},-Z^{-1},Y,-X)$$

for $T\in\mathrm{Sym}(n,\mathbb{R})^+$ and $X,Y\in\mathrm{M}(m,n,\mathbb{C})$.

Proof. Let $k=mn$. From the proof of Lemma 3.2.3 we have

$$\theta(iT',X',Y')=\det(T')^{-1/2}\theta(-(iT')^{-1},Y',-X') \tag{3.13}$$

for $T'\in\mathrm{Sym}(k,\mathbb{R})^+$ and $X',Y'\in\mathrm{M}(k,1,\mathbb{C})$. Let $T\in\mathrm{Sym}(n,\mathbb{R})^+$ and $X,Y\in\mathrm{M}(m,n,\mathbb{C})$. To prove the lemma we will substitute $T'=F\otimes T$, $X'=\tilde{X}$ and $Y'=\tilde{Y}$ in (3.13). Now

$$\theta(i(F\otimes T),\tilde{X},\tilde{Y})=\theta(F\otimes iT,\tilde{X},\tilde{Y})$$

$$= \theta(\widetilde{iT}, \tilde{X}, \tilde{Y})$$
$$= \theta(F, iT, X, Y). \qquad \text{(use Lemma 3.3.4)}$$

And

$$\theta((-(i(F \otimes T))^{-1}, \tilde{Y}, -\tilde{X})$$
$$= \theta(F^{-1} \otimes \big(-(iT)^{-1}\big), \tilde{Y}, -\tilde{X})$$
$$= \theta(F^{-1}, -(iT)^{-1}, Y, -X). \qquad \text{(use Lemma 3.3.4 with } F^{-1})$$

Finally,

$$\det(F \otimes T) = \det(F)^n \det(T)^m.$$

The equality (3.13) now implies that

$$\theta(F, iT, X, Y) = \det(F)^{-n/2} \det(T)^{-m/2} \theta(F^{-1}, -(iT)^{-1}, Y, -X),$$

or equivalently,

$$\theta(F, iT, X, Y) = \det(F)^{-n/2} \det((-i)iT)^{-m/2} \theta(F^{-1}, -(iT)^{-1}, Y, -X).$$

The assertion of the lemma follows now from Lemma 1.10.5. □

Lemma 3.4.3. *Let m and n be positive integers. Let $M, N \in \mathrm{M}(m,n,\mathbb{C})$, $E \in \mathrm{Sym}(n,\mathbb{C})$, and $F \in \mathrm{Sym}(m,\mathbb{C})$. Then*

$$\mathrm{tr}(E\, {}^{\mathrm{t}}MFN) = \mathrm{tr}(E\, {}^{\mathrm{t}}NFM).$$

Proof. Let $E = (e_{ij})$, $M = [M_1 \cdots M_n]$, and $N = [N_1, \cdots M_n]$. We have

$$\begin{aligned}
\mathrm{tr}(E\, {}^{\mathrm{t}}MFN) &= \mathrm{tr}(\begin{bmatrix} e_{11} & \cdots & e_{1n} \\ \vdots & & \vdots \\ e_{n1} & \cdots & e_{nn} \end{bmatrix} \begin{bmatrix} {}^{\mathrm{t}}M_1FN_1 & \cdots & {}^{\mathrm{t}}M_1FN_n \\ \vdots & & \vdots \\ {}^{\mathrm{t}}M_nFN_1 & \cdots & {}^{\mathrm{t}}M_nFN_n \end{bmatrix}) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} e_{ij}\, {}^{\mathrm{t}}M_jFN_i \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} e_{ji}\, {}^{\mathrm{t}}N_iFM_j \\
&= \mathrm{tr}(\begin{bmatrix} e_{11} & \cdots & e_{1n} \\ \vdots & & \vdots \\ e_{n1} & \cdots & e_{nn} \end{bmatrix} \begin{bmatrix} {}^{\mathrm{t}}N_1FM_1 & \cdots & {}^{\mathrm{t}}N_1FM_n \\ \vdots & & \vdots \\ {}^{\mathrm{t}}N_nFM_1 & \cdots & {}^{\mathrm{t}}N_nFM_n \end{bmatrix}) \\
&= \mathrm{tr}(E\, {}^{\mathrm{t}}NFM).
\end{aligned}$$

This completes the proof. □

Lemma 3.4.4. *Let m and n be positive integers, and let $F \in \mathrm{Sym}(m,\mathbb{R})^+$. Let $R \in \mathrm{M}(m,n,\mathbb{R})$. Then $\mathrm{tr}(F[R]) \geq 0$, and $\mathrm{tr}(F[R]) = 0$ if and only if $R = 0$.*

Proof. Write $R = [R_1 \cdots R_n]$. Then

$$\begin{aligned}\mathrm{tr}(F[R]) &= \mathrm{tr}(\begin{bmatrix} {}^{\mathrm{t}}R_1 \\ \vdots \\ {}^{\mathrm{t}}R_n \end{bmatrix} F \begin{bmatrix} R_1 & \cdots & R_n \end{bmatrix}) \\ &= \mathrm{tr}(\begin{bmatrix} {}^{\mathrm{t}}R_1 \\ \vdots \\ {}^{\mathrm{t}}R_n \end{bmatrix} \begin{bmatrix} FR_1 & \cdots & FR_n \end{bmatrix}) \\ &= \mathrm{tr}(\begin{bmatrix} {}^{\mathrm{t}}R_1FR_1 & \cdots & {}^{\mathrm{t}}R_1FR_n \\ \vdots & & \vdots \\ {}^{\mathrm{t}}R_nFR_1 & \cdots & {}^{\mathrm{t}}R_nFR_n \end{bmatrix}) \\ &= \sum_{i=1}^{n} F[R_i].\end{aligned}$$

Since F is positive-definite, we have $F[R_i] \geq 0$ for $1 \leq i \leq n$. It follows that $\mathrm{tr}(F[R]) \geq 0$. Assume that $\mathrm{tr}(F[R]) = 0$. Then $F[R_i] = 0$ for $1 \leq i \leq n$. Since F is positive-definite, $R_1 = \cdots = R_n = 0$. □

Lemma 3.4.5. *Let m and n be positive integers. Let $F \in \mathrm{Sym}(m, \mathbb{Z})$ be even. If $W \in \mathrm{M}(n, \mathbb{Z})$ and $N \in \mathrm{M}(m, n, \mathbb{Z})$, then $\mathrm{tr}(WF[N]) = \mathrm{tr}(F[N]W)$ is an even integer.*

Proof. Write $W = (w_{ij})$ and $N = [N_1 \cdots N_n]$. Then

$$\begin{aligned}\mathrm{tr}(WF[N]) &= \mathrm{tr}(\begin{bmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & & \vdots \\ w_{n1} & \cdots & w_{nn} \end{bmatrix} \begin{bmatrix} {}^{\mathrm{t}}N_1FN_1 & \cdots & {}^{\mathrm{t}}N_1FN_n \\ \vdots & & \vdots \\ {}^{\mathrm{t}}N_nFN_1 & \cdots & {}^{\mathrm{t}}N_nFN_n \end{bmatrix}) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}\, {}^{\mathrm{t}}N_jFN_i \\ &= \sum_{\substack{i,j \in \{1,\ldots,n\}, \\ i \neq j}} w_{ij}\, {}^{\mathrm{t}}N_jFN_i + \sum_{i=1}^{n} w_{ii}\, {}^{\mathrm{t}}N_iFN_i \\ &= \sum_{\substack{i,j \in \{1,\ldots,n\}, \\ i < j}} 2w_{ij}\, {}^{\mathrm{t}}N_jFN_i + \sum_{i=1}^{n} w_{ii}\, {}^{\mathrm{t}}N_iFN_i \\ &\equiv 0 \pmod 2\end{aligned}$$

because F is an even integral symmetric matrix (see Lemma 1.5.1). □

Lemma 3.4.6. *For every positive integer ℓ, let*

$$f_\ell : \mathrm{M}(m, n, \mathbb{Z}) \to \mathbb{C}$$

be a function, and assume that the limit $\lim_{\ell\to\infty} f_\ell(N)$ *exists for every* $N \in \mathrm{M}(m,n,\mathbb{C})$. *Define* $f : \mathrm{M}(m,n,\mathbb{Z}) \to \mathbb{C}$ *by*

$$f(N) = \lim_{\ell\to\infty} f_\ell(N)$$

for $N \in \mathrm{M}(m,n,\mathbb{Z})$. *Suppose that* $g : \mathrm{M}(m,n,\mathbb{Z}) \to \mathbb{R}_{\geq 0}$ *is a function such that*

$$|f_\ell(N)| \leq g(N)$$

for every $\ell \in \mathbb{Z}^+$ *and* $N \in \mathrm{M}(m,n,\mathbb{Z})$, *and* $\sum_{N\in\mathrm{M}(m,n,\mathbb{Z})} g(N)$ *converges. Then*

$$\sum_{N\in\mathrm{M}(m,n,\mathbb{Z})} f(N) \quad \text{and} \quad \sum_{N\in\mathrm{M}(m,n,\mathbb{Z})} f_\ell(N) \quad \text{for} \quad \ell \in \mathbb{Z}^+$$

converge absolutely, and

$$\lim_{\ell\to\infty} \sum_{N\in\mathrm{M}(m,n,\mathbb{Z})} f_\ell(N) = \sum_{N\in\mathrm{M}(m,n,\mathbb{Z})} f(N).$$

Proof. This is an application of Lebesgue's dominated convergence theorem (see the theorem on p. 26 of [24]). □

Lemma 3.4.7. *Let* m *and* n *be positive integers, and assume that* m *is even. Let* $F \in \mathrm{Sym}(m,\mathbb{Z})^+$ *be even, and let* N *be the level of* F. *Let* $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \Gamma_0(N)$. *Assume that* D *is invertible, and let* d *be a non-zero integer such that* dD^{-1} *is integral. Let* $\chi(M)$ *be as in Theorem 3.3.5. Then*

$$\chi(M) = d^{-mn} \det(D)^{m/2} \sum_{R\in\mathrm{M}(m,n,\mathbb{Z}/d\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(BD^{-1}F[R])\big).$$

Proof. For every positive integer ℓ, we define

$$T_\ell = \ell^{-1} \cdot 1_n.$$

Evidently, $T_\ell \in \mathrm{Sym}(n,\mathbb{R})^+$ for $\ell \in \mathbb{Z}^+$. Let $\ell \in \mathbb{Z}^+$. By Theorem 3.3.5

$$\begin{aligned} &\chi(M)\theta(F,Z,X,Y) \\ &\qquad = s(M,Z)^{-m}\theta(F, M\cdot Z, X\,{}^{\mathrm{t}}A + FY\,{}^{\mathrm{t}}B, F^{-1}X\,{}^{\mathrm{t}}C + Y\,{}^{\mathrm{t}}D) \end{aligned} \tag{3.14}$$

for $Z \in \mathbb{H}_n$ and $X, Y \in \mathrm{M}(m,n,\mathbb{C})$. Since m is even, we have

$$s(M,Z)^{-m} = \det(CZ+D)^{-m/2}$$

for $Z \in \mathbb{H}_n$. Let $Z = iT_\ell$ and $X = Y = 0$ in (3.14), we obtain

$$\chi(M)\theta(F,iT_\ell) = \det(iCT_\ell + D)^{-m/2}\theta(F, M\cdot iT_\ell) \tag{3.15}$$

where we write $\theta(F,Z) = \theta(F,Z,0,0)$ for $Z \in \mathbb{H}_n$. Multiplying this equation by $\det(T_\ell)^{m/2}$, we obtain:

$$\det(T_\ell)^{m/2}\chi(M)\theta(F,iT_\ell) \\ = \det(T_\ell)^{m/2}\det(iCT_\ell+D)^{-m/2}\theta(F,M\cdot iT_\ell). \quad (3.16)$$

To prove the lemma we will determine the limits of both sides of (3.16) as $\ell\to\infty$. Using Lemma 3.4.2, the left-hand side of (3.16) can be computed as:

$$\begin{aligned}\text{LHS of (3.16)} &= \det(T_\ell)^{m/2}\chi(M)\theta(F,iT_\ell)\\ &= \det(T_\ell)^{m/2}\chi(M)\det(F)^{-n/2}\det(T_\ell)^{-m/2}\theta(F^{-1},-(iT_\ell)^{-1})\\ &= \chi(M)\det(F)^{-n/2}\theta(F^{-1},-(iT_\ell)^{-1}).\end{aligned}$$

We claim that

$$\lim_{\ell\to\infty}\theta(F^{-1},-(iT_\ell)^{-1}) = 1. \quad (3.17)$$

To prove this, we first note that

$$\begin{aligned}\theta(F^{-1},-(iT_\ell)^{-1}) &= \sum_{R\in\mathrm{M}(m,n,\mathbb{Z})}\exp\big(\pi i\mathrm{tr}(-(iT_\ell)^{-1}F^{-1}[R])\big)\\ &= \sum_{R\in\mathrm{M}(m,n,\mathbb{Z})}\exp\big(-\pi\ell\mathrm{tr}(F^{-1}[R])\big).\end{aligned}$$

Since F^{-1} is positive-definite, it follows that for $R\in\mathrm{M}(m,n,\mathbb{Z})$ we have $\mathrm{tr}(F^{-1}[R])\geq 0$ with $\mathrm{tr}(F^{-1}[R])=0$ if and only if $R=0$ (see Lemma 3.4.4). It follows that

$$\lim_{\ell\to\infty}\exp\big(-\pi\ell\mathrm{tr}(F^{-1}[R])\big) = \begin{cases}0 & \text{if } R\neq 0,\\ 1 & \text{if } R=0.\end{cases}$$

We also have

$$|\exp\big(-\pi\ell\mathrm{tr}(F^{-1}[R])\big)| = \exp\big(-\pi\ell\mathrm{tr}(F^{-1}[R])\big) \leq \exp\big(-\pi\mathrm{tr}(F^{-1}[R])\big)$$

for $R\in\mathrm{M}(m,n,\mathbb{Z})$, and the series

$$\sum_{R\in\mathrm{M}(m,n,\mathbb{Z})}\exp\big(-\pi\mathrm{tr}(F^{-1}[R])\big)$$

converges absolutely by Proposition 3.1.8 (with $A=F^{-1}$, $Z=i1_n$, and $X=Y=0$). Lemma 3.4.6 now implies that

$$\begin{aligned}\lim_{\ell\to\infty}\theta(F^{-1},-(iT_\ell)^{-1}) &= \lim_{\ell\to\infty}\sum_{R\in\mathrm{M}(m,n,\mathbb{Z})}\exp\big(-\pi\ell\mathrm{tr}(F^{-1}[R])\big)\\ &= \sum_{R\in\mathrm{M}(m,n,\mathbb{Z})}\lim_{\ell\to\infty}\exp\big(-\pi\ell\mathrm{tr}(F^{-1}[R])\big)\\ &= \sum_{R\in\mathrm{M}(m,n,\mathbb{Z})}\left\{\begin{array}{ll}0 & \text{if } R\neq 0,\\ 1 & \text{if } R=0\end{array}\right\}\\ &= 1.\end{aligned}$$

It follows that

$$\lim_{\ell\to\infty} \text{LHS of (3.16)} = \chi(M)\det(F)^{-n/2}. \tag{3.18}$$

We now consider the right-hand side of (3.16). We first rewrite $M \cdot iT_\ell$. Let $Z \in \mathbb{H}_n$, and define

$$W = {}^{\mathrm{t}}D^{-1}Z(CZ+D)^{-1}.$$

We claim that

$$M \cdot Z = BD^{-1} + W. \tag{3.19}$$

To see this, we calculate:

$$\begin{aligned}
BD^{-1} + W &= BD^{-1} + {}^{\mathrm{t}}D^{-1}Z(CZ+D)^{-1} \\
&= (BD^{-1}(CZ+D) + {}^{\mathrm{t}}D^{-1}Z)(CZ+D)^{-1} \\
&= (BD^{-1}CZ + B + {}^{\mathrm{t}}D^{-1}Z)(CZ+D)^{-1} \\
&= ((BD^{-1}C + {}^{\mathrm{t}}D^{-1})Z + B)(CZ+D)^{-1} \\
&= ((BD^{-1}C\,{}^{\mathrm{t}}D + 1)\,{}^{\mathrm{t}}D^{-1}Z + B)(CZ+D)^{-1} \\
&= ((BD^{-1}D\,{}^{\mathrm{t}}C + 1)\,{}^{\mathrm{t}}D^{-1}Z + B)(CZ+D)^{-1} \\
&= ((B\,{}^{\mathrm{t}}C + 1)\,{}^{\mathrm{t}}D^{-1}Z + B)(CZ+D)^{-1} \\
&= (A\,{}^{\mathrm{t}}D\,{}^{\mathrm{t}}D^{-1}Z + B)(CZ+D)^{-1} \\
&= (AZ+B)(CZ+D)^{-1} \\
&= M \cdot Z.
\end{aligned}$$

In this calculation we used Lemma 1.9.2. We now define

$$T'_\ell = {}^{\mathrm{t}}D^{-1}T_\ell(C(iT_\ell)+D)^{-1}.$$

Multiplying by i, we obtain

$$iT'_\ell = {}^{\mathrm{t}}D^{-1}(iT_\ell)(C(iT_\ell)+D)^{-1}.$$

By the general identity (3.19) we have

$$M \cdot iT_\ell = BD^{-1} + iT'_\ell.$$

Since $BD^{-1} \in \mathrm{Sym}(n,\mathbb{R})$ by Lemma 1.9.2, and since $M \cdot iT_\ell \in \mathbb{H}_n$, it follows that $iT'_\ell \in \mathbb{H}_n$. We now have:

$$\begin{aligned}
\theta(F, M\cdot iT_\ell) &= \sum_{R\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}((M\cdot iT_\ell)F[R])\big) \\
&= \sum_{R\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}((BD^{-1}+iT'_\ell)F[R])\big) \\
&= \sum_{R\in \mathrm{M}(m,n,\mathbb{Z}/d\mathbb{Z})} \sum_{N\in d\mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}((BD^{-1}+iT'_\ell)F[R+N])\big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{R\in \mathrm{M}(m,n,\mathbb{Z}/d\mathbb{Z})} \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}((BD^{-1}+iT'_\ell)F[R+dN])\big)\\
&= \sum_{R\in \mathrm{M}(m,n,\mathbb{Z}/d\mathbb{Z})} \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}((BD^{-1}+iT'_\ell)\\
&\qquad \times (F[R]+d\,{}^{\mathrm{t}}NFR+d\,{}^{\mathrm{t}}RFN+d^2F[N]))\big)\\
&= \sum_{R\in \mathrm{M}(m,n,\mathbb{Z}/d\mathbb{Z})} \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(BD^{-1}F[R]) - \pi \mathrm{tr}(T'_\ell F[R])\big)\\
&\qquad \times \exp\big(-\pi d\mathrm{tr}(T'_\ell\,{}^{\mathrm{t}}NFR) - \pi d\mathrm{tr}(T'_\ell\,{}^{\mathrm{t}}RFN) - \pi d^2\mathrm{tr}(T'_\ell F[N])\big)\\
&\qquad \times \exp\big(\pi i \mathrm{tr}(BdD^{-1}({}^{\mathrm{t}}NFR+{}^{\mathrm{t}}RFN)\big)\exp\big(\pi i d\mathrm{tr}(BdD^{-1}F[N])\big)\\
&= \sum_{R\in \mathrm{M}(m,n,\mathbb{Z}/d\mathbb{Z})} \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(BD^{-1}F[R]) - \pi \mathrm{tr}(T'_\ell F[R])\big)\\
&\qquad \times \exp\big(-2\pi d\mathrm{tr}(T'_\ell\,{}^{\mathrm{t}}NFR) - \pi d^2\mathrm{tr}(T'_\ell F[N])\big)\\
&\qquad \times \exp\big(2\pi i \mathrm{tr}(BdD^{-1}({}^{\mathrm{t}}NFR)\big)\exp\big(\pi i d\mathrm{tr}(BdD^{-1}F[N])\big)\\
&= \sum_{R\in \mathrm{M}(m,n,\mathbb{Z}/d\mathbb{Z})} \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(BD^{-1}F[R]) - \pi \mathrm{tr}(T'_\ell F[R])\big)\\
&\qquad \times \exp\big(-2\pi d\mathrm{tr}(T'_\ell\,{}^{\mathrm{t}}NFR) - \pi d^2\mathrm{tr}(T'_\ell F[N])\big)\\
&\qquad \times \exp\big(\pi i d\mathrm{tr}(BdD^{-1}F[N])\big).
\end{aligned}
$$

For the last two equalities we used Lemma 3.4.3, along with the fact that the matrix BdD^{-1} is integral (by the definition of d) and symmetric (by Lemma 1.9.2). By Lemma 3.4.5 we also have $\exp\big(\pi i d\mathrm{tr}(BdD^{-1}F[N])\big) = 1$. Hence,

$$
\begin{aligned}
\theta(F, M\cdot iT_\ell) &= \sum_{R\in \mathrm{M}(m,n,\mathbb{Z}/d\mathbb{Z})} \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(BD^{-1}F[R]) - \pi \mathrm{tr}(T'_\ell F[R])\big)\\
&\qquad \times \exp\big(-2\pi d\mathrm{tr}(T'_\ell\,{}^{\mathrm{t}}NFR) - \pi d^2\mathrm{tr}(T'_\ell F[N])\big)\\
&= \sum_{R\in \mathrm{M}(m,n,\mathbb{Z}/d\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(BD^{-1}F[R]) - \pi \mathrm{tr}(T'_\ell F[R])\big)\\
&\qquad \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(-2\pi d\mathrm{tr}(T'_\ell\,{}^{\mathrm{t}}NFR) - \pi d^2\mathrm{tr}(T'_\ell F[N])\big)\\
&= \sum_{R\in \mathrm{M}(m,n,\mathbb{Z}/d\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(BD^{-1}F[R]) - \pi \mathrm{tr}(T'_\ell F[R])\big)\\
&\qquad \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(id^2T'_\ell F[N]) + 2\pi i \mathrm{tr}({}^{\mathrm{t}}NdFR(iT'_\ell))\big)\\
&= \sum_{R\in \mathrm{M}(m,n,\mathbb{Z}/d\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(BD^{-1}F[R]) - \pi \mathrm{tr}(T'_\ell F[R])\big)\\
&\qquad \times \theta(F, id^2T'_\ell, dFR(iT'_\ell), 0)\\
\theta(F, M\cdot iT_\ell) &= \sum_{R\in \mathrm{M}(m,n,\mathbb{Z}/d\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(BD^{-1}F[R])\big)
\end{aligned}
$$

$$\exp\big(-\pi\mathrm{tr}(T'_\ell F[R])\big)\theta(F, id^2T'_\ell, dFR(iT'_\ell), 0). \tag{3.20}$$

Let $R \in \mathrm{M}(m, n, \mathbb{Z})$. By Lemma 3.4.2 we have:

$$\begin{aligned}&\theta(F, id^2T'_\ell, dFR(iT'_\ell), 0)\\ &\quad= \det(F)^{-n/2}\det(d^2T'_\ell)^{-m/2}\theta(F^{-1}, -(id^2T'_\ell)^{-1}, 0, -dFR(iT'_\ell)).\end{aligned} \tag{3.21}$$

Now

$$\begin{aligned}&\theta(F^{-1}, -(id^2T'_\ell)^{-1}, 0, -dFR(iT'_\ell))\\ &\qquad= \sum_{N\in\mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i\mathrm{tr}(-(id^2T'_\ell)^{-1}F^{-1}[N + dFR(iT'_\ell)])\big).\end{aligned}$$

Let $N \in \mathrm{M}(m, n, \mathbb{Z})$. Then

$$\begin{aligned}
&\exp\big(\pi i\mathrm{tr}(-(id^2T'_\ell)^{-1}F^{-1}[N + dFR(iT'_\ell)])\big)\\
&\quad= \exp\big(-\pi d^{-2}\mathrm{tr}(T'^{-1}_\ell\,{}^{\mathrm{t}}(N + dFRiT'_\ell)F^{-1}(N + dFRiT'_\ell))\big)\\
&\quad= \exp\big(-\pi d^{-2}\mathrm{tr}(T'^{-1}_\ell({}^{\mathrm{t}}N + diT'_\ell\,{}^{\mathrm{t}}RF)(F^{-1}N + diRT'_\ell))\big)\\
&\quad= \exp\big(-\pi d^{-2}\mathrm{tr}((T'^{-1}_\ell\,{}^{\mathrm{t}}N + di\,{}^{\mathrm{t}}RF)(F^{-1}N + diRT'_\ell))\big)\\
&\quad= \exp\big(-\pi d^{-2}\mathrm{tr}(T'^{-1}_\ell F^{-1}[N] + diT'^{-1}_\ell\,{}^{\mathrm{t}}NRT'_\ell + di\,{}^{\mathrm{t}}RN - d^2\,{}^{\mathrm{t}}RFRT'_\ell)\big)\\
&\quad= \exp\big(-\pi d^{-2}\mathrm{tr}(T'^{-1}_\ell F^{-1}[N])\big)\exp\big(-2\pi id^{-1}\mathrm{tr}({}^{\mathrm{t}}RN)\big)\\
&\qquad\quad\times\exp\big(\pi\mathrm{tr}(T'_\ell F[R])\big)\\
&\quad= \exp\big(-\pi d^{-2}\mathrm{tr}((CiT_\ell + D)T_\ell^{-1}\,{}^{\mathrm{t}}DF^{-1}[N])\big)\exp\big(-2\pi id^{-1}\mathrm{tr}({}^{\mathrm{t}}RN)\big)\\
&\qquad\quad\times\exp\big(\pi\mathrm{tr}(T'_\ell F[R])\big)\\
&\quad= \exp\big(-\pi d^{-2}\mathrm{tr}(\ell(i\ell^{-1}C + D)\,{}^{\mathrm{t}}DF^{-1}[N])\big)\exp\big(-2\pi id^{-1}\mathrm{tr}({}^{\mathrm{t}}RN)\big)\\
&\qquad\quad\times\exp\big(\pi\mathrm{tr}(T'_\ell F[R])\big)\\
&\quad= \exp\big(-\pi id^{-2}\mathrm{tr}(C\,{}^{\mathrm{t}}DF^{-1}[N])\big)\exp\big(-\pi d^{-2}\ell\mathrm{tr}(D\,{}^{\mathrm{t}}DF^{-1}[N])\big)\\
&\qquad\quad\times\exp\big(-2\pi id^{-1}\mathrm{tr}({}^{\mathrm{t}}RN)\big)\exp\big(\pi\mathrm{tr}(T'_\ell F[R])\big)\\
&\quad= \exp\big(-\pi id^{-2}\mathrm{tr}(C\,{}^{\mathrm{t}}DF^{-1}[N])\big)\exp\big(-\pi d^{-2}\ell\mathrm{tr}(F^{-1}[ND])\big)\\
&\qquad\quad\times\exp\big(-2\pi id^{-1}\mathrm{tr}({}^{\mathrm{t}}RN)\big)\exp\big(\pi\mathrm{tr}(T'_\ell F[R])\big).
\end{aligned}$$

It follows that

$$\exp\big(-\pi\mathrm{tr}(T'_\ell F[R])\big)\theta(F^{-1}, -(id^2T'_\ell)^{-1}, 0, -dFR(iT'_\ell)) \tag{3.22}$$

$$\begin{aligned}&= \sum_{N\in\mathrm{M}(m,n,\mathbb{Z})}\exp\big(-\pi id^{-2}\mathrm{tr}(C\,{}^{\mathrm{t}}DF^{-1}[N])\big)\\ &\qquad\times\exp\big(-2\pi id^{-1}\mathrm{tr}({}^{\mathrm{t}}RN)\big)\exp\big(-\pi d^{-2}\ell\mathrm{tr}(F^{-1}[ND])\big).\end{aligned} \tag{3.23}$$

We claim that

$$\lim_{\ell\to\infty}\exp\big(-\pi\mathrm{tr}(T'_\ell F[R])\big)\theta(F^{-1}, -(id^2T'_\ell)^{-1}, 0, -dFR(iT'_\ell)) = 1. \tag{3.24}$$

To prove this we use (3.23) and Lemma 3.4.6. Since F^{-1} is positive-definite we have, for $N \in \mathrm{M}(m,n,\mathbb{Z})$, $\mathrm{tr}(F^{-1}[ND]) \geq 0$, and $\mathrm{tr}(F^{-1}[ND]) = 0$ if and only if $ND = 0$, that is, if and only $N = 0$ (see Lemma 3.4.4. This implies that for $N \in \mathrm{M}(m,n,\mathbb{Z})$,

$$\begin{aligned}
&\lim_{\ell\to\infty} \exp\big(-\pi i d^{-2}\mathrm{tr}(C\,{}^{\mathrm{t}}DF^{-1}[N])\big)\\
&\qquad\times \exp\big(-2\pi i d^{-1}\mathrm{tr}({}^{\mathrm{t}}RN)\big)\exp\big(-\pi d^{-2}\ell\,\mathrm{tr}(F^{-1}[ND])\big)\\
&= \exp\big(-\pi i d^{-2}\mathrm{tr}(C\,{}^{\mathrm{t}}DF^{-1}[N])\big)\exp\big(-2\pi i d^{-1}\mathrm{tr}({}^{\mathrm{t}}RN)\big) \qquad (3.25)\\
&\qquad\times \lim_{\ell\to\infty}\exp\big(-\pi d^{-2}\ell\,\mathrm{tr}(F^{-1}[ND])\big)\\
&= \begin{cases} 1 & \text{if } N = 0,\\ 0 & \text{if } N \neq 0.\end{cases} \qquad (3.26)
\end{aligned}$$

We also have

$$\begin{aligned}
&|\exp\big(-\pi i d^{-2}\mathrm{tr}(C\,{}^{\mathrm{t}}DF^{-1}[N])\big)\exp\big(-2\pi i d^{-1}\mathrm{tr}({}^{\mathrm{t}}RN)\big)\\
&\quad\times\exp\big(-\pi d^{-2}\ell\,\mathrm{tr}(F^{-1}[ND])\big)|\\
&\quad\leq \exp\big(-\pi d^{-2}\ell\,\mathrm{tr}(F^{-1}[ND])\big)\\
&\quad\leq \exp\big(-\pi d^{-2}\mathrm{tr}(F^{-1}[ND])\big),
\end{aligned}$$

and the series

$$\sum_{N\in\mathrm{M}(m,n,\mathbb{Z})}\exp\big(-\pi d^{-2}\mathrm{tr}(F^{-1}[ND])\big)$$

converges by Proposition 3.1.8. We now may apply Lemma 3.4.6 and conclude that (3.24) holds. Going back, we have

$$\begin{aligned}
\text{RHS of (3.16)} &= \det(T_\ell)^{m/2}\det(iCT_\ell + D)^{-m/2}\theta(F, M\cdot iT_\ell)\\
&= \det(T_\ell)^{m/2}\det(iCT_\ell + D)^{-m/2}\det(F)^{-n/2}\det(d^2T'_\ell)^{-m/2}\\
&\qquad\sum_{R\in\mathrm{M}(m,n,\mathbb{Z}/d\mathbb{Z})}\exp\big(\pi i\mathrm{tr}(BD^{-1}F[R])\big)\\
&\qquad\exp\big(-\pi\mathrm{tr}(T'_\ell F[R])\big)\theta(F^{-1}, -(id^2T'_\ell)^{-1}, 0, -dFR(iT'_\ell))\\
&= \det(F)^{-n/2}d^{-mn}\det(iCT_\ell + D)^{-m/2}\det(T_\ell T'^{-1}_\ell)^{m/2}\\
&\qquad\sum_{R\in\mathrm{M}(m,n,\mathbb{Z}/d\mathbb{Z})}\exp\big(\pi i\mathrm{tr}(BD^{-1}F[R])\big)\\
&\qquad\exp\big(-\pi\mathrm{tr}(T'_\ell F[R])\big)\theta(F^{-1}, -(id^2T'_\ell)^{-1}, 0, -dFR(iT'_\ell))\\
&= \det(F)^{-n/2}d^{-mn}\det(i\ell^{-1}C + D)^{-m/2}\det((i\ell^{-1}C + D)\,{}^{\mathrm{t}}D)^{m/2}\\
&\qquad\sum_{R\in\mathrm{M}(m,n,\mathbb{Z}/d\mathbb{Z})}\exp\big(\pi i\mathrm{tr}(BD^{-1}F[R])\big)\\
&\qquad\exp\big(-\pi\mathrm{tr}(T'_\ell F[R])\big)\theta(F^{-1}, -(id^2T'_\ell)^{-1}, 0, -dFR(iT'_\ell))\\
&= \det(F)^{-n/2}d^{-mn}\det(D)^{m/2}
\end{aligned}$$

$$\sum_{R\in \mathrm{M}(m,n,\mathbb{Z}/d\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(BD^{-1}F[R])\big)$$
$$\exp\big(-\pi\mathrm{tr}(T'_\ell F[R])\big)\theta(F^{-1}, -(id^2T'_\ell)^{-1}, 0, -dFR(iT'_\ell)).$$

By (3.26) we now have

$$\begin{aligned}&\lim_{\ell\to\infty} \text{RHS of (3.16)}\\ &= \det(F)^{-n/2} d^{-mn}\det(D)^{m/2} \sum_{R\in \mathrm{M}(m,n,\mathbb{Z}/d\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(BD^{-1}F[R])\big).\end{aligned} \quad (3.27)$$

A comparison of (3.18) and (3.27) completes the proof. □

Let n and N be positive integers. We have the subgroup $\Gamma_0(N)$ of $\mathrm{Sp}(2n,\mathbb{Z})$. Sometimes, to indicate the dependence of $\Gamma_0(N)$ we will write $\Gamma_0^{(n)}(N)$ for $\Gamma_0(N)$. Let K be the subgroup of $\Gamma_0^{(n)}(N)$ generated by the matrices of the form

$$\begin{bmatrix} {}^{\mathrm{t}}U^{-1} & \\ & U\end{bmatrix}, \qquad U\in \mathrm{SL}(n,\mathbb{Z}), \quad (3.28)$$

$$\begin{bmatrix} 1 & S \\ & 1\end{bmatrix}, \qquad S\in \mathrm{Sym}(n,\mathbb{Z}), \quad (3.29)$$

$$\begin{bmatrix} 1 & \\ T & 1\end{bmatrix}, \qquad T\in \mathrm{Sym}(n,\mathbb{Z}) \quad \text{and} \quad T\equiv 0 \pmod N. \quad (3.30)$$

Let $M_1, M_2\in \Gamma_0^{(n)}(N)$. We will say that M_1 and M_2 are equivalent, and write $M_1\sim M_2$, if there exist $k_1,k_2\in K$ such that $k_1M_1k_2 = M_2$. Clearly, $\sim$ is an equivalence relation on $\Gamma_0^{(n)}(N)$.

Lemma 3.4.8. *Let n and N be positive integers with $N>1$. Let $k\in K$. Then $\chi(k)=1$.*

Proof. Since χ is a character by Lemma 3.4.1, we may assume that k is of the form (3.28), (3.29), or (3.30). We now use the formula from Lemma 3.4.7 to conclude that $\chi(k)=1$. □

Lemma 3.4.9. *Let n and N be positive integers with $N>1$. Let*

$$M_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1\end{bmatrix}, \quad M_2 = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2\end{bmatrix} \in \Gamma_0(N)\subset \mathrm{Sp}(2n,\mathbb{Z}).$$

If $M_1\sim M_2$, then $\det(D_1)\equiv \det(D_2) \pmod N$.

Proof. Let g be one of the generators for K, so that g is of the form (3.28), (3.29), or (3.30). It suffices to verify that if $gM_1 = M_2$ or $M_1g = M_2$, then $\det(D_1)\equiv\det(D_2) \pmod N$. This follows by direct computations. □

Lemma 3.4.10. *Let n and N be positive integers with $N > 1$. Let $M \in \Gamma_0^{(n)}(N)$. Then M is equivalent to*

$$\left[\begin{array}{cccc|cccc} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & a & & & & b \\ \hline & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \\ & & & c & & & & d \end{array}\right] \tag{3.31}$$

for some $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0^{(1)}(N)$.

Proof. We will prove the lemma by induction on n. If $n = 1$, the lemma is trivially true. Assume that $n \geq 2$ and that the lemma hold for $n - 1$; we will prove that it holds for n.

We will first prove the following claim: The element M is equivalent to an element of the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where D has the form

$$\begin{bmatrix} 1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}, \qquad d_2|d_3, \quad \dots, \quad d_{n-1}|d_n. \tag{3.32}$$

To begin the proof of the claim, let $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Since $N > 1$ and ${}^{\mathrm{t}}AD - {}^{\mathrm{t}}CB = 1$ (see Lemma 1.9.2), we have ${}^{\mathrm{t}}AD \equiv 1 \pmod{N}$; this implies that D is non-zero. By the theorem on elementary divisors, Theorem 1.12.1, there exist $g_1, g_2 \in \mathrm{SL}(n, \mathbb{Z})$, and positive integers $d_1, \dots, d_n$ such that

$$d_1|d_2, \quad d_2|d_3, \quad \dots, \quad d_{n_1}|d_n$$

and

$$g_1 D g_2 = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}.$$

Moreover, d_1 is the greatest common divisor of the entries of D. It follows that

$$\begin{bmatrix} {}^{\mathrm{t}}g_1^{-1} & \\ & g_1 \end{bmatrix} M \begin{bmatrix} {}^{\mathrm{t}}g_2^{-1} & \\ & g_2 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$$

where

$$D_1 = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}.$$

Since

$$\begin{bmatrix} {}^{\mathrm{t}}g_1{}^{-1} & \\ & g_1 \end{bmatrix}, \quad \begin{bmatrix} {}^{\mathrm{t}}g_2{}^{-1} & \\ & g_2 \end{bmatrix} \in K$$

we have

$$M \sim \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}.$$

By Lemma 1.9.2 we have $A_1\,{}^{\mathrm{t}}D_1 - B_1\,{}^{\mathrm{t}}C_1 = 1$. Taking the transpose of this equation, and letting $A_1 = (a_{ij})$, $B_1 = (b_{ij})$, $C_1 = (c_{ij})$, we obtain:

$$\begin{aligned} 1 &= D_1\,{}^{\mathrm{t}}A_1 - C_1\,{}^{\mathrm{t}}B_1 \\ &= \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix} - \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{n1} \\ \vdots & & \vdots \\ b_{1n} & \cdots & b_{nn} \end{bmatrix} \\ &= \begin{bmatrix} d_1 a_{11} - c_{11} b_{11} - \cdots - c_{1n} b_{1n} & * \\ * & * \end{bmatrix}. \end{aligned}$$

Thus,

$$1 = d_1 a_{11} - c_{11} b_{11} - \cdots - c_{1n} b_{1n}. \tag{3.33}$$

This equation implies that one of $c_{11}, \ldots, c_{1n}$ is non-zero; let c be their common divisor. Equation (3.33) also implies that d_1 and c are relatively prime. Let $s_1, \ldots, s_n$ be integers such that

$$c = c_{11} s_1 + \cdots + c_{1n} s_n.$$

Define $S \in \mathrm{Sym}(n, \mathbb{Z})$ by

$$S = \begin{bmatrix} & s_1 & & \\ s_1 & s_2 & \cdots & s_n \\ & \vdots & & \\ & s_n & & \end{bmatrix},$$

and define

$$\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} 1 & S \\ & 1 \end{bmatrix}$$

Since

$$\begin{bmatrix} 1 & S \\ & 1 \end{bmatrix} \in K$$

we have

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \sim \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}.$$

Moreover,

$$\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} = \begin{bmatrix} A_1 & A_1 S + B_1 \\ C_1 & C_1 S + D_1 \end{bmatrix}$$

with

$$\begin{aligned} D_2 &= C_1 S + D_1 \\ &= \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} + \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} & s_1 & & \\ s_1 & s_2 & \cdots & s_n \\ & \vdots & & \\ & s_n & & \end{bmatrix} \\ &= \begin{bmatrix} d_1 + c_{12} s_1 & c & * \\ * & * & * \end{bmatrix}. \end{aligned}$$

Since d_1 and c are relatively prime, and c is the greatest common divisor of $c_{11}, c_{12}, \ldots, c_{1n}$, it follows that $d_1 + c_{12}s_1$ and c are relatively prime. As a consequence of this, the greatest common divisor of the entries of D_2 is 1. An application of the theorem on elementary divisors to D_2 similar to the first application above then proves that

$$\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \sim \begin{bmatrix} A_3 & B_3 \\ C_3 & D_3 \end{bmatrix}$$

where D_3 has the form (3.32); the key point is that the greatest common divisor of the entries of D_2 is 1. This proves the claim.

Thanks to the claim, we may assume that $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ with D having the form (3.32). Define

$$S = \begin{bmatrix} -b_{11} & -b_{21} & \cdots & -b_{n1} \\ -b_{21} & & & \\ \vdots & & & \\ -b_{n1} & & & \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} -c_{11} & -c_{12} & \cdots & -c_{1n} \\ -c_{12} & & & \\ \vdots & & & \\ -c_{1n} & & & \end{bmatrix}.$$

Let

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} 1 & S \\ & 1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1 & \\ T & 1 \end{bmatrix}.$$

Since

$$\begin{bmatrix} 1 & S \\ & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & \\ T & 1 \end{bmatrix} \in K$$

we have

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \sim \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}.$$

Explicitly,

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} A + SC + BT + SDT & B + SD \\ C + DT & D \end{bmatrix}.$$

By the choice of S and T and the fact that D as the form (3.32), the first column of B_1 is zero, and the first row of C_1 is zero; of course, $D_1 = D$, so that D_1 has the form (3.32). By Lemma 1.9.2 we have ${}^{\mathrm{t}}D_1 B_1 = {}^{\mathrm{t}}B_1 D_1$ and $C_1\, {}^{\mathrm{t}}D_1 = D_1\, {}^{\mathrm{t}}C_1$. Therefore, letting $B_1 = (b_{ij})$,

$$\begin{bmatrix} 1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 0 & b_{12} & \cdots & b_{1n} \\ \vdots & \vdots & & \vdots \\ 0 & b_{n2} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \\ b_{12} & \cdots & b_{n2} \\ \vdots & & \vdots \\ b_{1n} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$$

$$\begin{bmatrix} 0 & b_{12} & \cdots & b_{1n} \\ \vdots & \vdots & & \vdots \\ 0 & d_n b_{n2} & \cdots & d_n b_{nn} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \\ b_{12} & \cdots & d_n b_{n2} \\ \vdots & & \vdots \\ b_{1n} & \cdots & d_n b_{nn} \end{bmatrix}.$$

This equality implies that the first row of B_1 is also zero. Similarly, the first column of C_1 is zero, so that B_1 and C_1 have the form

$$B_1 = \begin{bmatrix} 0 & 0 \\ 0 & B_2 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 0 & 0 \\ 0 & C_2 \end{bmatrix}$$

for some $B_2 \in \mathrm{M}(n-1, \mathbb{Z})$ and $C_2 \in N\mathrm{M}(n-1, \mathbb{Z})$. By Lemma 1.9.2 we have $1 = A_1\, {}^{\mathrm{t}}D_1 - B_1\, {}^{\mathrm{t}}C_1$. Writing this in terms of matrices, we find that A_1 has the form

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & A_2 \end{bmatrix}$$

for some $A_2 \in \mathrm{M}(n-1, \mathbb{Z})$. Clearly, D_1 has the form

$$D_1 = \begin{bmatrix} 1 & 0 \\ 0 & D_2 \end{bmatrix}$$

for some $D_2 \in \mathrm{M}(n-1, \mathbb{Z})$. We now have

$$M \sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & B_2 \\ \hline 0 & 0 & 1 & 0 \\ 0 & C_2 & 0 & D_2 \end{array}\right].$$

By Lemma 1.9.2, the matrix $\left[\begin{smallmatrix} A_2 & B_2 \\ C_2 & D_2 \end{smallmatrix}\right]$ is contained in $\mathrm{Sp}(2(n-1), \mathbb{Z})$; since $C_2 \equiv 0 \pmod N$ we have

$$\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \in \Gamma_0^{(n-1)}(N).$$

Applying the induction hypothesis to $\left[\begin{smallmatrix} A_2 & B_2 \\ C_2 & D_2 \end{smallmatrix}\right]$ now completes the proof. $\square$

Theorem 3.4.11. *Let m and n be positive integers, and assume that m is even. Let $F \in \mathrm{Sym}(m,\mathbb{Z})^+$ be even, and let N be the level of F. Let $\chi : \Gamma_0(N) \to \mu_8$ be as in Theorem 3.3.5. If $N = 1$, then χ is the trivial character of $\Gamma_0(N) = \mathrm{Sp}(2n,\mathbb{Z})$. Assume that $N > 1$. We recall from Lemma 1.5.4 that N divides $\det(F)$, and that $\det(F)$ and N have the same set of prime divisors. Let $\Delta = \Delta(F) = (-1)^{m/2}\det(F)$ be the discriminant of F. Let $\left(\frac{\Delta}{\cdot}\right)$ be the Kronecker symbol from section 1.4, which is a Dirichlet character modulo $\det(F)$ by Proposition 1.4.2 and Lemma 1.5.2. Define $\chi_F : \mathbb{Z} \to \mathbb{C}$ as in Lemma 2.7.7; by this lemma, χ_F is a Dirichlet character modulo N. The function χ takes values in $\{\pm 1\}$, and the diagram*

$$\begin{array}{ccccc}
(\mathbb{Z}/\det(A)\mathbb{Z})^\times & \longrightarrow & (\mathbb{Z}/N\mathbb{Z})^\times & \longleftarrow & \Gamma_0(N) \\
& \searrow^{\left(\frac{\Delta}{\cdot}\right)} & \downarrow{\scriptstyle \chi_F} & \swarrow_{\chi} & \\
& & \{\pm 1\} & &
\end{array}$$

commutes. Here, the map $\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times$ is defined by $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \mapsto \det(D)$. Consequently,

$$\chi(\begin{bmatrix} A & B \\ C & D \end{bmatrix}) = \Big(\frac{\Delta}{\det(D)}\Big) = \Big(\frac{(-1)^k\det(F)}{\det(D)}\Big) \tag{3.34}$$

for $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \Gamma_0(N)$.

Proof. Assume first that $N = 1$. By Lemma 1.5.4 we have $\det(F) = 1$. By Theorem 3.3.5 we have

$$\chi(M)\theta(F,Z) = s(M,Z)^{-m}\theta(F, M\cdot Z) \tag{3.35}$$

for $M \in \mathrm{Sp}(2n,\mathbb{Z})$ and $Z \in \mathbb{H}_n$. In particular, for $Z \in \mathbb{H}_n$,

$$\begin{aligned}
\chi(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix})\theta(F,Z) &= s(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, Z)^{-m}\theta(F, \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\cdot Z) \\
\chi(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix})\theta(F,Z) &= \det(-Z)^{-m/2}\theta(F, -Z^{-1}).
\end{aligned} \tag{3.36}$$

On the other hand, by Lemma 3.4.2 we have

$$\theta(F,Z) = \det(-iZ)^{-m/2}\theta(F^{-1}, -Z^{-1})$$

for $Z \in \mathbb{H}_n$. Now for $Z \in \mathbb{H}_n$,

$$\begin{aligned}
\theta(F^{-1},Z) &= \sum_{R\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(F^{-1}[N]Z\big) \\
&= \sum_{R\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}({}^{\mathrm{t}}NF^{-1}NZ\big) \\
&= \sum_{R\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}({}^{\mathrm{t}}NF^{-1}FF^{-1}NZ\big)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{R\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}({}^{\mathrm{t}}F^{-1}NF(F^{-1}N)Z\big)\\
&= \sum_{R\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}({}^{\mathrm{t}}NFNZ)\big)\\
&= \theta(F,Z).
\end{aligned}$$

Therefore,

$$\theta(F,Z) = \det(-iZ)^{-m/2}\theta(F,-Z^{-1}) \tag{3.37}$$

for $Z \in \mathbb{H}_n$. Comparing (3.36) and (3.37), we obtain

$$\chi(\begin{bmatrix} & 1\\ -1 & \end{bmatrix}) = i^{-mn/2}.$$

By Proposition 2.5.1, m is divisible by 8. This implies that $i^{-mn/2}=1$. Hence,

$$\chi(\begin{bmatrix} & 1\\ -1 & \end{bmatrix}) = 1. \tag{3.38}$$

Next, by (3.35), we have for $Z \in \mathbb{H}_n$,

$$\begin{aligned}
\chi(\begin{bmatrix} 1 & B\\ & 1\end{bmatrix})\theta(F,Z) &= s(\begin{bmatrix} 1 & B\\ & 1\end{bmatrix},Z)^{-m}\theta(F,\begin{bmatrix} 1 & B\\ & 1\end{bmatrix}\cdot Z)\\
&= j(\begin{bmatrix} 1 & B\\ & 1\end{bmatrix},Z)^{-m}\theta(F,Z+B)\\
&= \theta(F,Z+B)\\
&= \sum_{R\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(F[N](Z+B))\big)\\
&= \sum_{R\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(F[N]Z)\big)\exp\big(\pi i \mathrm{tr}(F[N]B)\big)\\
&= \sum_{R\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(F[N]Z)\big)\\
&= \theta(F,Z).
\end{aligned}$$

Here, the penultimate step follows from Lemma 3.4.5. It follows that

$$\chi(\begin{bmatrix} 1 & B\\ & 1\end{bmatrix}) = 1. \tag{3.39}$$

We now have $\chi(M)=1$ for all $M \in \mathrm{Sp}(2n,\mathbb{Z})$ by Theorem 1.9.6.

Next, assume that $N>1$. The commutativity of the left side of the diagram was proven in Lemma 2.7.9. To prove the commutativity of right side of the diagram, let

$$M = \begin{bmatrix} A & B\\ C & D\end{bmatrix} \in \Gamma_0(N).$$

By Lemma 3.4.10, M is equivalent to

$$M_1 = \left[\begin{array}{cccc|cccc} 1 &&&&&&& \\ & \ddots &&&&&& \\ && 1 &&&&& \\ &&& a &&&& b \\ \hline &&&& 1 &&& \\ &&&&& \ddots && \\ &&&&&& 1 & \\ &&& c &&&& d \end{array}\right]$$

for some $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0^{(1)}(N)$. By Lemma 3.4.8 we have $\chi(M) = \chi(M_1)$. Also, by Lemma 3.4.9, we have $\det(D) \equiv d \pmod N$. Define the function $\alpha : \Gamma_0^{(1)}(N) \to \mathbb{C}$ as in (2.19) and (2.20). We claim that

$$\chi(M) = \chi(M_1) = \alpha(\begin{bmatrix} a & b \\ c & d \end{bmatrix}).$$

Assume first that $d > 0$. By Lemma 3.4.7,

$$\chi(M) = \chi(M_1) = d^{-mn+m/2} \sum_{R \in \mathrm{M}(m,n,\mathbb{Z}/d\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(bd^{-1}F[R_n])\big),$$

where we write $R = [R_1 \cdots R_n]$ for $R \in \mathrm{M}(m,n,\mathbb{Z}/d\mathbb{Z})$. Hence,

$$\begin{aligned} \chi(M) &= d^{-mn+m/2+mn-m} \sum_{q \in \mathrm{M}(m,1,\mathbb{Z}/d\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(bd^{-1}F[q])\big) \\ &= d^{-m/2} \sum_{q \in \mathrm{M}(m,1,\mathbb{Z}/d\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(bd^{-1}F[q])\big) \\ &= \alpha(\begin{bmatrix} a & b \\ c & d \end{bmatrix}). \end{aligned}$$

Assume next that $d < 0$. We have $M_1 = M_2 M_3$, where

$$M_2 = \left[\begin{array}{cccc|cccc} 1 &&&&&&& \\ & \ddots &&&&&& \\ && 1 &&&&& \\ &&& -1 &&&& \\ \hline &&&& 1 &&& \\ &&&&& \ddots && \\ &&&&&& 1 & \\ &&&&&&& -1 \end{array}\right]$$

and

$$M_3 = \left[\begin{array}{cccc|cccc} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & -a & & & & -b \\ \hline & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \\ & & & -c & & & & -d \end{array}\right].$$

The formula from Lemma 3.4.7 implies that $\chi(M_2) = (-1)^{m/2}$, and by an argument as in the case $d > 0$, we have

$$\chi(M_3) = \alpha(\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}).$$

Then

$$\begin{aligned} \chi(M) &= \chi(M_1) \\ &= \chi(M_2 M_3) \\ &= \chi(M_2)\chi(M_3) \\ &= (-1)^{m/2}\alpha(\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}) \\ &= \alpha(\begin{bmatrix} a & b \\ c & d \end{bmatrix}), \end{aligned}$$

where the last step follows from the definition of α (see (2.20)). Next, by (2.22), we have

$$\alpha(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \chi_F(d),$$

where χ_F is the Dirichlet character mod N defined in Lemma 2.7.7. Since $\det(D) \equiv d \pmod{N}$, we obtain

$$\chi(M) = \chi_F(\det(D)).$$

This proves the commutativity of the right side of the diagram. Finally, by Lemma 2.7.9 we have

$$\chi_F(\det(D)) = \Big(\frac{(-1)^{m/2}\det(F)}{\det(D)}\Big).$$

This completes the proof. □

3.5 Spherical harmonics

Lemma 3.5.1. *Let m and n be positive integers. Assume that $1 \le n < m$. Let $\eta \in \mathrm{M}(m, n, \mathbb{C})$ be such that*

$${}^{\mathrm{t}}\eta\eta = 0.$$

Let $\xi_{\alpha\beta}$ for $1 \le \alpha \le m$ and $1 \le \beta \le n$ be variables. Define $\xi = (\xi_{\alpha\beta})$, and let $\partial = (\partial/\partial\xi_{\alpha\beta})$. Define

$$L = \det({}^{\mathrm{t}}\eta\partial).$$

We have

$$\begin{aligned} &L^r\Big(\exp\big(\pi i \mathrm{tr}(P\,{}^{\mathrm{t}}\xi\xi + 2\,{}^{\mathrm{t}}Q\xi + R)\big)\Big) \\ &\qquad = \det(2\pi i(P\,{}^{\mathrm{t}}\xi + {}^{\mathrm{t}}Q)\eta)^r \exp\big(\pi i \mathrm{tr}(P\,{}^{\mathrm{t}}\xi\xi + 2\,{}^{\mathrm{t}}Q\xi + R)\big) \quad (3.40) \end{aligned}$$

for positive integers r, $R \in \mathrm{M}(n, \mathbb{C})$, $P \in \mathrm{Sym}(n, \mathbb{C})$, and $Q \in \mathrm{M}(m, n, \mathbb{C})$.

Proof. Let $\alpha \in \{1, \dots, m\}$ and $\beta \in \{1, \dots, n\}$. We begin by proving

$$\frac{\partial}{\partial\xi_{\alpha\beta}}\big(\mathrm{tr}(P\,{}^{\mathrm{t}}\xi\xi + 2\,{}^{\mathrm{t}}Q\xi)\big) = 2(\xi P + Q)_{\alpha\beta} \tag{3.41}$$

$$\frac{\partial}{\partial\xi_{\gamma\delta}}\frac{\partial}{\partial\xi_{\alpha\beta}}\big(\mathrm{tr}(P\,{}^{\mathrm{t}}\xi\xi + 2\,{}^{\mathrm{t}}Q\xi)\big) = 0 \qquad \text{if } \gamma \neq \alpha, \tag{3.42}$$

$$\frac{\partial}{\partial\xi_{\gamma\delta}}\big((\xi P + Q)_{\alpha\beta}\big) = \begin{cases} 0 & \text{if } \gamma \neq \alpha \\ P_{\beta\delta} = P_{\delta\beta} & \text{if } \gamma = \alpha. \end{cases} \tag{3.43}$$

Write $\xi = [\xi_1 \cdots \xi_n]$, $P = (P_{ij})$ and $Q = (Q_{ij})$. Then

$$\begin{aligned} \mathrm{tr}(P\,{}^{\mathrm{t}}\xi\xi + 2\,{}^{\mathrm{t}}Q) &= \mathrm{tr}(\begin{bmatrix} P_{11} & \cdots & P_{1n} \\ \vdots & & \vdots \\ P_{n1} & \cdots & P_{nn} \end{bmatrix} \begin{bmatrix} {}^{\mathrm{t}}\xi_1 \\ \vdots \\ {}^{\mathrm{t}}\xi_n \end{bmatrix} \begin{bmatrix} \xi_1 & \cdots & \xi_n \end{bmatrix} \\ &\quad + 2 \begin{bmatrix} Q_{11} & \cdots & Q_{m1} \\ \vdots & & \vdots \\ Q_{1n} & \cdots & Q_{mn} \end{bmatrix} \begin{bmatrix} \xi_{11} & \cdots & \xi_{1n} \\ \vdots & & \vdots \\ \xi_{m1} & \cdots & \xi_{mn} \end{bmatrix}) \\ &= \mathrm{tr}(\begin{bmatrix} P_{11} & \cdots & P_{1n} \\ \vdots & & \vdots \\ P_{n1} & \cdots & P_{nn} \end{bmatrix} \begin{bmatrix} {}^{\mathrm{t}}\xi_1\xi_1 & \cdots & {}^{\mathrm{t}}\xi_1\xi_n \\ \vdots & & \vdots \\ {}^{\mathrm{t}}\xi_n\xi_1 & \cdots & {}^{\mathrm{t}}\xi_n\xi_n \end{bmatrix}) \\ &\quad + 2\mathrm{tr}(\begin{bmatrix} \sum_{i=1}^m Q_{i1}\xi_{i1} & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & \sum_{i=1}^m Q_{in}\xi_{in} \end{bmatrix}) \\ &= \mathrm{tr}(\begin{bmatrix} \sum_{j=1}^n P_{1j}\,{}^{\mathrm{t}}\xi_j\xi_1 & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & \sum_{j=1}^n P_{nj}\,{}^{\mathrm{t}}\xi_j\xi_n \end{bmatrix}) \\ &\quad + 2\mathrm{tr}(\begin{bmatrix} \sum_{i=1}^m Q_{i1}\xi_{i1} & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & \sum_{i=1}^m Q_{in}\xi_{in} \end{bmatrix}) \end{aligned}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} P_{ij}\ {}^{\mathrm{t}}\xi_j\xi_i + 2\sum_{j=1}^{n}\sum_{i=1}^{m} Q_{ij}\xi_{ij}$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m} P_{ij}\xi_{ki}\xi_{kj} + 2\sum_{j=1}^{n}\sum_{i=1}^{m} Q_{ij}\xi_{ij}.$$

It follows that:

$$\begin{aligned}
&\frac{\partial}{\partial\xi_{\alpha\beta}}\big(\mathrm{tr}(P\ {}^{\mathrm{t}}\xi\xi + 2\ {}^{\mathrm{t}}Q\xi)\big)\\
&\quad= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m} P_{ij}\frac{\partial}{\partial\xi_{\alpha\beta}}\big(\xi_{ki}\xi_{kj}\big)\\
&\qquad + 2\sum_{j=1}^{n}\sum_{i=1}^{m} Q_{ij}\frac{\partial}{\partial\xi_{\alpha\beta}}\big(\xi_{ij}\big)\\
&\quad= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m} P_{ij}(\xi_{ki}\frac{\partial}{\partial\xi_{\alpha\beta}}\big(\xi_{kj}\big) + \xi_{kj}\frac{\partial}{\partial\xi_{\alpha\beta}}\big(\xi_{ki}\big))\\
&\qquad + 2\sum_{j=1}^{n}\sum_{i=1}^{m} Q_{ij}\frac{\partial}{\partial\xi_{\alpha\beta}}\big(\xi_{ij}\big)\\
&\quad= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m}(\left\{\begin{array}{rl} P_{i\beta}\xi_{\alpha i} & \text{if } k=\alpha,\ j=\beta,\\ 0 & \text{if } k\neq\alpha \text{ or } j\neq\beta \end{array}\right\}\\
&\qquad + \left\{\begin{array}{rl} P_{\beta j}\xi_{\alpha j} & \text{if } k=\alpha,\ i=\beta,\\ 0 & \text{if } k\neq\alpha \text{ or } i\neq\beta \end{array}\right\})\\
&\qquad + 2\sum_{j=1}^{n}\sum_{i=1}^{m} Q_{ij}\frac{\partial}{\partial\xi_{\alpha\beta}}\big(\xi_{ij}\big)\\
&\quad= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m}\left\{\begin{array}{rl} 2P_{\beta\beta}\xi_{\alpha\beta} & \text{if } k=\alpha,\ i=j=\beta,\\ P_{\beta j}\xi_{\alpha j} & \text{if } k=\alpha,\ i=\beta,\ j\neq\beta,\\ P_{i\beta}\xi_{\alpha i} & \text{if } k=\alpha,\ i\neq\beta,\ j=\beta\\ 0 & \text{if } k\neq\alpha \text{ or } \beta\notin\{i,j\} \end{array}\right\}\\
&\qquad + 2Q_{\alpha\beta}\\
&\quad= \sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\begin{array}{rl} 2P_{\beta\beta}\xi_{\alpha\beta} & \text{if } i=j=\beta,\\ P_{\beta j}\xi_{\alpha j} & \text{if } i=\beta,\ j\neq\beta,\\ P_{i\beta}\xi_{\alpha i} & \text{if } i\neq\beta,\ j=\beta\\ 0 & \beta\notin\{i,j\} \end{array}\right\}\\
&\qquad + 2Q_{\alpha\beta}\\
&\quad= \sum_{i=1}^{n} P_{i\beta}\xi_{\alpha i} + \sum_{j=1}^{n} P_{\beta j}\xi_{\alpha j} + 2Q_{\alpha\beta}\\
&\quad= 2\sum_{\ell=1}^{n}\xi_{\alpha\ell}P_{\ell\beta} + 2Q_{\alpha\beta}
\end{aligned}$$

$$= 2(\xi P + Q)_{\alpha\beta}.$$

This proves (3.41). Since we proved above that

$$\frac{\partial}{\partial \xi_{\alpha\beta}}\big(\mathrm{tr}(P\,{}^{\mathrm{t}}\xi\xi + 2\,{}^{\mathrm{t}}Q\xi)\big) = 2\sum_{\ell=1}^{n} P_{\ell\beta}\xi_{\alpha\ell} + 2Q_{\alpha\beta}$$

we also see that (3.42) holds. Finally, (3.43) follows from the identity

$$(\xi P + Q)_{\alpha\beta} = \sum_{\ell=1}^{n} P_{\ell\beta}\xi_{\alpha\ell} + Q_{\alpha\beta}$$

which we have already noted.

Let I be the set of all n-tuples $G = (g_1, \dots, g_n)$ where $g_1, \dots, g_n$ are integers such that $1 \le g_1 < g_2 \le \cdots < g_n \le m$. Let $G = (g_1, \dots, g_n) \in I$, and let X be an $m \times n$ matrix with entries from some commutative ring R. Write

$$X = \begin{bmatrix} X_1 \\ \vdots \\ X_m \end{bmatrix}$$

where each $X_i \in \mathrm{M}(1, n, R)$. Then

$$\begin{bmatrix} X_{g_1} \\ \cdots \\ X_{g_n} \end{bmatrix}$$

is an $n \times n$ matrix, and we define

$$X_G = \det(\begin{bmatrix} X_{g_1} \\ \cdots \\ X_{g_n} \end{bmatrix}).$$

By the Cauchy-Binet formula, we have

$$\det({}^{\mathrm{t}}\eta\partial) = \sum_{G\in I} \eta_G \partial_G.$$

We may further write, for $G \in I$,

$$\partial_G = \sum_{\sigma} \mathrm{sign}(\sigma) \frac{\partial}{\partial \xi_{g_1\sigma(g_1)}} \cdots \frac{\partial}{\partial \xi_{g_n\sigma(g_n)}},$$

where σ ranges over the permutations of the set $\{g_1, \dots, g_n\}$. The differential operator L is now given by the following formula:

$$L = \sum_{G\in I} \eta_G \sum_{\sigma} \mathrm{sign}(\sigma) \frac{\partial}{\partial \xi_{g_1\sigma(g_1)}} \cdots \frac{\partial}{\partial \xi_{g_n\sigma(g_n)}}.$$

It follows that:

$$
\begin{aligned}
L\Big(& \exp\big(\pi i \mathrm{tr}(P\,{}^{\mathrm{t}}\xi\xi + 2\,{}^{\mathrm{t}}Q\xi + R)\big)\Big) \\
&= \sum_{G\in I} \eta_G \sum_{\sigma} \mathrm{sign}(\sigma) \\
&\times \frac{\partial}{\partial \xi_{g_1\sigma(g_1)}} \cdots \frac{\partial}{\partial \xi_{g_n\sigma(g_n)}} \Big(\exp\big(\pi i \mathrm{tr}(P\,{}^{\mathrm{t}}\xi\xi + 2\,{}^{\mathrm{t}}Q\xi + R)\big)\Big) \\
&= 2\pi i \sum_{G\in I} \eta_G \sum_{\sigma} \mathrm{sign}(\sigma) \frac{\partial}{\partial \xi_{g_1\sigma(g_1)}} \cdots \frac{\partial}{\partial \xi_{g_{n-2}\sigma(g_{n-2})}} \\
&\times \frac{\partial}{\partial \xi_{g_{n-1}\sigma(g_{n-1})}} \Big((\xi P + Q)_{g_n\sigma(g_n)} \exp\big(\pi i \mathrm{tr}(P\,{}^{\mathrm{t}}\xi\xi + 2\,{}^{\mathrm{t}}Q\xi + R)\big)\Big)
\end{aligned}
$$

where we have used (3.41). Next, taking into account that $g_{n-1} \neq g_n$, using (3.42), and also (3.41) again, we have by the product rule:

$$
\begin{aligned}
L\Big(& \exp\big(\pi i \mathrm{tr}(P\,{}^{\mathrm{t}}\xi\xi + 2\,{}^{\mathrm{t}}Q\xi + R)\big)\Big) \\
&= (2\pi i)^2 \sum_{G\in I} \eta_G \sum_{\sigma} \mathrm{sign}(\sigma) \frac{\partial}{\partial \xi_{g_1\sigma(g_1)}} \cdots \frac{\partial}{\partial \xi_{g_{n-2}\sigma(g_{n-2})}} \\
&\Big((\xi P + Q)_{g_{n-1}\sigma(g_{n-1})} (\xi P + Q)_{g_n\sigma(g_n)} \exp\big(\pi i \mathrm{tr}(P\,{}^{\mathrm{t}}\xi\xi + 2\,{}^{\mathrm{t}}Q\xi + R)\big)\Big).
\end{aligned}
$$

Continuing, we obtain:

$$
\begin{aligned}
L\Big(& \exp\big(\pi i \mathrm{tr}(P\,{}^{\mathrm{t}}\xi\xi + 2\,{}^{\mathrm{t}}Q\xi + R)\big)\Big) \\
&= (2\pi i)^n \sum_{G\in I} \eta_G \sum_{\sigma} \mathrm{sign}(\sigma) \prod_{j=1}^{n} (\xi P + Q)_{g_j\sigma(g_j)} \\
&\times \exp\big(\pi i \mathrm{tr}(P\,{}^{\mathrm{t}}\xi\xi + 2\,{}^{\mathrm{t}}Q\xi + R)\big) \\
&= (2\pi i)^n \exp\big(\pi i \mathrm{tr}(P\,{}^{\mathrm{t}}\xi\xi + 2\,{}^{\mathrm{t}}Q\xi + R)\big) \\
&\times \sum_{G\in I} \eta_G \sum_{\sigma} \mathrm{sign}(\sigma) \prod_{j=1}^{n} (\xi P + Q)_{g_j\sigma(g_j)} \\
&= (2\pi i)^n \exp\big(\pi i \mathrm{tr}(P\,{}^{\mathrm{t}}\xi\xi + 2\,{}^{\mathrm{t}}Q\xi + R)\big) \sum_{G\in I} \eta_G (\xi P + Q)_G \\
&= (2\pi i)^n \exp\big(\pi i \mathrm{tr}(P\,{}^{\mathrm{t}}\xi\xi + 2\,{}^{\mathrm{t}}Q\xi + R)\big) \det({}^{\mathrm{t}}\eta(\xi P + Q)) \\
&= \det(2\pi i\,{}^{\mathrm{t}}\eta(\xi P + Q)) \exp\big(\pi i \mathrm{tr}(P\,{}^{\mathrm{t}}\xi\xi + 2\,{}^{\mathrm{t}}Q\xi + R)\big) \\
&= \det(2\pi i(P\,{}^{\mathrm{t}}\xi + {}^{\mathrm{t}}Q)\eta) \exp\big(\pi i \mathrm{tr}(P\,{}^{\mathrm{t}}\xi\xi + 2\,{}^{\mathrm{t}}Q\xi + R)\big).
\end{aligned}
$$

This proves (3.40) in the case $r = 1$. To prove that (3.40) holds for all positive integers r it will suffice to prove that if $f : \mathrm{M}(m, n, \mathbb{C}) \to \mathbb{C}$ is a smooth function, then

$$
L\Big(\det((P\,{}^{\mathrm{t}}\xi + {}^{\mathrm{t}}Q)\eta) f(\xi)\Big) = \det((P\,{}^{\mathrm{t}}\xi + {}^{\mathrm{t}}Q)\eta) L\big(f(\xi)\big). \tag{3.44}
$$

We first assert that if $\beta, \gamma, \mu, \lambda \in \{1, \dots, n\}$, then

$$(\sum_{i=1}^{m} \eta_{i\beta} \frac{\partial}{\partial \xi_{i\gamma}})\big(\sum_{\ell=1}^{m} (\xi P + Q)_{\ell\mu} \eta_{\ell\lambda}\big) = 0. \tag{3.45}$$

To see this, we calculate as follows:

$$\begin{aligned}
(\sum_{i=1}^{m} \eta_{i\beta} \frac{\partial}{\partial \xi_{i\gamma}})\big(\sum_{\ell=1}^{m} (\xi P + Q)_{\ell\mu} \eta_{\ell\lambda}\big) &= \sum_{i=1}^{m} \sum_{\ell=1}^{m} \eta_{i\beta} \eta_{\ell\lambda} \frac{\partial}{\partial \xi_{i\gamma}} \big((\xi P + Q)_{\ell\mu}\big) \\
&= \sum_{i=1}^{m} \eta_{i\beta} \eta_{\ell\lambda} P_{\gamma\mu} \qquad \text{(by (3.43))} \\
&= P_{\gamma\mu} \sum_{i=1}^{m} \eta_{i\beta} \eta_{i\lambda} \\
&= P_{\gamma\mu} ({}^{\mathrm{t}}\eta\eta)_{\beta\lambda} \\
&= 0
\end{aligned}$$

because ${}^{\mathrm{t}}\eta\eta = 0$ by assumption. We may write L as:

$$\begin{aligned}
L &= \det({}^{\mathrm{t}}n\partial) \\
&= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) ({}^{\mathrm{t}}\eta\partial)_{\sigma(1)1} \cdots ({}^{\mathrm{t}}\eta\partial)_{\sigma(n)n} \\
&= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{j=1}^{n} ({}^{\mathrm{t}}\eta\partial)_{\sigma(j)j} \\
&= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{j=1}^{n} \sum_{i=1}^{m} \eta_{i\sigma(j)} \frac{\partial}{\partial \xi_{ij}}.
\end{aligned}$$

We will apply this expression for L to $\det((P\,{}^{\mathrm{t}}\xi + {}^{\mathrm{t}}Q)\eta) f(\xi)$. To do this, we note first that $\det((P\,{}^{\mathrm{t}}\xi + {}^{\mathrm{t}}Q)\eta)$ is a sum of products of terms of the form

$$\sum_{\ell=1}^{m} (\xi P + Q)_{\ell\mu} \eta_{\ell\lambda}$$

for $\lambda, \mu \in \{1, \dots, n\}$. By (3.45), any such term is annihilated by

$$\sum_{i=1}^{m} \eta_{i\beta} \frac{\partial}{\partial \xi_{i\gamma}}$$

for any $\beta, \gamma \in \{1, \dots, n\}$. By this fact, and the product rule, we have

$$\Big(\sum_{i=1}^{m} \eta_{i\sigma(j)} \frac{\partial}{\partial \xi_{ij}}\Big)\big(\det((P\,{}^{\mathrm{t}}\xi + {}^{\mathrm{t}}Q)\eta) f(\xi)\big)$$

$$= \det((P\,{}^{\mathrm{t}}\xi + {}^{\mathrm{t}}Q)\eta)\Big(\sum_{i=1}^{m} \eta_{i\sigma(j)} \frac{\partial}{\partial \xi_{ij}}\Big)\big(f(\xi)\big).$$

We now find that

$$\begin{aligned} &L\Big(\det((P\,{}^{\mathrm{t}}\xi + {}^{\mathrm{t}}Q)\eta) f(\xi)\Big) \\ &\qquad = \det((P\,{}^{\mathrm{t}}\xi + {}^{\mathrm{t}}Q)\eta) \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \big(\prod_{j=1}^{n} \sum_{i=1}^{m} \eta_{i\sigma(j)} \frac{\partial}{\partial \xi_{ij}} \big)(f(\xi)) \\ &\quad = \det((P\,{}^{\mathrm{t}}\xi + {}^{\mathrm{t}}Q)\eta) L(f(\xi)). \end{aligned}$$

This proves (3.44), and thus completes the proof. ☐

Let m and n be positive integers, let r be a non-negative integer, and let $F \in \mathrm{Sym}(m, \mathbb{R})^+$. For r a non-negative integer, we let $\mathcal{H}_{r,n}(F)$ be the $\mathbb{C}$ vector space spanned by the polynomials

$$\det({}^{\mathrm{t}}XF\zeta)^r$$

where X is an $m \times n$ matrix of variables, and $\zeta \in \mathrm{M}(m, n, \mathbb{C})$ is such that

$${}^{\mathrm{t}}\zeta F \zeta = 0.$$

We refer to the elements of $\mathcal{H}_{r,n}(F)$ as spherical functions of degree n and weight r with respect to F.

Lemma 3.5.2. *Let m and n be positive integers, let r be a non-negative integer, and let $F \in \mathrm{Sym}(m, \mathbb{R})^+$. If $n > m$, then $\mathcal{H}_{r,n}(F) = 0$.*

Proof. Assume that $m > n$. Let $\zeta \in \mathrm{M}(m, n, \mathbb{C})$ be such that ${}^{\mathrm{t}}\zeta F \zeta = 0$. It will suffice to prove that the function $\mathrm{M}(m, n, \mathbb{C}) \to \mathbb{C}$ defined by $X \mapsto \det({}^{\mathrm{t}}XF\zeta)^r$ is identically zero. Let $X \in \mathrm{M}(m, n, \mathbb{C})$. The product ${}^{\mathrm{t}}XF\zeta$ is the matrix of the composition

$$\mathbb{C}^n \xrightarrow{\zeta} \mathbb{C}^m \xrightarrow{F} \mathbb{C}^m \xrightarrow{{}^{\mathrm{t}}X} \mathbb{C}^n.$$

Since $n > m$, the first operator in the composition is has a non-trivial kernel; hence, the composition also has a non-trivial kernel. This implies that $\det({}^{\mathrm{t}}XF\zeta) = 0$. ☐

Theorem 3.5.3. *Let m and n be positive inters, let r be a non-negative integer, and let $F \in \mathrm{Sym}(m, \mathbb{Z})^+$ be even. Let $\Phi \in \mathcal{H}_{r,n}(F)$. For $Z \in \mathbb{H}_n$ define*

$$\theta(F, Z, \Phi) = \sum_{N \in \mathrm{M}(m,n,\mathbb{Z})} \Phi(N) \exp\big(\pi i \mathrm{tr}(ZF[N])\big).$$

If D is a product of closed disks in $\mathbb{C}$ such that $D \subset \mathbb{H}_n$, then the series $\theta(F, Z, \Phi)$ converges absolutely and uniformly on D. The resulting function on $\mathbb{H}_n$ is analytic in each complex variable, and satisfies the equation

$$\det(CZ + D)^{-r} s(M, Z)^{-m} \theta(F, M \cdot Z, \Phi) = \chi(M)\theta(F, Z, \Phi)$$

for $Z \in \mathbb{H}_n$ and $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \Gamma_0(N)$. Here, $\chi : \Gamma_0(N) \to \mu_8$ is as in Theorem 3.3.5.

Proof. By Lemma 3.5.2 we may assume that $m \geq n$. We may also assume that $\Phi(X) = \det({}^{\mathrm{t}}XF\zeta)^r$ for some $\zeta \in \mathrm{M}(m,n,\mathbb{C})$ such that ${}^{\mathrm{t}}\zeta F\zeta = 0$. Let $E \in \mathrm{Sym}(m,\mathbb{R})^+$ be such that $E^2 = F$. Define $\eta = E\zeta$. Then ${}^{\mathrm{t}}\eta\eta = {}^{\mathrm{t}}\zeta E^2\zeta = {}^{\mathrm{t}}\zeta F\zeta = 0$. Also,

$$\begin{aligned}\Phi(X) &= \det({}^{\mathrm{t}}XF\zeta)^r \\ &= \det({}^{\mathrm{t}}XFE^{-1}\eta) \\ \Phi(X) &= \det({}^{\mathrm{t}}XE\eta). \end{aligned} \tag{3.46}$$

By Theorem 3.3.5 we have

$$\begin{aligned}&\theta(F, M\cdot Z, X\,{}^{\mathrm{t}}A + FY\,{}^{\mathrm{t}}B, F^{-1}X\,{}^{\mathrm{t}}C + Y\,{}^{\mathrm{t}}D) \\ &\qquad\qquad = \chi(M)s(M,Z)^m\theta(F,Z,X,Y)\end{aligned}$$

for $X, Y \in \mathrm{M}(m,n,\mathbb{C})$, $Z \in \mathbb{H}_n$, and $M = \left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right] \in \Gamma_0(N)$. Let $\xi \in \mathrm{M}(m,n,\mathbb{C})$ and $M = \left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right] \in \Gamma_0(N)$. Letting $X = 0$ and $Y = E^{-1}\xi$ in the last equation yields

$$\theta(F, M\cdot Z, E\xi\,{}^{\mathrm{t}}B, E^{-1}\xi\,{}^{\mathrm{t}}D) = \chi(M)s(M,Z)^m\theta(F,Z,0,E^{-1}\xi). \tag{3.47}$$

We consider each side of this equation. First of all,

$$\begin{aligned}
&\theta(F, M\cdot Z, E\xi\,{}^{\mathrm{t}}B, E^{-1}\xi\,{}^{\mathrm{t}}D) \\
&\quad = \sum_{N\in\mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i\mathrm{tr}((M\cdot Z)F[N - E^{-1}\xi\,{}^{\mathrm{t}}D]) \\
&\qquad + 2\pi i\mathrm{tr}({}^{\mathrm{t}}NE\xi\,{}^{\mathrm{t}}B) - \pi i\mathrm{tr}({}^{\mathrm{t}}(E\xi\,{}^{\mathrm{t}}B)E^{-1}\xi\,{}^{\mathrm{t}}D)\big) \\
&\quad = \sum_{N\in\mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i\mathrm{tr}((M\cdot Z)F[N - E^{-1}\xi\,{}^{\mathrm{t}}D]) \\
&\qquad + 2\mathrm{tr}({}^{\mathrm{t}}NE\xi\,{}^{\mathrm{t}}B) - \mathrm{tr}(B\,{}^{\mathrm{t}}\xi\xi\,{}^{\mathrm{t}}D)\big) \\
&\quad = \sum_{N\in\mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i\mathrm{tr}((M\cdot Z)\,{}^{\mathrm{t}}(N - E^{-1}\xi\,{}^{\mathrm{t}}D)F(N - E^{-1}\xi\,{}^{\mathrm{t}}D)) \\
&\qquad + 2\pi i\mathrm{tr}({}^{\mathrm{t}}NE\xi\,{}^{\mathrm{t}}B) - \pi i\mathrm{tr}(B\,{}^{\mathrm{t}}\xi\xi\,{}^{\mathrm{t}}D)\big) \\
&\quad = \sum_{N\in\mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i\mathrm{tr}((M\cdot Z)({}^{\mathrm{t}}NFN - {}^{\mathrm{t}}NE\xi\,{}^{\mathrm{t}}D - D\,{}^{\mathrm{t}}\xi EN + D\,{}^{\mathrm{t}}\xi\xi\,{}^{\mathrm{t}}D)) \\
&\qquad + 2\pi i\mathrm{tr}({}^{\mathrm{t}}NE\xi\,{}^{\mathrm{t}}B) - \pi i\mathrm{tr}(B\,{}^{\mathrm{t}}\xi\xi\,{}^{\mathrm{t}}D)\big) \\
&\quad = \sum_{N\in\mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i\mathrm{tr}((M\cdot Z)D\,{}^{\mathrm{t}}\xi\xi\,{}^{\mathrm{t}}D) - \pi i\mathrm{tr}(B\,{}^{\mathrm{t}}\xi\xi\,{}^{\mathrm{t}}D) \\
&\qquad - \pi i\mathrm{tr}((M\cdot Z)\,{}^{\mathrm{t}}NE\xi\,{}^{\mathrm{t}}D) - \pi i\mathrm{tr}((M\cdot Z)D\,{}^{\mathrm{t}}\xi EN) + 2\pi i\mathrm{tr}({}^{\mathrm{t}}NE\xi\,{}^{\mathrm{t}}B) \\
&\qquad + \pi i\mathrm{tr}((M\cdot Z)\,{}^{\mathrm{t}}NFN)\big) \\
&\quad = \sum_{N\in\mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i\mathrm{tr}({}^{\mathrm{t}}D(M\cdot Z)D\,{}^{\mathrm{t}}\xi\xi) - \pi i\mathrm{tr}({}^{\mathrm{t}}DB\,{}^{\mathrm{t}}\xi\xi)
\end{aligned}$$

$$
\begin{aligned}
&- \pi i \mathrm{tr}({}^{\mathrm{t}}D(M\cdot Z)\,{}^{\mathrm{t}}NE\xi) - \pi i \mathrm{tr}({}^{\mathrm{t}}NE\xi\,{}^{\mathrm{t}}D(M\cdot Z)) + 2\pi i \mathrm{tr}({}^{\mathrm{t}}B\,{}^{\mathrm{t}}NE\xi) \\
&+ \pi i \mathrm{tr}((M\cdot Z)\,{}^{\mathrm{t}}NFN)\big) \\
= &\sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(({}^{\mathrm{t}}D((M\cdot Z)D-B)\,{}^{\mathrm{t}}\xi\xi) \\
&- \pi i \mathrm{tr}({}^{\mathrm{t}}D(M\cdot Z)\,{}^{\mathrm{t}}NE\xi) - \pi i \mathrm{tr}({}^{\mathrm{t}}D(M\cdot Z)\,{}^{\mathrm{t}}NE\xi) + 2\pi i \mathrm{tr}({}^{\mathrm{t}}B\,{}^{\mathrm{t}}NE\xi) \\
&+ \pi i \mathrm{tr}((M\cdot Z)\,{}^{\mathrm{t}}NFN)\big) \\
= &\sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(({}^{\mathrm{t}}D((M\cdot Z)D-B)\,{}^{\mathrm{t}}\xi\xi) \\
&- 2\pi i \mathrm{tr}(({}^{\mathrm{t}}D(M\cdot Z) - {}^{\mathrm{t}}B)\,{}^{\mathrm{t}}NE\xi) + \pi i \mathrm{tr}((M\cdot Z)\,{}^{\mathrm{t}}NFN)\big).
\end{aligned}
$$

Now

$$
\begin{aligned}
{}^{\mathrm{t}}D((M\cdot Z)D-B) &= {}^{\mathrm{t}}D(M\cdot Z)D - {}^{\mathrm{t}}DB \\
&= {}^{\mathrm{t}}D(AZ+B)(CZ+D)^{-1}D - {}^{\mathrm{t}}BD \\
&= \big({}^{\mathrm{t}}D(AZ+B)(CZ+D)^{-1} - {}^{\mathrm{t}}B\big)D \\
&= \big({}^{\mathrm{t}}D(AZ+B) - {}^{\mathrm{t}}B(CZ+D)\big)(CZ+D)^{-1}D \\
&= \big({}^{\mathrm{t}}DAZ + {}^{\mathrm{t}}DB - {}^{\mathrm{t}}BCZ - {}^{\mathrm{t}}BD\big)(CZ+D)^{-1}D \\
&= \big(({}^{\mathrm{t}}DA - {}^{\mathrm{t}}BC)Z + {}^{\mathrm{t}}DB - {}^{\mathrm{t}}BD\big)(CZ+D)^{-1}D \\
&= Z(CZ+D)^{-1}D.
\end{aligned}
$$

We also note that $Z(CZ+D)^{-1}D$ is symmetric because it is equal to the symmetric matrix ${}^{\mathrm{t}}D(M\cdot Z)D - {}^{\mathrm{t}}DB$. And

$$
\begin{aligned}
{}^{\mathrm{t}}D(M\cdot Z) - {}^{\mathrm{t}}B &= {}^{\mathrm{t}}D(AZ+B)(CZ+D)^{-1} - {}^{\mathrm{t}}B \\
&= \big({}^{\mathrm{t}}D(AZ+B) - {}^{\mathrm{t}}B(CZ+D)\big)(CZ+D)^{-1} \\
&= \big({}^{\mathrm{t}}DAZ + {}^{\mathrm{t}}DB - {}^{\mathrm{t}}BCZ - {}^{\mathrm{t}}BD\big)(CZ+D)^{-1} \\
&= Z(CZ+D)^{-1}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\theta(F, M\cdot Z, E\xi\,{}^{\mathrm{t}}B, E^{-1}\xi\,{}^{\mathrm{t}}D) \\
&\quad = \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(Z(CZ+D)^{-1}D\,{}^{\mathrm{t}}\xi\xi) \\
&\quad\quad - 2\pi i \mathrm{tr}(Z(CZ+D)^{-1}\,{}^{\mathrm{t}}NE\xi) + \pi i \mathrm{tr}((M\cdot Z)\,{}^{\mathrm{t}}NFN)\big) \\
&\quad = \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(Z(CZ+D)^{-1}D\,{}^{\mathrm{t}}\xi\xi \\
&\quad\quad - 2Z(CZ+D)^{-1}\,{}^{\mathrm{t}}NE\xi + (M\cdot Z)\,{}^{\mathrm{t}}NFN)\big).
\end{aligned}
$$

Next,

$$
\theta(F, Z, 0, E^{-1}\xi)
$$

$$\begin{aligned}
&= \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(ZF[N - E^{-1}\xi])\big) \\
&= \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(Z\,{}^{\mathrm{t}}\xi\xi - Z\,{}^{\mathrm{t}}NE\xi - Z\,{}^{\mathrm{t}}\xi EN + Z\,{}^{\mathrm{t}}NFN)\big) \\
&= \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(Z\,{}^{\mathrm{t}}\xi\xi) - \pi i \mathrm{tr}(Z\,{}^{\mathrm{t}}NE\xi) - \pi i \mathrm{tr}(Z\,{}^{\mathrm{t}}\xi EN) \\
&\quad + \pi i \mathrm{tr}(Z\,{}^{\mathrm{t}}NFN)\big) \\
&= \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(Z\,{}^{\mathrm{t}}\xi\xi) - \pi i \mathrm{tr}(Z\,{}^{\mathrm{t}}NE\xi) - \pi i \mathrm{tr}({}^{\mathrm{t}}\xi ENZ) \\
&\quad + \pi i \mathrm{tr}(Z\,{}^{\mathrm{t}}NFN)\big) \\
&= \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(Z\,{}^{\mathrm{t}}\xi\xi) - \pi i \mathrm{tr}(Z\,{}^{\mathrm{t}}NE\xi) - \pi i \mathrm{tr}(Z\,{}^{\mathrm{t}}NE\xi) \\
&\quad + \pi i \mathrm{tr}(Z\,{}^{\mathrm{t}}NFN)\big) \\
&= \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \exp\big(\pi i \mathrm{tr}(Z\,{}^{\mathrm{t}}\xi\xi - 2Z\,{}^{\mathrm{t}}NE\xi + Z\,{}^{\mathrm{t}}NFN)\big).
\end{aligned}$$

We will now apply the differential operator L^r from Lemma 3.5.1 to both sides of (3.47). Because of the convergence properties of Proposition 3.1.8 we may exchange differentiation and summation (see p. 162 of [17]). By Lemma 3.5.1 we have

$$\begin{aligned}
&L^r\Big(\theta(F, M\cdot Z, E\xi\,{}^{\mathrm{t}}B, E^{-1}\xi\,{}^{\mathrm{t}}D)\Big) \\
&\quad= \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} L^r\Big(\exp\big(\pi i \mathrm{tr}(Z(CZ+D)^{-1}D\,{}^{\mathrm{t}}\xi\xi \\
&\quad\quad - 2Z(CZ+D)^{-1}\,{}^{\mathrm{t}}NE\xi + (M\cdot Z)\,{}^{\mathrm{t}}NFN)\big)\Big) \\
&\quad= \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \det(2\pi i(Z(CZ+D)^{-1}D\,{}^{\mathrm{t}}\xi - Z(CZ+D)^{-1}\,{}^{\mathrm{t}}NE)\eta)^r \\
&\quad\quad \times \exp\big(\pi i \mathrm{tr}(Z(CZ+D)^{-1}D\,{}^{\mathrm{t}}\xi\xi \\
&\quad\quad - 2Z(CZ+D)^{-1}\,{}^{\mathrm{t}}NE\xi + (M\cdot Z)\,{}^{\mathrm{t}}NFN)\big).
\end{aligned}$$

Evaluating at $\xi = 0$, we get

$$\begin{aligned}
&L^r\Big(\theta(F, M\cdot Z, E\xi\,{}^{\mathrm{t}}B, E^{-1}\xi\,{}^{\mathrm{t}}D)\Big)|_{\xi=0} \\
&\quad= \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \det(2\pi i(-Z(CZ+D)^{-1}\,{}^{\mathrm{t}}NE)\eta)^r \\
&\quad\quad \times \exp\big(\pi i \mathrm{tr}((M\cdot Z)\,{}^{\mathrm{t}}NFN)\big) \\
&\quad= \det(-2\pi i Z(CZ+D)^{-1})^r \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \det({}^{\mathrm{t}}NE\eta)^r \\
&\quad\quad \times \exp\big(\pi i \mathrm{tr}((M\cdot Z)F[N])\big).
\end{aligned}$$

And

$$\begin{aligned}
&L^r\Big(\theta(F,Z,0,E^{-1}\xi)\Big) \\
&\quad = \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} L^r\Big(\exp\big(\pi i\mathrm{tr}(Z\,{}^{\mathrm{t}}\xi\xi - 2Z\,{}^{\mathrm{t}}NE\xi + Z\,{}^{\mathrm{t}}NFN)\big)\Big) \\
&\quad = \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \det(2\pi i(Z\,{}^{\mathrm{t}}\xi - Z\,{}^{\mathrm{t}}NE)\eta)^r \\
&\qquad\quad \times \exp\big(\pi i\mathrm{tr}(Z\,{}^{\mathrm{t}}\xi\xi - 2Z\,{}^{\mathrm{t}}NE\xi + Z\,{}^{\mathrm{t}}NFN)\big).
\end{aligned}$$

Evaluating at $\xi = 0$, we obtain:

$$\begin{aligned}
&L^r\Big(\theta(F,Z,0,E^{-1}\xi)\Big)|_{\xi=0} \\
&\quad = \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \det(2\pi i(-Z\,{}^{\mathrm{t}}NE)\eta)^r \exp\big(\pi i\mathrm{tr}(Z\,{}^{\mathrm{t}}NFN)\big) \\
&\quad = \det(-2\pi i Z)^r \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \det({}^{\mathrm{t}}NE\eta)^r \exp\big(\pi i\mathrm{tr}(ZF[N])\big).
\end{aligned}$$

By (3.47) we now have

$$\begin{aligned}
&\det(-2\pi i Z(CZ+D)^{-1})^r \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \det({}^{\mathrm{t}}NE\eta)^r \exp\big(\pi i\mathrm{tr}((M\cdot Z)F[N])\big) \\
&\quad = \det(-2\pi i Z)^r \chi(M) s(M,Z)^m \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \det({}^{\mathrm{t}}NE\eta)^r \exp\big(\pi i\mathrm{tr}(ZF[N])\big)
\end{aligned}$$

so that by (3.46),

$$\begin{aligned}
&\sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \Phi(N) \exp\big(\pi i\mathrm{tr}((M\cdot Z)F[N])\big) \\
&\quad = \chi(M)\det(CZ+D)^r s(M,Z)^m \sum_{N\in \mathrm{M}(m,n,\mathbb{Z})} \Phi(N)\exp\big(\pi i\mathrm{tr}(ZF[N])\big).
\end{aligned}$$

This proves the theorem. $\square$

Appendix A

Some tables

A.1 Tables of fundamental discriminants

$-3 = -3$	$-35 = (-7) \cdot 5$	$-68 = (-4) \cdot 17$
$-4 = -4$	$-39 = (-3) \cdot 13$	$-71 = -71$
$-7 = -7$	$-40 = (-8) \cdot 5$	$-79 = -79$
$-8 = -8$	$-43 = -43$	$-83 = -83$
$-11 = -11$	$-47 = -47$	$-84 = (-4) \cdot (-3) \cdot (-7)$
$-15 = (-3) \cdot 5$	$-51 = (-3) \cdot 17$	$-87 = (-3) \cdot 29$
$-19 = -19$	$-52 = (-4) \cdot 13$	$-88 = (-11) \cdot 8$
$-20 = (-4) \cdot 5$	$-55 = (-11) \cdot 5$	$-91 = (-7) \cdot 13$
$-23 = -23$	$-56 = (-7) \cdot 8$	$-95 = (-19) \cdot 5$
$-24 = (-3) \cdot 8$	$-59 = -59$	
$-31 = -31$	$-67 = -67$	

Table A.1: Negative fundamental discriminants between -1 and -100, factored into products of prime fundamental discriminants.

$1 = 1$	$37 = 37$	$73 = 73$
$5 = 1$	$40 = 8 \cdot 5$	$76 = (-4) \cdot (-19)$
$8 = 8$	$41 = 41$	$77 = (-7) \cdot (-11)$
$12 = (-4)(-3)$	$44 = (-4) \cdot (-11)$	$85 = 5 \cdot 15$
$13 = 13$	$53 = 53$	$88 = (-8) \cdot (-11)$
$17 = 17$	$56 = (-8) \cdot (-7)$	$89 = 89$
$21 = (-3)(-7)$	$57 = 57$	$92 = (-4) \cdot (-23)$
$24 = (-8)(-3)$	$60 = (-4) \cdot (-3) \cdot 5$	$93 = (-3) \cdot (-31)$
$28 = (-4)(-7)$	$61 = 61$	$97 = 97$
$29 = 29$	$65 = (-8) \cdot (-7)$	
$33 = 33$	$69 = (-3)(-23)$	

Table A.2: Positive fundamental discriminants between 1 and 100, factored into products of prime fundamental discriminants.

Index

Symbols

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